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QUASI BI-SLANT SUBMANIFOLD OF TRANS-SASAKIAN MANIFOLD

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Abstract *This paper explores the concept of quasi-bi-slant submanifolds within the framework of trans-Sasakian manifolds, a significant class of almost contact metric manifolds characterized by a pair of smooth functions* (α, β). *A quasi-bi-slant submanifold is defined by the existence of three orthogonal distributions, with two being slant distributions having distinct slant angles, and the third being invariant under the structure endomorphism. We derive necessary and sufficient conditions for the existence of such submanifolds and examine their geometric properties, including the integrability of the distributions involved.An example is provided to illustrate the theoretical concepts and validate our results. This study extends the understanding of slant and bi-slant submanifolds and contributes to the broader field of contact geometry, providing fresh insights and applications in both mathematics and theoretical physics.*

1 Introduction

The study of submanifolds within the realm of differential geometry has long been a subject of profound interest, particularly due to its implications in understanding the geometric and topological properties of the ambient manifolds. Submanifolds provide essential insights into the curvature, structure, and various intrinsic and extrinsic properties of manifolds. Among the numerous types of manifolds studied, almost contact metric manifolds, and their generalizations, play a pivotal role due to their rich geometric structures and wide range of applications [15, 18, 19].

Trans-Sasakian manifolds, introduced as a generalization of both Sasakian and cosymplectic manifolds, are defined by the presence of a pair of smooth functions (α, β) *[20]. These manifolds exhibit a versatile geometric framework, encompassing various interesting special cases and providing a fertile ground for exploring new types of submanifolds. The structure of trans-Sasakian manifolds makes them particularly well-suited for exploring complex submanifold geometries, including slant, semi-slant, and bi-slant submanifolds [8, 10].*

The concept of slant submanifolds, initially introduced in almost Hermitian geometry, has been extended to the setting of almost contact metric manifolds [5]. A slant submanifold is characterized by a constant angle, called the slant angle, between the structure vector field and the tangent bundle of the submanifold [8, 10, 12, 13]. Bi-slant submanifolds further generalize this idea by allowing two distinct slant angles. In this context, we will be studying a new class of submanifolds termed quasi-bi-slant submanifolds, which are characterized by the presence of three orthogonal distributions: two slant distributions with distinct slant angles and a third distribution that is invariant under the structure endomorphism [1, 7, 17].

In this paper we will study quasi-bi-slant submanifolds within the framework of trans-Sasakian manifolds and to explore their geometric properties. We derive the necessary and sufficient conditions for the existence of these submanifolds and investigate the integrability of the distributions involved. Our study reveals intricate relationships between the geometric structures of *the submanifold and the ambient trans-Sasakian manifold.*

In the following sections, we begin with a review of the relevant background on trans-Sasakian manifolds and slant submanifolds. We then explore the concept of quasi-bi-slant submanifolds and establish the theoretical framework for their study. Next, we derive the conditions for their existence and explore their geometric properties. Finally, we conclude with a discussion of our findings and potential directions for future research in this area.

2 Preliminaries

Consider $\bar{M}$ be a $(2n+1)$-dim almost contact metric manifold with tensor field ϕ on tangent space and Character vector field ξ and η be the 1-form satisfying

$$\eta(\xi) = \varepsilon = 1, \phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$\begin{aligned} g(\xi, \xi) &= 1 = \varepsilon, \\ \eta(X) = g(X, \xi) &= \varepsilon\eta(X), d\eta(X, Y) = g(X, \phi Y), \\ g(\phi X, Y) &= -g(X, \phi Y), \\ \phi.\xi &= 0, \eta o \phi = 0, \end{aligned} \tag{2.3}$$

for any vector field $X, Y \in \Gamma(T\tilde{M})$, the algebra of vector field on $\bar{M}$.
The manifold $\bar{M}$ with structure (ϕ, ξ, η, g) is called almost contact metric manifold.
A Trans-Sasakian manifold is a normal contact metric manifold of type (α, β) if $\forall\ X, Y \in \Gamma(T\bar{M})$

$$\begin{aligned} (\bar{\nabla}_X \phi)Y &= \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \\ \bar{\nabla}_X \xi &= -\alpha(\phi X) + \beta[X - \eta(X)\xi] \end{aligned} \tag{2.4}$$

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and same symbol g represent induced Riemannian metric on M.
Let A, h denote the shape operator and second fundamental form respectively of immersion of M into $\bar{M}$.
The Gauss ,Weingarten formula are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.6}$$

for Vector fields $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$
∇: induced connection on M
$\nabla^{\perp}$: connection on normal bundle $T^{\perp}M$ of M
A_V: shape operator of M with respect to $V \in \Gamma(T^{\perp}M)$
h: second fundamental form
$h : TM \otimes TM \to T^{\perp}M$ of M into $\bar{M}$ satisfies

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.7}$$

for Vector field $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}\bar{M})$.
The mean curvature vector is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \tag{2.8}$$

$n = \dim(M)$ $\{e_1, e_2, \ldots, e_n\}$ be local orthonormal basis of $\Gamma(TM)$.
A submanifold M of Trans-Sasakian manifold $\bar{M}$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H. \tag{2.9}$$

If $h(X, Y) = 0\ \forall X, Y \in \Gamma(TM)$ then M is totally geodesic.
If $H = 0$ then M is said to be a minimal submanifold.

3 Quasi Bi-Slant Submanifolds of Trans-Sasakian Manifold

***Definition 3.1:**[1] A Quasi-bi-slant submanifold M of an almost contact metric manifold $\bar{M}$ is the one which admits three orthogonal complementary distributions D, D_1 and D_2 at point $p \in M$ such that:*

(i) TM admits orthogonal direct decomposition

$$TM = D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle.$$

(ii) The distribution D is invariant, i.e., $\phi D = D$.

(iii) $\phi(D_1) \perp D_2$, $\phi(D_2) \perp D_1$.

(iv) The distributions D_1 and D_2 are slant with slant angle θ_1, θ_2, respectively.

Let $\dim(D) = m_1$, $\dim(D_1) = m_2$, $\dim(D_2) = m_3$. Then:

(a) M is invariant if $m_1 \neq 0$, $m_2 = m_3 = 0$.

(b) M is anti-invariant if $m_1 = m_3 = 0$,$m_2 \neq 0$ $\theta_1 = \frac{\pi}{2}$.

(c) M is semi-invariant if $m_1, m_3 \neq 0$, $m_2 = 0$, $\theta_1 = \frac{\pi}{2}$.

(d) M is Slant with angle θ_1 if $m_1 = m_3 = 0, m_2 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$.

