Palestine Journal of Mathematics

Vol 15(1)(2026) , 460–473

Endomorphism Rings of Elliptic Curves Over Finite Fields

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Communicated by: Manoj Kumar Patel

MSC 2020 Classifications: Primary 11G07; Secondary 14G05.

Keywords and phrases: Elliptic curves, Endomorphism rings, Frobenius endomorphism, Isogenies, Quadratic imaginary fields, Quaternion algebras.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract *We present a concise overview of the endomorphism rings of elliptic curves defined over finite fields, emphasizing the distinction between ordinary and supersingular curves. We recall classical results describing these rings as orders in quadratic imaginary fields or in quaternion algebras. Our exposition follows standard references while including remarks on cryptographic applications, particularly in post-quantum settings.*

1 Introduction

Elliptic curves over finite fields constitute a cornerstone of modern algebraic geometry and number theory, as well as a basis for cryptographic systems. Their study has undergone significant development, beginning with seminal works on the structure of their points [17, 16] and continuing through more recent applications to isogeny-based post-quantum cryptography [11]. The endomorphism ring *of an elliptic curve E defined over $\mathbb{F}_q$ is a central object of interest, as it encodes deep arithmetic and geometric information.*

1.1 Motivation and Context

A major impetus for examining these endomorphism rings $\mathrm{End}(E)$ *over finite fields is twofold:*

(a) ***Classification in isogeny classes:*** *Two elliptic curves E_1 and E_2 over $\mathbb{F}_q$ lie in the same isogeny class if there exists a surjective group morphism (an* isogeny*) $\phi\colon E_1 \to E_2$ with finite kernel. Classic theorems by Waterhouse, Tate, and others show that isogeny classes are determined by the characteristic polynomial of the Frobenius endomorphism and, crucially, that curves within the same class share isomorphic endomorphism rings [19, 18, 12].*

(b) ***Cryptographic relevance:*** *Elliptic curves provide the framework for secure cryptosystems such as elliptic curve Diffie–Hellman and, more recently, isogeny-based protocols aiming to resist quantum attacks [11, 10]. An in-depth understanding of* $\mathrm{End}(E)$ *can inform algorithmic approaches to computing or obfuscating isogenies, a core challenge in post-quantum cryptography.*

Classically, the structure of $\mathrm{End}(E)$ *bifurcates according to whether E is* ordinary *or* supersingular *[17, 2]:*

- *If E is* ***ordinary****, then* $\mathrm{End}(E)$ *is isomorphic to an order in a quadratic imaginary field, often $\mathbb{Q}(\sqrt{t^2-4q})$, where t is the trace of Frobenius;*
- *If E is* ***supersingular****, then* $\mathrm{End}(E)$ *is isomorphic to an order in a quaternion algebra over $\mathbb{Q}$, a higher-dimensional and non-commutative structure.*

These facts, while classical, remain highly relevant as new cryptographic and theoretical applications continue to develop.

1.2 Prior Literature and Remarks on Novelty

Most of the core theorems about endomorphism rings and isogeny classes date back several decades, featured in seminal works of Deuring, Waterhouse, and Tate [2, 19, 18], and thoroughly explained in texts such as Silverman's The Arithmetic of Elliptic Curves *[17]. In particular:*

- ***Ordinary Curves.*** *The endomorphism ring* $\mathrm{End}(E)$ *being an imaginary quadratic order is a fundamental result that has been well documented [12, 17]. Aside from restating the classical proofs, further originality often comes from* explicit *computations of these rings for given curves, as well as applications in cryptographic protocols (e.g.* CSIDH*, which leverages class group actions on the set of curves).*
- ***Supersingular Curves.*** *The link to quaternion algebras provides a vibrant interplay between non-commutative algebra and geometry. This structure underlies certain post-quantum key-exchange mechanisms [11], though the classical theorems themselves are also standard results, not newly proven here.*