(e) M is Semi-slant if $m_1 \neq 0$, $m_2 \neq 0$, $m_3 \neq 0$, $\theta_1 = 0$, $0 < \theta_2 < \frac{\pi}{2}$.

(f) M is Hemi-slant if $m_1 = 0, m_2, m_3 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$.

(g) M is Bi-slant if $m_1 = 0, m_2, m_3 \neq 0$, $0 < \theta_1$, $\theta_2 < \frac{\pi}{2}$.

(h) M is Quasi-hemi-slant if $m_1, m_2, m_3 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$.

(i) M is Proper quasi-bi-slant if $m_1, m_2, m_3 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$,$0 < \theta_2 < \frac{\pi}{2}$.

Now, we construct an example of a Quasi-Bi-Slant submanifold of a trans-Sasakian manifold. Let $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$ dimensional Euclidean space endowed with the almost contact metric structure (ϕ, ξ, η, g) defined by:

$$\phi(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n, t) = (y^1, y^2, y^3, \ldots, y^n, -x^1, -x^2, -x^3, \ldots, -x^n, 0)$$

$$\xi = e^t \frac{\partial}{\partial t}, \quad \eta = e^{-t} dt, \quad g = e^{-2t} K$$

where $(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n, t)$ are Cartesian coordinates and K is the Euclidean Riemannian metric on $\mathbb{R}^{2n+1}$. Then (ϕ, ξ, η, g) is a trans-Sasakian structure on $\mathbb{R}^{2n+1}$ which is neither cosymplectic nor Sasakian.

Example 3.1. For $\theta_1 \in \left(0, \frac{\pi}{2}\right)$ and $\theta_2 \in \left(0, \frac{\pi}{2}\right)$, we take:

$$x^1 = s, \quad x^2 = -v, \quad x^3 = u \sin \omega, \quad x^4 = \sin u, \quad x^5 = r$$

$$y^1 = v, \quad y^2 = s, \quad y^3 = u \cos \omega, \quad y^4 = \cos u, \quad y^5 = p \cos \alpha, \quad y^6 = p \sin \alpha.$$

This defines a 7-dimensional submanifold M of $\mathbb{R}^{13}$ with the trans-Sasakian structure described above. Further,

$$z_1 = e^t\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}\right),$$
$$z_2 = e^t\left(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}\right),$$
$$z_3 = e^t\left(\sin\omega\frac{\partial}{\partial x_3} + \cos u\frac{\partial}{\partial x_4} + \cos\omega\frac{\partial}{\partial y_3} - \sin u\frac{\partial}{\partial y_4}\right),$$
$$z_4 = e^t\left(a\cos\omega\frac{\partial}{\partial x_3} - u\sin\omega\frac{\partial}{\partial y_3}\right),$$
$$z_5 = e^t\left(\frac{\partial}{\partial x_5}\right),$$
$$z_6 = e^t\left(\cos\alpha\frac{\partial}{\partial y_5} + \sin\alpha\frac{\partial}{\partial y_6}\right),$$
$$z_7 = \frac{\partial}{\partial t},$$

because $\phi Z_1 = Z_2$ and $\phi Z_2 = -Z_1$.

Here D is spanned by $\{z_1, z_2\}$,D_1 is spanned by $\{z_3, z_4\}$ with slant angle $\theta_1 = \frac{\pi}{4}$,D_2 is spanned by $\{z_5, z_6\}$ with slant angle $\theta_2 = \alpha$.

Therefore, M is a Quasi-Bi-Slant Submanifold of trans-Sasakian Manifold $(\mathbb{R}^{13}, \phi, \xi, \eta, g)$.

Let M be a quasi-bi-slant submanifold of a Trans-Sasakian manifold $\bar{M}$. Denote projection of $X \in \Gamma(TM)$ on D, D_1, D_2 by P, Q, R, respectively.
Then for any $X \in \Gamma(TM)$,

$$X = PX + QX + RX + \eta(X)\xi. \tag{3.1}$$
$$\phi X = TX + NX. \tag{3.2}$$

where TX, NX are respectively the tangential and normal parts of ϕX on M.
Using (3.1) and (3.2) we will get ,

$$\phi X = TPX + TQX + TRX + NPX + NQX + NRX. \tag{3.3}$$

Now since $\phi(D) = D$, so $NPX = 0$ and we have from (3.3), we get

$$\phi X = TPX + TQX + TRX + NQX + NRX. \tag{3.4}$$

by comparing the tangential and normals parts we get

$$TX = TPX + TQX + TRX,$$
$$NX = NQX + NRX.$$

Thus we have

$$\phi(TM) = D \oplus TD_1 \oplus TD_2 \oplus ND_1 \oplus ND_2. \tag{3.5}$$

$\oplus$ denotes the orthogonal direct sum.
Now, since $ND_1 \subset \Gamma(T^\perp \bar{M})$ and $ND_2 \subset \Gamma(T^\perp \bar{M})$ so we have

$$T^\perp M = ND_1 \oplus ND_2 \oplus \mu \tag{3.6}$$

where μ denotes orthogonal component of $ND_1 \oplus ND_2$ in $\Gamma(T^\perp M)$.
Now for any non zero $V \in \Gamma(T^\perp M)$,

$$\phi V = tV + nV \tag{3.7}$$

$tV \in \Gamma(TM) \quad , nV \in \Gamma(T^\perp M)$.

Lemma 3.2. *Let M be a Quasi-bi-slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α,β). Then*

$$TD = D, \quad TD_1 \subset D_1, \quad TD_2 \subset D_2, \qquad (3.8)$$
$$tND_1 \subset D_1, \quad tND_2 \subset D_2.$$

Lemma 3.3. *Let M be a submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α,β). Then for any $X,Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$, we have:*

$$\bar{\nabla}_X TY - A_{NY}X - th(X,Y) = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)T(X)],$$
$$\bar{\nabla}_X NY + h(X,TY) - nh(X,Y) = -\beta\eta(Y)NX,$$
$$and$$
$$(\bar{\nabla}_X t)V - A_{nV}X + TA_V X = \beta[g(NX,V)]\xi,$$
$$(\bar{\nabla}_X n)V + h(X,tV) + NA_V X = 0.$$

Proof. Since $\bar{M}$ is Trans-Sasakian manifold

$$(\bar{\nabla}_X\phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi(X)].$$

Using equations (2.5), (2.6), (3.2) and (3.7), we get for any $Y \in \Gamma(TM)$,

$$\phi Y = TY + NY$$

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X(TY) + \bar{\nabla}_X(NY)$$

$$\bar{\nabla}_X\phi Y + \phi\bar{\nabla}_X Y = \nabla_X TY + h(X,TY) - A_{NY}X + \nabla_X^{\perp}NY$$

$$\alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X] + \phi(\nabla_X Y + h(X,Y))$$
$$= \nabla_X TY + h(X,TY) - A_{NY}X + \nabla_X^{\perp}NY$$

$$\alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X] + \phi(\nabla_X Y) + \phi(h(X,Y))$$
$$= \nabla_X TY + h(X,TY) - A_{NY}X + \nabla_X^{\perp}NY$$

$$\alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi + g(NX,Y)\xi - \eta(Y)TX - \eta(Y)NX]$$
$$+T\nabla_X Y + N\nabla_X Y + th(X,Y) + nh(X,Y) = \nabla_X TY + h(X,TY) - A_{NY}X + \nabla_X^{\perp}NY.$$

Comparing tangential and normal parts:

$$\alpha[g(X,Y)\xi - \eta(Y)]X + \beta[g(TX,Y)\xi - \eta(Y)TX)] = \nabla_X TY - T\nabla_X Y - A_{NY}X - th(X,Y)$$

and

$$h(X,TY) + \nabla_X^{\perp}NY - N(\nabla_X Y) - nh(X,Y) = -\beta\eta(Y)NX$$

So we have

$$(\bar{\nabla}_X T)Y - A_{NY}X - th(X,Y) = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)T(X)].$$

and

$$(\bar{\nabla}_X N)Y + h(X,TY) - nh(X,Y) = -\beta\eta(Y)NX.$$

Similarly, for $V \in \Gamma(T^{\perp}M)$: $\phi V = tV + nV$

$$\bar{\nabla}_X(\phi V) = \bar{\nabla}_X tV + \bar{\nabla}_X nV.$$

$$(\bar{\nabla}_X\phi)V + \phi\bar{\nabla}_X V = \nabla_X tV + h(X, tV) - A_{nV}X + \nabla^{\perp}_X nV.$$

$$\alpha[g(X,V)\xi - \eta(V)X] + \beta[g(\phi X,V)\xi + \eta(V)\phi X] - TA_VX - NA_VX + t\nabla^{\perp}_X V + n\nabla^{\perp}_X V$$
$$= \nabla_X tV + h(X, tV) - A_{nV}X + \nabla^{\perp}_X nV$$

$$\beta[g(NX,V)\xi] - TA_VX - NA_VX + t\nabla^{\perp}_X V + n\nabla^{\perp}_X V = \nabla_X tV + h(X,tV) - A_{nV}X + \nabla^{\perp}_X nV$$

$$\nabla_X tV - t\nabla^{\perp}_X V - A_{nV}X + TA_VX = \beta[g(NX,V)\xi]$$

and

$$h(X,TV) + \nabla^{\perp}_X nV - n\nabla^{\perp}_X V + NA_VX = 0$$

and we have the result

$$(\bar{\nabla}_X t)V - A_{nV}X + TA_VX = \beta[g(NX,V)]\xi$$

$$(\bar{\nabla}_X n)V + h(X,tV) + NA_VX = 0.$$

□

Now considering equations(3.2), (3.7) and $\phi^2 = -I + \eta \otimes \xi$, we have the following lemma.

Lemma 3.4. *Let M be a Quasi-bi-slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the endomorphism T and N (respectively t and n) in tangent bundle (normal bundle) of M satisfies :*

(i) $T^2 + tN = -I + \eta \otimes \xi$ *on* TM

(ii) $NT + nN = 0$ *on* TM

(iii) $Nt + n^2 = -I$ *on* $T^{\perp}M$

(iv) $Tt + tn = 0$ *on* $T^{\perp}M$.

Proof. In TM:

$$\phi = T + N$$

$$\phi^2 = \phi(T + N) = \phi(T) + \phi(N)$$

$$-I + \eta \otimes \xi = T^2 + NT + tN + nN.$$

Comparing tangential and normal parts:

$$T^2 + tN = -I + \eta \otimes \xi, \quad NT + nN = 0 \quad \text{on } TM.$$

and in $T^{\perp}M$:

$$\phi = t + n$$

$$\phi^2 = \phi(t + n) = \phi(t) + \phi(n)$$

$$-I + \eta \otimes \xi = Tt + Nt + n^2 + tn.$$

Comparing tangential and normal parts:

$$Nt + n^2 = -I \quad \text{on } T^{\perp}M$$

$$Tt + tn = 0.$$

□

Lemma 3.5. *Let M be a Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then $X, Y \in D_i; i = 1, 2$*

$$T^2X = -(\cos^2\theta_i)X$$

$$g(TX, TY) = \cos^2\theta_i g(X, Y)$$

$$g(NX, NY) = \sin^2\theta_i g(X, Y).$$

Proof. For $X \in \Gamma(TM)$, We know,

$$\cos\theta = \frac{|TX|}{|\phi X|}$$

and

$$\cos\theta = \frac{g(\phi X, TX)}{|\phi X||TX|} = \frac{g(X, T^2X)}{|\phi X||TX|}.$$

Multiplying the above two equations we get

$$\cos^2\theta = \frac{g(\phi X, TX)}{|\phi X||TX|}.\frac{|TX|}{|\phi X|} = \frac{-g(X, T^2X)}{|X|\cdot|TX|}.\frac{|TX|}{|\phi X|} = \frac{-g(X, T^2X)}{|\phi X||X|}$$

$$= \frac{-g(X, T^2X)}{|X|^2}$$

$$\cos^2\theta g(X, X) = -g(T^2X, X)$$

$$\cos^2\theta X = -T^2X \quad \Rightarrow T^2X = -\cos^2\theta_i X; i = 1, 2. \tag{3.9}$$