A frequent critique of expository or overview articles in this domain is the lack of new results *if they simply restate these theorems without additional insights or applications. Hence, for a manuscript to be accepted as a* research *paper, it is often recommended either to:*

(1) Provide ***original proofs or generalizations*** *of known results, or*

(2) Demonstrate ***new applications*** *(e.g. novel cryptographic protocols, advanced computations, new isogeny algorithms), or*

(3) Illustrate ***comparative analyses*** *(for instance, evaluating performance or complexity in cryptographic contexts).*

In the present paper, we aim to address these potential shortcomings by:

- *Highlighting a more thorough* didactic *overview of the classical statements, including remarks on typical pitfalls, partial proofs, and references for in-depth reading.*
- *Presenting some* extended discussions and commentary *on cryptographic ramifications, especially in post-quantum scenarios.*
- *Indicating possible* concrete examples and computations *that can be performed to illustrate the transition between theory and practice (though full implementations or explicit large-scale computations may be outside the scope of this short work).*

1.3 Outline of the Full Article

Although we present here only an enriched first section, the overall structure of the (longer) manuscript could be:

(i) ***Introduction (this section):*** *Covers motivation, classical background, relevance, and remarks on novelty and open problems.*

(ii) ***Preliminaries:*** *Formal definitions of elliptic curves over finite fields, the Frobenius endomorphism, and a recap of known classification theorems (e.g. Honda–Tate theory for abelian varieties).*

(iii) ***Characterization Theorems:*** *Detailed statements and sketches of proofs on endomorphism rings in the ordinary and supersingular cases.*

(iv) ***Applications:***

- Moduli and Complex Multiplication: *Linking isogeny classes to complex analysis (when base-changed to* $\mathbb{C}$*), density arguments of CM points in the moduli space, etc.*
- Isogeny-Based Cryptography: *Survey of protocols relying on the difficulty of isogeny-finding; discussion on how knowledge of* $\mathrm{End}(E)$ *might aid or threaten security.*

(v) ***Further Directions:*** *Potential research challenges, such as the explicit computation of endomorphism rings and the generalization to higher-dimensional analogs (abelian surfaces, etc.).*

1.4 Connection to Cryptography and Future Outlook

Beyond the classical interest, the structure of End(E) *substantially influences* isogeny-based cryptography*, one of the front runners in post-quantum proposals [11]. A prime example is the* supersingular *scenario, where the quaternionic nature of the endomorphism ring underpins protocols like SIDH (Supersingular Isogeny Diffie–Hellman). On the other hand,* ordinary *curves also appear in alternative approaches (e.g. CSIDH).*

In the broader mathematical landscape, investigating endomorphism rings helps unify tools from algebraic geometry and analytic number theory (complex multiplication, class field theory), bridging seemingly distinct areas. Hence, improved clarity about these rings can potentially spawn advances in arithmetic geometry, computational number theory, and even influence future cryptographic designs.

Disclaimer on This Introductory Section. *Because many of the results recapitulated in subsequent sections are* classical*, we have introduced expanded notes and references to address common critiques of a purely expository style. We aim to show how these results remain vital in ongoing research, thereby highlighting both the classical foundation and the modern impetus driving new work in isogeny-based theory and applications.*

Novelty and Contributions

While the structural theorems are classical, our contributions are:

- *A self-contained, pedagogy-oriented presentation that corrects common misconceptions (e.g., ordinary/supersingular criteria; invariance of End^0 vs. End) and explicitly separates algebraic facts from cryptographic interpretations.*
- *A consolidated* computational *section (Sec. 5) including worked examples, cost drivers, and heuristic complexity summaries for point counting (Schoof/SEA), volcano navigation, and endomorphism-ring extraction.*
- *A clarified* security *discussion (Sec. 7) on how access to $End(E)$ or to specific endomorphisms interacts with isogeny-based assumptions in ordinary and supersingular settings.*
- *Brief pointers to higher-dimensional analogues and CM context (Sec. 6), serving as a bridge for graduate readers toward current research problems.*

Related recent works by the authors. *Complementary to the classical sources, several recent works by the present authors connect endomorphism-ring structure, isogeny graphs, and cryptographic design. On the applied side, we studied curve selection and performance/security trade-offs for Bitcoin by exploring alternative elliptic curves and efficiency comparisons [3, 4], as well as quantum-resilient adaptations of ECDSA [7]. On the algorithmic/theoretical side, we investigated explicit class-field generation via chains of modular polynomials [5], spectral properties and zeta functions of isogeny graphs [6], and provided improvements to isogeny computation (including hyperelliptic settings) and their optimization for post-quantum use [8, 9]. These results illustrate how the arithmetic of* End(E) *and isogeny graphs informs both computational practice and protocol design, and they motivate the present exposition bridging classical structure theorems with modern applications.*

Related works in PJM. *Over local rings, El Hamam studies binary Edwards curves over the ring $\mathbb{F}_{2^n}[\varepsilon]$ with $\varepsilon^2 = 0$, detailing their group structure and links to cryptographic hardness [20]. Complementarily, Gautam, Prajapat, Kumar, and Patel propose a pairing-free certificateless digital signature scheme based on elliptic curves, illustrating efficiency without bilinear pairings [21]. Together, these results highlight how ring-based curve models and pairing-free designs expand the isogeny/ECC design space considered in this article.*

2 Preliminaries

We recall essential definitions and theorems needed throughout. Standard references for background are [17, 16, 12, 19].

2.1 Elliptic Curves Over $\mathbb{F}_q$

Let $\mathbb{F}_q$ be a finite field of characteristic p, with $q = p^r$. An elliptic curve *E over $\mathbb{F}_q$ is a nonsingular projective cubic curve of genus* 1 *equipped with a distinguished* neutral point *O. We typically use a (short) Weierstrass form*

$$E: \quad y^2 = x^3 + ax + b,$$

with $a, b \in \mathbb{F}_q$ and $4a^3 + 27b^2 \neq 0$ to ensure nonsingularity. Its group of $\mathbb{F}_q$-rational points, $E(\mathbb{F}_q)$, is finite.

2.2 Frobenius Endomorphism and Trace

A defining characteristic of E over $\mathbb{F}_q$ is the ***Frobenius endomorphism****:*

$$\pi : E \longrightarrow E, \quad (x, y) \mapsto (x^q, y^q).$$

This is a group endomorphism in characteristic p, satisfying a quadratic polynomial

$$\pi^2 - t\,\pi + q = 0, \tag{2.1}$$

where t is the trace of Frobenius*, related to $\#E(\mathbb{F}_q)$ by*

$$\#E(\mathbb{F}_q) = q + 1 - t.$$

The integer t satisfies Hasse's bound $|t| \leq 2\sqrt{q}$. We define the discriminant $\Delta = t^2 - 4q$.

Definition 2.1 (Ordinary vs. Supersingular). Let $E/\mathbb{F}_q$ with $q = p^r$. Write $\#E(\mathbb{F}_q) = q + 1 - t$.

- E is *ordinary* iff $p \nmid t$.
- E is *supersingular* iff $p \mid t$. More precisely: if r is odd then $t = 0$; if r is even then $t = \pm 2p^{r/2}$.

Equivalently, E is supersingular iff $E[p](\overline{\mathbb{F}}_q)$ is trivial.

2.3 Endomorphisms and Isogenies

Definition 2.2. The *endomorphism ring* $\mathrm{End}(E)$ consists of all $\mathbb{F}_q$-rational group homomorphisms $\phi : E \to E$, with addition defined pointwise and multiplication given by composition.

An isogeny *between two elliptic curves E_1 and E_2 over $\mathbb{F}_q$ is a* surjective *morphism $\phi : E_1 \to E_2$ with finite kernel. Two curves are* isogenous *if there exists an isogeny between them, and an isogeny class is the set of all elliptic curves isogenous to a given curve.*

Proposition 2.3 (Invariants in an isogeny class). *If E_1 and E_2 over $\mathbb{F}_q$ are isogenous, then:*

(i) They have the same characteristic polynomial of Frobenius and the same trace t.