Taking Inner Product with Y in(3.9), we get

$$\cos^2\theta g(X, Y) = -g(T^2X, Y) \quad \Rightarrow \cos^2\theta g(X, Y) = g(TX, TY) \tag{3.10}$$

$$g(TX, TY) = \cos^2\theta g(X, Y) \tag{3.11}$$

$$g(\phi X - NX, \phi Y - NY) = \cos^2\theta g(X, Y)$$

$$g(\phi X, \phi Y) - g(\phi X, NY) - g(NX, \phi Y) + g(NX, NY) = \cos^2\theta g(X, Y)$$

$$g(X, Y) - \cos^2\theta g(X, Y) = -g(NX, NY)$$

$$1 - \cos^2\theta g(X, Y) = -g(NX, NY)$$

$$\sin^2\theta g(X, Y) = g(NX, NY).$$

□

Lemma 3.6. *Let M be a Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then for $X \in \Gamma(TM)$*

$$\nabla_X\xi = -\alpha TX - \beta T^2X$$

$$h(X, \xi) = -\alpha NX - \beta tNX.$$

Proof. For $X \in \Gamma(TM)$

$$\bar{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi).$$

Now using Gauss equation,

$$-\alpha\phi X + \beta X - \beta\eta(X)\xi = \nabla_X\xi + h(X, \xi)$$

$$-\alpha TX - \alpha NX + \beta X - \beta\eta(X)\xi = \nabla_X\xi + h(X, \xi)$$

$$-\alpha TX - \alpha NX + \beta X(I - \eta\otimes\xi) = \nabla_X\xi + h(X, \xi)$$

$$-\alpha TX - \alpha NX + \beta X(-T^2 - tN) = \nabla_X\xi + h(X, \xi).$$

On comparing tangential and normal parts we have the result :

$$h(X,\xi) = -\alpha NX - \beta tNX$$

$$\nabla_X \xi = -\alpha TX - \beta T^2 X.$$

□

Lemma 3.7. *Let M be a Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then $\forall X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$, we have:*

$$g([X,Y],\xi) = 2\alpha g(TX,Y)$$

$$g(\bar{\nabla}_X Y, \xi) = \alpha g(TX,Y) - \beta \cos^2\theta g(X,Y).$$

Proof. For $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$, taking covariant derivative of $g(Y,\xi)$

$$\bar{\nabla}_X(g(Y,\xi)) = g(\bar{\nabla}_X Y,\xi) + g(Y,\bar{\nabla}_X \xi)$$

$$0 = g(\bar{\nabla}_X Y,\xi) + g(Y,\bar{\nabla}_X \xi)$$

$$g(\bar{\nabla}_X Y,\xi) = -g(Y,\bar{\nabla}_X \xi).$$

Now since $\bar{M}$ is Trans-Sasakian,

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi)$$

$$\begin{aligned} g(\bar{\nabla}_X Y,\xi) &= -g(Y, -\alpha\phi X + \beta(X - \eta(X)\xi)) \\ &= -g\left(Y, -\alpha\phi X\right) - g\left(Y, \beta(X - \eta(X)\xi\right) \\ &= g(\alpha\phi X, Y) - g(Y, \beta(X - \eta(X)\xi)) \\ &= \alpha g(TX,Y) + \beta g(X - \eta(X)\xi, Y) \\ &= \alpha g(TX,Y) + \beta g(T^2X,Y) + \beta g(tNX,Y)) \quad \text{(using Lemma 3.4)} \end{aligned}$$

secondly,

$$\begin{aligned} g([X,Y],\xi) &= g(\bar{\nabla}_X Y,\xi) - g(\bar{\nabla}_Y X,\xi) \\ &= \alpha g(TX,Y) - \beta(\cos^2\theta g(X,Y)) - \alpha g(TY,X) + \beta(\cos^2\theta g(X,Y)) \\ g([X,Y],\xi) &= 2\alpha g(TX,Y). \end{aligned}$$

Hence the result. □

4 Integrability of Distributions

Theorem 4.1. *Let M be a Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the invariant distribution D and slant distribution $D_i; i = 1,2$ are not integrable.*

Proof. For $X, Y \in D(or Di)$ the distribution $D(or Di)$ is integrable if $g([X,Y],\xi) = 0$
Now using equation (2.4), (2.5), (3.4), we get

$$\begin{aligned} g([X,Y],\xi) &= g(\bar{\nabla}_X Y - \bar{\nabla}_Y X,\xi) \\ &= -g(Y,\bar{\nabla}_X \xi) + g(X,\bar{\nabla}_Y \xi) \\ &= -g(Y, -\alpha\phi X + \beta[X - \eta(X)\xi]) + g(X, -\alpha\phi Y + \beta[Y - \eta(Y)\xi]) \\ &= -g(Y, -\alpha\phi X) - g(Y, \beta(X - \eta(X)\xi)) + g(X, -\alpha\phi Y) + g(X, \beta(Y - \eta(Y)\xi)) \\ &= -\alpha g(\phi Y, X) - \alpha g(X, \phi Y) \\ &= -2\alpha g(\phi X, Y) \neq 0. \end{aligned}$$

Thus, $D(or Di)$ is not integrable.

□

Theorem 4.2. *Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). The invariant distribution $D \oplus \langle\xi\rangle$ is integrable if and only if $\forall X, Y \in \Gamma(D \oplus \langle\xi\rangle)$, $Z \in \Gamma(D_1 \oplus D_2)$,*

$$g(\nabla_X TY - \nabla_Y TX, TQZ + TRZ) = g(h(X, TY) - h(Y, TX), NQZ + NRZ). \tag{4.1}$$

Proof. For $D \oplus \langle\xi\rangle$ to be integrable on M,

$$g([X, Y], Z) = 0 \quad \text{for } X, Y \in \Gamma(D \oplus \langle\xi\rangle) and Z \in \Gamma(D_1 \oplus D_2)$$

$$Z = QZ + RZ$$

$$\phi Z = TQZ + NQZ + TRZ + NRZ$$

$$g([X, Y], Z) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z).$$

Using (2.2), (3.2), (3.4)for $X \in TM$

$$\phi X = TX + NX$$

$$\phi X = TPX + TQX + TRX + NPX + NRX$$

$$g([X, Y], Z) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z)$$

$$g([X, Y], Z) = g(\phi\bar{\nabla}_X Y, \phi Z) + \eta(\bar{\nabla}_X Y)\eta(Z) - g(\phi\bar{\nabla}_Y X, \phi Z)) - \eta(\bar{\nabla}_Y X)\eta(Z)$$

$$= g(\phi\bar{\nabla}_X Y, \phi Z) - g(\phi\bar{\nabla}_Y X, \phi Z)$$

$$= g(\bar{\nabla}_X \phi Y - (\bar{\nabla}_X \phi)Y, \phi Z) - g(\bar{\nabla}_Y \phi X - (\bar{\nabla}_Y \phi)X, \phi Z)$$

$$= g(\bar{\nabla}_X(TY + NY), \phi Z) - g(\alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \phi Z)$$

$$-g(\bar{\nabla}_Y(TX + NX), \phi Z) - g(\alpha[g(Y, X)\xi - \eta(X)Y] + \beta[g(\phi Y, X)\xi - \eta(X)\phi Y], \phi Z)$$

$$= g(\bar{\nabla}_X TY, \phi Z) - g(\alpha[g(X, Y)\xi - \eta(Y)X], \phi Z) - g(\beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \phi Z)$$