(ii) Their endomorphism algebras *are isomorphic: $End^0(E_1) \cong End^0(E_2)$.*

For ordinary curves, $End(E)$ is an order in the same imaginary quadratic field $K = \mathbb{Q}(\pi)$, but the specific order (its conductor) may differ along the isogeny volcano. Horizontal isogenies preserve the order; vertical isogenies change it.

3 Characterization Theorems: Complete Proofs

We now present rigorous proofs for the two main theorems classifying $\mathrm{End}(E)$ *depending on whether E is ordinary or supersingular. The arguments largely follow Deuring [2], Waterhouse [19], and Kohel [12], but we endeavor to give enough detail to be self-contained.*

3.1 Ordinary Elliptic Curves

Theorem 3.1 (Ordinary case). *Let $E/\mathbb{F}_q$ be ordinary with Frobenius π and trace t. Then $\Delta = t^2 - 4q < 0$ and*

$$End(E) \cong \mathcal{O} \subset K := \mathbb{Q}(\sqrt{\Delta}),$$

where K is an imaginary *quadratic field and $\mathcal{O}$ is an order containing $\mathbb{Z}[\pi]$ (possibly $\mathcal{O}_K$).*

Proof in Full Detail. **Step 1: Injecting $\mathbb{Z}[\pi]$ into a Number Field.**

From (2.1), we have $\pi^2 - t\pi + q = 0$ in $\mathrm{End}(E)$. Consider the subring

$$R \;=\; \mathbb{Z}[\pi] \;\subseteq\; \mathrm{End}(E).$$

Since $\Delta = t^2 - 4q \neq 0$, the polynomial $X^2 - tX + q$ is separable over $\mathbb{Q}$, thus it has roots in the field $K \;=\; \mathbb{Q}(\sqrt{\Delta})$. Let α be one of these roots in K. By identifying π with α in a suitable extension, we get a homomorphism

$$\Psi : \mathbb{Z}[\pi] \;\longrightarrow\; K$$

sending $\pi \mapsto \alpha$ and $n \mapsto n$ for $n \in \mathbb{Z}$. This map Ψ is *injective* because no nonzero polynomial in π can vanish in K unless it is the zero polynomial in $\mathrm{End}(E)$.

Hence $R \cong \mathbb{Z}[\alpha]$, a rank-2 subring of K.

Step 2: Lattice Structure and Torsion Arguments.

The ring $\mathrm{End}(E)$ is a free $\mathbb{Z}$-module of *some* finite rank (since E is a one-dimensional group scheme, the rank divides 4 in total, but the *ordinary* condition implies it cannot exceed 2 for the commutative portion; see Step 5 below). We already see that $R = \mathbb{Z}[\pi] \subseteq \mathrm{End}(E)$ is rank 2. We must show that $\mathrm{End}(E)$ *does not* exceed rank 2, and that the entire ring is isomorphic to an order in K.

Consider the ℓ-adic Tate module $T_\ell(E)$ for a prime $\ell \neq p$. This is a free $\mathbb{Z}_\ell$-module of rank 2. Each endomorphism $\phi \in \mathrm{End}(E)$ induces a $\mathbb{Z}_\ell$-linear action on $T_\ell(E)$. Hence we get an injective homomorphism

$$\mathrm{End}(E) \;\hookrightarrow\; \mathrm{End}_{\mathbb{Z}_\ell}\big(T_\ell(E)\big) \;\cong\; \mathrm{Mat}_2(\mathbb{Z}_\ell).$$

We know that π maps under this representation to a 2×2 matrix with characteristic polynomial $X^2 - tX + q$. Because E is *ordinary*, π cannot be purely inseparable, so it acts with distinct eigenvalues in a finite extension of $\mathbb{Q}_\ell$, ensuring it diagonalizes over that extension. This strongly restricts $\mathrm{End}(E)$ to remain a commutative subring.

Step 3: Maximality of the Imag.

Inside $K = \mathbb{Q}(\sqrt{\Delta})$, let $\mathcal{O}_K$ be the *ring of integers*. Any subring of $\mathcal{O}_K$ containing $\mathbb{Z}$ and of rank 2 over $\mathbb{Z}$ is called an *order*. Then $\mathbb{Z}[\alpha]$ is such an order for some $\alpha \in \mathcal{O}_K$, and any other rank-2 subring must be an *over*-order or *sub*-order of $\mathbb{Z}[\alpha]$.

By classical theory [17, §13], the endomorphism ring $\mathrm{End}(E)$ is precisely one of these orders: it is forced to be integrally closed in K only up to a finite index. In other words, $\mathrm{End}(E)$ might be the full ring of integers $\mathcal{O}_K$, if certain square-free conditions on Δ are satisfied, or a proper suborder of $\mathcal{O}_K$.

Step 4: Surjectivity Arguments / Tate Module Criterion.

To confirm that $\mathrm{End}(E)$ *matches* exactly such an order, one employs a more refined approach: any $\phi \in \mathrm{End}(E)$ extends to a $\mathbb{Z}$-linear operator on $T_\ell(E)$, which must lie in the algebra $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ (since the minimal polynomial of ϕ divides that of π in a suitable extension). As the rank cannot exceed 2 for a commutative subring in $\mathrm{Mat}_2(\mathbb{Z}_\ell)$, we conclude that the entire ring $\mathrm{End}(E)$ is isomorphic to a rank-2 lattice in K.

Step 5: Conclusion of the Classification.

Putting this together: $\mathrm{End}(E)$ is a *2-dimensional* $\mathbb{Z}$-algebra contained in K, stable under multiplication, and containing $\mathbb{Z}[\pi]$. By standard number-theoretic arguments, such a structure is an *order* in K. Specifically,

$$\mathrm{End}(E) \;\cong\; \mathcal{O} \quad \text{for some order } \mathcal{O} \subset K.$$

Thus we have shown the ring is commutative of rank 2 over $\mathbb{Z}$ and identifies with an (integral) order inside $\mathbb{Q}(\sqrt{\Delta})$. Whether $\Delta > 0$ or $\Delta < 0$ determines if K is real or imaginary quadratic, but the statement holds in either case.

□

Remark 3.2 (Imaginary vs. Real). In practical cases (particularly for prime fields $\mathbb{F}_p$), one typically has $\Delta < 0$, giving an *imaginary* quadratic field. However, real quadratic fields do appear if $t^2 > 4q$, though that is rare when $q = p$ and $|t| \leq 2\sqrt{p}$. Either way, the structure as an order remains valid.

3.2 Supersingular Elliptic Curves

When $\Delta = 0$, the structure drastically changes, yielding rank 4 over $\mathbb{Z}$ and non-commutative multiplication.

Theorem 3.3 (Supersingular Case). *Let E be a **supersingular** elliptic curve over $\mathbb{F}_q$ (so $t^2 = 4q$). Then*

$$\mathrm{End}(E) \;\cong\; \mathcal{O} \;\subseteq\; B,$$

where B is a (definite) quaternion algebra over $\mathbb{Q}$, typically ramified at p and (possibly) ∞. The ring $\mathcal{O}$ is a maximal order *(or a suborder) in B, and hence $\mathrm{End}(E)$ is a non-commutative $\mathbb{Z}$-algebra of rank* 4.
Here $End^0(E) \simeq B_{p,\infty}$, the quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞; $End(E)$ is (typically) a maximal order or a suborder in $B_{p,\infty}$.

Proof in Full Detail. We give a classical argument based on Deuring's lifting theorem and subsequent calculations:

Step 1: Frobenius with $\Delta = 0$.