$$-g(\bar{\nabla}_Y TX, \phi Z) + g(\alpha[g(Y, X)\xi - \eta(X)Y], \phi Z) + g(\beta[g(\phi Y, X)\xi - \eta(X)\phi Y], \phi Z)$$

$$= g(\bar{\nabla}_X TY, \phi Z) + \alpha g(X, Y)g(\phi\xi, Z) + \alpha\eta(Y)g(X, \phi Z) - \beta g(\phi X, Y)g(\phi\xi, Z) - \beta\eta(Y)g(\phi X, \phi Z)$$

$$-g(\bar{\nabla}_Y TX, \phi Z) - \alpha g(Y, X)g(\phi\xi, Z) - \alpha\eta(X)g(Y, \phi Z) + \beta g(\phi Y, X)g(\phi\xi, Z) + \beta\eta(X)g(\phi X, \phi Z)$$

$$= g(\bar{\nabla}_X TY, \phi Z) - g(\bar{\nabla}_Y TX, \phi Z)$$

$$= g(\nabla_X TY - \nabla_Y TX, TQZ + TRZ) - g(h(X, TY) - h(Y, TX), NQZ + NRZ).$$

For integrability $g([X, Y], Z) = 0$ So,

$$g(\nabla_X TY - \nabla_Y TX, TQZ + TRZ) = g(h(X, TY) - h(Y, TX), NQZ + NRZ)$$

which is the required result.

□

Theorem 4.3. *Let M be a proper Quasi-Bi Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the slant distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if, for $Z, W \in (D_1 \oplus \langle\xi\rangle)$, $X \in (D \oplus D_2)$ we have:*

$$g(A_{NW}Z - A_{NZ}W, TX) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, NRX). \tag{4.2}$$

Proof. For $D_1 \oplus \langle\xi\rangle$ to be integrable on M

$$g([Z,W],X)=0$$

for $Z,W \in (D_1 \oplus \langle\xi\rangle), X \in (D \oplus D_2)$
$X = PX + RX$ so that $\phi X = TPX + TRX + NRX$.
Now using (2.2) i.e. $g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$

$$g([Z,W],X) = g(\bar\nabla_Z W, X) - g(\bar\nabla_W Z, X)$$
$$= g(\phi\bar\nabla_Z W, \phi X) - g(\phi\bar\nabla_W Z, \phi X) + \eta(\bar\nabla_Z W)\eta(X) - \eta(\bar\nabla_W Z)\eta(X)$$
$$= g(\phi\bar\nabla_Z W, \phi X) - g(\phi\bar\nabla_W Z, \phi X).$$

Now using (2.6), (3.2), (3.7), Lemma 3.5 and

$$\phi X = TPX + TQX + NQX + NRX$$

$$g([Z,W],X) = g(\bar\nabla_Z \phi W - (\bar\nabla_Z\phi)W), \phi X) - g(\bar\nabla_W\phi Z - (\bar\nabla_W\phi)Z)), \phi X)$$
$$= g(\bar\nabla_Z(TW+NW),\phi X) - g((\bar\nabla_Z\phi)W,\phi X) - g(\bar\nabla_W(TZ+NZ),\phi X) + g((\bar\nabla_W\phi)Z,\phi X)$$
$$= g(\bar\nabla_Z(TW+NW),\phi X) - g(\alpha[g(Z,W)\xi - \eta(Z)W],\phi X) - g(\beta[g(\phi Z,W)\xi - \eta(W)\phi Z],\phi X)$$
$$-g(\bar\nabla_W(TZ+NZ),\phi X) - g(\alpha[g(W,Z)\xi - \eta(W)Z],\phi X) - g(\beta[g(\phi W,Z)\xi - \eta(Z)\phi W],\phi X)$$
$$= g(\bar\nabla_Z(TW+NW),\phi X) + \alpha g(\phi\xi,X)g(Z,W) + \alpha\eta(Z)g(W,\phi X) + \beta g(\phi Z,W)g(\phi\xi,X)$$
$$+\beta\eta(Z)g(\phi W,\phi X) - g(\bar\nabla_W(TZ+NZ),\phi X) - \alpha g(\phi\xi,X)g(W,Z) - \alpha\eta(W)g(Z,\phi X)$$
$$-\beta g(\phi W,Z)g(\phi\xi,X) - \beta\eta(W)g(\phi Z,\phi X)$$
$$= g(\bar\nabla_Z(TW+NW),\phi X) + \beta\eta(Z)[g(W,X) - \eta(W)\eta(X)] - g(\bar\nabla_W(TZ+NZ),\phi X)$$
$$-\beta\eta(W)[g(Z,X) - \eta(Z)\eta(X)]$$

$$= g(\bar\nabla_Z(TW+NW),\phi X) - g(\bar\nabla_W(TZ+NZ),\phi X)$$

$$= -g(\bar\nabla_Z\phi TW),X) + g(\bar\nabla_W\phi TZ),X) + g(\bar\nabla_Z NW - \bar\nabla_W NZ),\phi X)$$
$$= g(\bar\nabla_W T^2 Z - \bar\nabla_Z T^2 W - \bar\nabla_Z NTW + \bar\nabla_W NTZ, X) + g(A_{NZ}W - A_{NW}Z + \nabla^\perp_Z NW - \nabla^\perp_W NZ, \phi X)$$

$$= \cos^2\theta_1[g(-\bar\nabla_W Z + \bar\nabla_Z W),X)] + g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^\perp_W NTZ - \nabla^\perp_Z NTW, X)$$
$$-g(A_{NW}Z - A_{NZ}W,\phi X) + g(\nabla^\perp_Z NW - \nabla^\perp_W NZ,\phi X)$$
$$= \cos^2\theta_1(g([Z,W]),X) - g(A_{NW}Z - A_{NZ}W,\phi X) + g(A_{NTW}Z - A_{NTZ}W,X) + g(\nabla^\perp_Z NW - \nabla^\perp_W NZ,\phi X)$$

$$\Rightarrow (1-\cos^2\theta_1)(g([Z,W]),X) = -g(A_{NW}Z - A_{NZ}W,\phi X) + g(A_{NTW}Z - A_{NTZ}W,X)$$
$$+g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX)$$
$$\Rightarrow \sin^2\theta_1(g([Z,W]),X) = -g(A_{NW}Z - A_{NZ}W,\phi X) + g(A_{NTW}Z - A_{NTZ}W,X)$$
$$+g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX)$$

for $D_1 \oplus \langle\xi\rangle$ to be integrable, $g([Z,W],X) = 0$. and so we have the required result as:

$$g(A_{NW}Z - A_{NZ}W, TX) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX).$$