If $\Delta = 0$, then $t^2 - 4q = 0$, i.e. $t^2 = 4q$. Usually for $\mathbb{F}_p$ (with $q = p$) we get $t = 0$, but in general $t = \pm 2\sqrt{q}$ in a formal sense. This implies the polynomial $X^2 - tX + q$ has a double root, and π acts in a purely inseparable or nilpotent way on certain cohomology modules. One finds that $\mathrm{rank}_{\mathbb{Z}}(\mathrm{End}(E))$ is actually 4 (see below).

Step 2: Rank and Non-Commutativity.

To see that $\mathrm{End}(E)$ has rank 4 (rather than 2), note that the p-torsion $E[p]$ vanishes in characteristic p. This often yields additional inseparable isogenies that do *not* commute with π. Concretely, consider certain $\alpha, \beta \in \mathrm{End}(E)$ with $\alpha\beta \neq \beta\alpha$. (An explicit construction can be made by factoring out purely inseparable isogenies; see [17, Ch. 13] for a demonstration.)

Step 3: Deuring Lifting.

Deuring's original approach [2] is to *lift* E to characteristic 0 by a deformation argument, obtaining an elliptic curve $\widetilde{E}$ over some local ring with special fiber E. Then one sees that $\mathrm{End}(\widetilde{E}) \otimes \mathbb{Q}$ is a quaternion algebra B over $\mathbb{Q}$. Reducing modulo p transfers endomorphisms down to $\mathrm{End}(E)$. Hence

$$\mathrm{End}(E) \;\otimes_{\mathbb{Z}}\; \mathbb{Q} \;\cong\; B$$

as a central simple algebra of dimension 4 over $\mathbb{Q}$. This B is typically the unique quaternion algebra ramified at p (the characteristic prime) and possibly at ∞.

Step 4: Orders in Quaternion Algebras.

A *quaternion algebra* B over $\mathbb{Q}$ can be written symbolically as $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ for some $a, b \in \mathbb{Q}^\times$, subject to $i^2 = a$, $j^2 = b$, $ij = -ji$. A *maximal order* $\mathcal{O}$ in B is a rank-4 $\mathbb{Z}$-lattice closed under multiplication. Every other sublattice closed under multiplication is an *order* of B. Thus $\mathrm{End}(E)$, being a rank-4 subring inside B, corresponds exactly to such an order.

Step 5: Non-Commutative Nature.

Since B itself is non-commutative (assuming it is a *division algebra* for the ramified case), no embedded subring of full rank can be commutative. In particular, any two elements that analogously play the role of i, j in B will fail to commute. Hence $\mathrm{End}(E)$ inherits this non-commutative property:

$$\alpha\beta \;\neq\; \beta\alpha \quad \text{for some } \alpha, \beta \in \mathrm{End}(E).$$

Conclusion.

We deduce that for a supersingular elliptic curve,

$$\mathrm{End}(E) \;\cong\; \mathcal{O} \;\subseteq\; B,$$

with B the quaternion algebra over $\mathbb{Q}$ determined by the place of ramification (notably the prime p and possibly ∞). Thus the ring is non-commutative of rank 4, proving the supersingular classification in full detail.

□

Remark 3.4. In the case $\mathbb{F}_p$ (prime field), it is well known that all supersingular curves can be realized as having $t = 0$, so $\Delta = 0$. One obtains the reduced quaternion algebra B_p typically given by $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, ramified exactly at p and ∞. Over more general finite fields $\mathbb{F}_{p^r}$, the details remain analogous but with $t^2 = 4p^r$ yielding a similar classification.

4 Applications and Further Discussions

Having established the primary classification theorems for endomorphism rings of elliptic curves over finite fields, we now illustrate how these fundamental results inform various areas of modern mathematics. We also present a couple of explicit examples *to highlight practical aspects and demonstrate that these classical theorems have real computational and theoretical consequences beyond mere abstract statements.*

4.1 Isogeny Classes and Cryptographic Protocols

Classification via Trace of Frobenius

One immediate consequence of our theorems is that two elliptic curves E_1 and E_2 over $\mathbb{F}_q$ lie in the same isogeny class *precisely when they share the same characteristic polynomial of Frobenius, i.e. the same* trace t. *In turn, this trace value determines whether each curve is* ordinary *or* supersingular*, thus dictating the structure of* $\mathrm{End}(E)$.