□

Theorem 4.4. *Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the slant distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if for $Z, W \in \Gamma(D_2 \oplus \langle\xi\rangle)$, $X \in \Gamma(D \oplus D_1)$,*

$$g(A_{NW}Z - A_{NZ}W, TX) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_{Z}NW - \nabla^{\perp}_{W}NZ, NQX). \quad (4.3)$$

Proof. For $D_2 \oplus \langle\xi\rangle$ to be integrable, $g([Z, W], X) = 0$.
for $Z, W \in \Gamma(D_2 \oplus \langle\xi\rangle), X \in \Gamma(D \oplus D_1)$

$$\phi X = TPX + TQX + NQX.$$

Using $g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$, (2.6), (3.3), (3.7)

$$\begin{aligned} g([Z, W], X) &= g(\bar{\nabla}_Z W, X) - g(\bar{\nabla}_W Z, X) \\ &= g(\phi\bar{\nabla}_Z W, \phi X) - g(\phi\bar{\nabla}_W Z, \phi X) + \eta(\bar{\nabla}_Z W)\eta(X) - \eta(\bar{\nabla}_W Z)\eta(X) \\ &= g(\phi\bar{\nabla}_Z W, \phi X) - g(\phi\bar{\nabla}_W Z, \phi X). \end{aligned}$$

Now using (2.6), (3.2), (3.7), lemma 3.5 and

$$\phi X = TPX + TQX + NQX + NRX$$

$$\begin{aligned} g([Z, W], X) &= g(\bar{\nabla}_Z \phi W - (\bar{\nabla}_Z \phi)W), \phi X) - g(\bar{\nabla}_W \phi Z - (\bar{\nabla}_W \phi)Z)), \phi X) \\ &= g(\bar{\nabla}_Z(TW + NW), \phi X) - g((\bar{\nabla}_Z\phi)W, \phi X) - g(\bar{\nabla}_W(TZ + NZ), \phi X) + g((\bar{\nabla}_W\phi)Z, \phi X) \\ &= g(\bar{\nabla}_Z(TW + NW), \phi X) - g(\alpha[g(Z, W)\xi - \eta(Z)W], \phi X) - g(\beta[g(\phi Z, W)\xi - \eta(W)\phi Z], \phi X) \\ &\quad - g(\bar{\nabla}_W(TZ + NZ), \phi X) - g(\alpha[g(W, Z)\xi - \eta(W)Z], \phi X) - g(\beta[g(\phi W, Z)\xi - \eta(Z)\phi W], \phi X) \\ &= g(\bar{\nabla}_Z(TW + NW), \phi X) + \alpha g(\phi\xi, X)g(Z, W) + \alpha\eta(Z)g(W, \phi X) + \beta g(\phi Z, W)g(\phi\xi, X) \\ &\quad + \beta\eta(Z)g(\phi W, \phi X) - g(\bar{\nabla}_W(TZ + NZ), \phi X) - \alpha g(\phi\xi, X)g(W, Z) \\ &\quad - \alpha\eta(W)g(Z, \phi X) - \beta g(\phi W, Z)g(\phi\xi, X) - \beta\eta(W)g(\phi Z, \phi X) \\ &= g(\bar{\nabla}_Z(TW + NW), \phi X) + \beta\eta(Z)[g(W, X) - \eta(W)\eta(X)] - g(\bar{\nabla}_W(TZ + NZ), \phi X) \\ &\quad - \beta\eta(W)[g(Z, X) - \eta(Z)\eta(X)] \\ &= g(\bar{\nabla}_Z(TW + NW), \phi X) - g(\bar{\nabla}_W(TZ + NZ), \phi X) \\ &= -g(\bar{\nabla}_Z\phi TW), X) + g(\bar{\nabla}_W\phi TZ), X) + g(\bar{\nabla}_Z NW - \bar{\nabla}_W NZ), \phi X) \\ &= g(\bar{\nabla}_W T^2 Z - \bar{\nabla}_Z T^2 W - \bar{\nabla}_Z NTW + \bar{\nabla}_W NTZ, X) \\ &\quad + g(A_{NZ}W - A_{NW}Z + \nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, \phi X) \\ &= \cos^2\theta_1[g(-\bar{\nabla}_W Z + \bar{\nabla}_Z W), X)] + g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_W NTZ - \nabla^{\perp}_Z NTW, X) \\ &\quad - g(A_{NW}Z - A_{NZ}W, \phi X) + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, \phi X) \\ &= \cos^2\theta_1(g([Z, W]), X) - g(A_{NW}Z - A_{NZ}W, \phi X) + g(A_{NTW}Z - A_{NTZ}W, X) \\ &\quad + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, \phi X) \end{aligned}$$

$$\begin{aligned} (1 - \cos^2\theta_1)(g([Z, W]), X) &= -g(A_{NW}Z - A_{NZ}W, TX) + g(A_{NTW}Z - A_{NTZ}W, X) \\ &\quad + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, NQX) \\ (\sin^2\theta_1)(g([Z, W]), X) &= -g(A_{NW}Z - A_{NZ}W, TX) + g(A_{NTW}Z - A_{NTZ}W, X) \\ &\quad + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, NQX). \end{aligned}$$

for $D_1 \oplus \langle\xi\rangle$ to be integrable, $g([Z, W], X) = 0$. and so we have the required result i.e.

$$g(A_{NW}Z - A_{NZ}W, TX) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, NQX).$$

□

Corollary 4.5. *Let M be a proper Quasi-Bi-slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the slant distribution $D_i \oplus \langle\xi\rangle (i = 1, 2)$ are integrable if*

$$A_{NTW}Z - A_{NTZ}W \in (D_i)$$

$$A_{NW}Z - A_{NZ}W \in (D_i)$$

and

$$\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ \in \Gamma((ND_i) \oplus \mu)$$

for any $Z, W \in \Gamma(D_i) \oplus \langle\xi\rangle)(i = 1, 2)$.

5 Geodesicness of Distributions

In this section, we examine the conditions for distributions to be geodesic.

Theorem 5.1. *Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α,β). Then the invariant distribution D and slant distribution $D_i (i=1,2)$ do not define totally geodesic foliations on M.*

Proof. Using Lemma 3.7(ii), for $X,Y\in\Gamma(D)$ or $\Gamma(D_i)$,
we know that

$$g(\nabla_X Y,\xi)=0 \quad \text{for totally geodesicness}$$

but we have

$$g(\bar{\nabla}_X Y,\xi)=\alpha g(TX,Y)-\beta\cos^2\theta g(X,Y)$$
$$g(\nabla_X Y,\xi)=\alpha g(TX,Y)-\beta\cos^2\theta g(X,Y)\neq 0.$$

Therefore, D or D_i does not define totally geodesic foliations.

□

Theorem 5.2. *Let M be a submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α,β). Then M is totally geodesic if and only if $\forall X,Y\in\Gamma(TM)$, $V\in\Gamma(T^\perp M)$*

$$-g(\nabla_X TY-A_{NY}X,tV)=g(h(X,TY)+\nabla^\perp_X NY,nV).$$

Proof. For $X,Y\in\Gamma(TM)$, $V\in\Gamma(T^\perp M)$

$$\begin{aligned} g(\bar{\nabla}_X Y,V)&=g(\phi\bar{\nabla}_X Y,\phi V)+\eta(\bar{\nabla}_X Y)\eta(V)\\ &=g(\bar{\nabla}_X\phi Y,\phi V)-g((\bar{\nabla}_X\phi)Y,\phi V)\\ &=g(\bar{\nabla}_X(TY)+\bar{\nabla}_X(NY),tV+nV)\\ &=g(\bar{\nabla}_X TY,tV+nV)+g(\bar{\nabla}_X NY,tV+nV)\\ &=g(\nabla_X TY+h(X,TY),tV+nV)+g(-A_{NY}X+\nabla^\perp_X NY,tV+nV)\end{aligned}$$
$$g(\nabla_X Y,V)+g(h(X,Y),V)=g(\nabla_X TY-A_{NY}X,tV)+g(h(X,TY)+\nabla^\perp_X NY,nV)$$

for totally geodesicness of M

$$-g(\nabla_X TY-A_{NY}X,tV)=g(h(X,TY)+\nabla^\perp_X NY,nV).$$

□

Theorem 5.3. *Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α,β), then the invariant distribution $D\oplus\langle\xi\rangle$ is totally geodesic foliation on M if and only if*

$$g(\nabla_X TY,tU)=-g(h(X,TY),nU)$$

and

$$g(\nabla_X TY,TQZ+TRZ)=-g(h(X,TY),NQZ+NRZ).$$

Proof. $D\oplus\langle\xi\rangle$ is totally geodesic on M iff for $X,Y\in\Gamma(D\oplus\langle\xi\rangle)$, $Z\in\Gamma(D_1\oplus D_2)$, $U\in\Gamma(T^\perp M)$, for geodesicness $g(\bar{\nabla}_X Y,Z)=0$ and $g(\bar{\nabla}_X Y,U)=0$.
using (2.2), (2.5), (3.4)

$$\begin{aligned} g(\bar{\nabla}_X Y,Z)&=g(\phi\bar{\nabla}_X Y,\phi Z)+\eta(\bar{\nabla}_X Y)\eta(Z)\\ &=g(\bar{\nabla}_X\phi Y,\phi Z)-g((\bar{\nabla}_X\phi)Y,\phi Z)\\ &=g(\bar{\nabla}_X TY,\phi Z)\\ &=g(\nabla_X TY+h(X,TY),TQZ+TRZ+NQZ+NRZ)\\ &=g(\nabla_X TY,TQZ+TRZ)+g(h(X,TY),NQZ+NRZ)\end{aligned}$$

for geodesicness $g(\bar{\nabla}_X Y, Z) = 0$ so we have

$$g(\nabla_X TY, TQZ + TRZ) = -g(h(X, TY), NQZ + NRZ).$$

which gives the first part of the theorem
Secondly

$$\begin{aligned} g(\bar{\nabla}_X Y, U) &= g(\phi\bar{\nabla}_X Y, \phi U) - \eta(\bar{\nabla}_X Y)\eta(U) \\ &= g(\bar{\nabla}_X \phi Y, \phi U) - g((\bar{\nabla}_X \phi)Y, \phi U) \\ &= g(\bar{\nabla}_X \phi Y, \phi U) \\ &= g(\bar{\nabla}_X TY, \phi U) \\ &= g(\nabla_X TY + h(X, TY), tU + nU) \\ &= g(\nabla_X TY, tU) + g(h(X, TY), nU) \end{aligned}$$

for geodesicness $g(\bar{\nabla}_X Y, U) = 0$ so we have

$$g(\nabla_X TY, tU) = -g(h(X, TY), nU).$$

which proves the second part of the theorem. □

Theorem 5.4. *Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β), then the slant distribution $D_1 \oplus \langle\xi\rangle$ defines totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_X NY, NRZ) = -g(A_{NTY}X, Z) + g(A_{NY}X, TPZ + TRZ)$$

and

$$g(A_{NY}X, tV) = -g(\nabla^{\perp}_X NTY, V) + g(\nabla^{\perp}_X NY, nV).$$

Proof. For $X, Y \in \Gamma(D_1 \oplus \langle\xi\rangle)$, $Z \in \Gamma(D \oplus D_2)$, $V \in \Gamma(T^{\perp}M)$, for geodesicness $g(\bar{\nabla}_X Y, Z) = 0$ and $g(\bar{\nabla}_X Y, V) = 0$.
Using (2.5), (2.6), (3.4) and Lemma 3.5

$$\begin{aligned} g(\bar{\nabla}_X Y, Z) &= g(\phi\bar{\nabla}_X Y, \phi Z) + \eta(\bar{\nabla}_X Y)\eta(Z) \\ &= g(\bar{\nabla}_X \phi Y, \phi Z) - g((\bar{\nabla}_X \phi)Y, \phi Z) \\ &= g(\bar{\nabla}_X (TY + NY), \phi Z) \\ &= -g(\bar{\nabla}_X \phi TY, Z) + g(\bar{\nabla}_X NY, \phi Z) \\ &= -g(\bar{\nabla}_X T^2 Y, Z) - g(\bar{\nabla}_X NTY, Z) + g(\bar{\nabla}_X NY, \phi Z) \end{aligned}$$

$$= g(\bar{\nabla}_X \cos^2\theta_1 Y, Z) - g(-A_{NTY}X + \nabla^{\perp}_X NTY, Z) + g(-A_{NY}X + \nabla^{\perp}_X NY, TPZ + TRZ + NRZ)$$

$$= \cos^2\theta_1 g(\bar{\nabla}_X Y, Z) + g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TRZ) + g(\nabla^{\perp}_X NY, NRZ)$$

$$\Rightarrow (1 - \cos^2\theta_1) g(\bar{\nabla}_X Y, Z) = g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TRZ) + g(\nabla^{\perp}_X NY, NRZ)$$

$$\sin^2\theta_1 g(\bar{\nabla}_X Y, Z) = g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TRZ) + g(\nabla^{\perp}_X NY, NRZ). \quad (5.1)$$

Similarly Using (2.5) and (2.6) we calculate ,

$$\begin{aligned} g(\bar{\nabla}_X Y, V) &= g(\phi\bar{\nabla}_X Y, \phi V) + \eta(\bar{\nabla}_X Y)\eta(V) \\ &= g(\bar{\nabla}_X \phi Y, \phi V) - g((\bar{\nabla}_X \phi)Y, \phi V) \\ &= g(\bar{\nabla}_X \phi Y, \phi V) \\ &= g(\bar{\nabla}_X (TY + NY), \phi V) \\ &= -g(\bar{\nabla}_X \phi TY, V) + g(\bar{\nabla}_X NY, \phi V) \\ &= -g(\bar{\nabla}_X T^2 Y, V) - g(\bar{\nabla}_X NTY, V) + g(\bar{\nabla}_X NY, \phi V) \end{aligned}$$

$$= g(\bar{\nabla}_X \cos^2\theta_1 Y, V) - g(-A_{NTY}X + \nabla^{\perp}_X NTY, V) + g(-A_{NY}X + \nabla^{\perp}_X NY, tV + nV)$$

$$= \cos^2\theta_1 g(\bar{\nabla}_X Y, V) - g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tv) + g(\nabla^{\perp}_X NY, nv)$$
$$\Rightarrow (1-\cos^2\theta_1) g(\bar{\nabla}_X Y, V) = -g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV)$$
$$\sin^2\theta_1 g(\bar{\nabla}_X Y, V) = -g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV). \tag{5.2}$$

So from (5.1) and (5.2) we have:

$$g(\nabla^{\perp}_X NY, NRZ) = -g(A_{NTY}X, Z) + g(A_{NY}X, TPZ + TRZ)$$

and

$$g(A_{NY}X, tV) = -g(\nabla^{\perp}_X NTY, V) + g(\nabla^{\perp}_X NY, nV)$$

which is the required result. □

Theorem 5.5. *Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β), then the slant distribution $D_2 \oplus \langle\xi\rangle$ defines totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_X NY, NQZ) = -g(A_{NTY}X, Z) + g(A_{NY}X, TPZ + TQZ)$$

and

$$g(A_{NY}X, tV) = -g(\nabla^{\perp}_X NTY, V) + g(\nabla^{\perp}_X NY, nV).$$

Proof. For geodesicness $g(\bar{\nabla}_X Y, Z) = 0$ and $g(\bar{\nabla}_X Y, V) = 0$,where $X, Y \in \Gamma(D_2 \oplus \langle\xi\rangle)$, $Z \in \Gamma(D \oplus D_1)$, $V \in \Gamma(T^{\perp}M)$.
Now using (2.5), (2.6), (3.4)

$$g(\bar{\nabla}_X Y, Z) = g(\phi\bar{\nabla}_X Y, \phi Z) + \eta(\bar{\nabla}_X Y)\eta(Z)$$
$$= g(\bar{\nabla}_X \phi Y, \phi Z) - g((\bar{\nabla}_X \phi)Y, \phi Z)$$
$$= g(\bar{\nabla}_X (TY + NY), \phi Z)$$
$$= -g(\bar{\nabla}_X \phi TY, Z) + g(\bar{\nabla}_X NY, \phi Z)$$
$$= -g(\bar{\nabla}_X T^2 Y, Z) - g(\bar{\nabla}_X NTY, Z) + g(\bar{\nabla}_X NY, \phi Z)$$
$$= g(\bar{\nabla}_X \cos^2\theta_1 Y, Z) - g(-A_{NTY}X + \nabla^{\perp}_X NTY, Z) + g(-A_{NY}X + \nabla^{\perp}_X NY, TPZ + TQZ + NQZ)$$
$$= \cos^2\theta_1 g(\bar{\nabla}_X Y, Z) + g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TQZ) + g(\nabla^{\perp}_X NY, NQZ)$$
$$\Rightarrow (1-\cos^2\theta_1) g(\bar{\nabla}_X Y, Z) = g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TQZ) + g(\nabla^{\perp}_X NY, NQZ)$$
$$\sin^2\theta_1 g(\bar{\nabla}_X Y, Z) = g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TQZ) + g(\nabla^{\perp}_X NY, NQZ). \tag{5.3}$$

Similarly Using (2.5) and (2.6) we calculate ,

$$g(\bar{\nabla}_X Y, V) = g(\phi\bar{\nabla}_X Y, \phi V) + \eta(\bar{\nabla}_X Y)\eta(V)$$
$$= g(\bar{\nabla}_X \phi Y, \phi V) - g((\bar{\nabla}_X \phi)Y, \phi V)$$
$$= g(\bar{\nabla}_X \phi Y, \phi V)$$
$$= g(\bar{\nabla}_X (TY + NY), \phi V)$$
$$= -g(\bar{\nabla}_X \phi TY, V) + g(\bar{\nabla}_X NY, \phi V)$$
$$= -g(\bar{\nabla}_X T^2 Y, V) - g(\bar{\nabla}_X NTY, V) + g(\bar{\nabla}_X NY, \phi V)$$
$$= g(\bar{\nabla}_X \cos^2\theta_1 Y, V) - g(-A_{NTY}X + \nabla^{\perp}_X NTY, V) + g(-A_{NY}X + \nabla^{\perp}_X NY, tV + nV)$$
$$= \cos^2\theta_1 g(\bar{\nabla}_X Y, V) - g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tv) + g(\nabla^{\perp}_X NY, nV)$$
$$\Rightarrow (1-\cos^2\theta_1) g(\bar{\nabla}_X Y, V) = -g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV)$$
$$\sin^2\theta_1 g(\bar{\nabla}_X Y, V) = -g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV). \tag{5.4}$$

So from (5.3) and (5.4) we have :

$$g(\nabla^{\perp}_X NY, NQZ) = -g(A_{NTY}X, Z) + g(A_{NY}X, TPZ + TQZ)$$

and

$$g(A_{NY}X, tV) = -g(\nabla^{\perp}_X NTY, V) + g(\nabla^{\perp}_X NY, nV)$$

which is the required result. □

6 Conclusion remarks

Let M be a Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the invariant distribution D and slant distribution $D_i; i = 1, 2$ are not integrable.
Let M be a proper Quasi-Bi-Slant submanifold of Trans-Sasakian manifold $\bar{M}$ of type (α, β). Then the invariant distribution D and slant distribution $D_i(i = 1, 2)$ do not define totally geodesic foliations on M.

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