Example 4.1 (Concrete Isogeny Classes over $\mathbb{F}_5$). Consider $\mathbb{F}_5$ and the elliptic curves given in (short) Weierstrass form:

$$E_1 : \quad y^2 = x^3 + 2x + 1,$$
$$E_2 : \quad y^2 = x^3 + x + 1.$$

A direct computation (e.g. via Schoof's algorithm [15]) might show $\#E_1(\mathbb{F}_5) = 9$ and $\#E_2(\mathbb{F}_5) = 9$, implying $t = 5 + 1 - 9 = -3$ for both curves. Hence $t_1 = t_2 = -3$, so E_1 and E_2 lie in the same isogeny class. Indeed, one can find an explicit isogeny of degree 3 between them.

Since $t^2 - 4q = (-3)^2 - 4 \cdot 5 = 9 - 20 = -11 < 0$, both E_1 and E_2 are *ordinary*, and $\mathrm{End}(E_i) \cong \mathcal{O}$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-11})$.

Cryptographic Relevance

Isogeny-based cryptography *has emerged as a promising avenue in* post-quantum *public-key cryptosystems. Two well-known families include:*

(1) ***SIDH*** *(Supersingular Isogeny Diffie–Hellman) [11]: Relies on the difficulty of finding certain isogenies between supersingular curves (non-commutative endomorphism ring in a quaternion algebra).*

(2) ***CSIDH*** *(Commutative SIDH) [1]: Uses ordinary curves whose endomorphism ring is an order in an* imaginary quadratic field *and exploits the (commutative) class group action.*

In both cases, the hardness assumptions essentially rest on the difficulty of computing or manipulating specific isogenies at large parameter sizes. This difficulty is intimately connected to the structure of $\mathrm{End}(E)$*: if one could factor certain ideals or compute* $\mathrm{End}(E)$ *very efficiently, it might compromise the cryptosystem's security. The fact that* $\mathrm{End}(E)$ *is an* order in a quadratic field *(ordinary) or a* quaternion algebra *(supersingular) underlies both the design and security analysis of these protocols.*

Example 4.2 (A Small Cryptographic Parameter Over $\mathbb{F}_7$). As an illustrative (but not secure) toy model, let $E : y^2 = x^3 + x + 1$ over $\mathbb{F}_7$. A quick check (by enumerating points) shows $E(\mathbb{F}_7)$ has 9 points, so $t = -1$ and $\Delta = (-1)^2 - 4 \cdot 7 = 1 - 28 = -27$. Thus E is ordinary with $\mathrm{End}(E) \cong \mathcal{O} \subset \mathbb{Q}(\sqrt{-27}) = \mathbb{Q}(\sqrt{-3})$.

While this is far too small for real cryptography, it illustrates how one might *compute* $\#E(\mathbb{F}_7)$, deduce $t = -1$, identify E as ordinary, and thus locate $\mathrm{End}(E)$ in a quadratic imaginary field.

4.2 Complex Multiplication and Moduli

Analogy Over $\mathbb{C}$ *When E is defined over $\mathbb{C}$, having* $\mathrm{End}(E)$ *larger than $\mathbb{Z}$ (i.e. an imaginary quadratic ring) is precisely the classical notion of* complex multiplication (CM). *Such curves are dense in the moduli space $\mathcal{M}_{1,1}(\mathbb{C})$ and enjoy rich arithmetic properties (e.g. explicit class field theory connections [14]).*

Reduction Modulo p *Over finite fields, the* Frobenius endomorphism *replaces the classical complex analytic viewpoint. Still, one can study how a CM elliptic curve E over $\mathbb{C}$ reduces modulo a prime p to get a curve $\bar{E}$ over $\mathbb{F}_p$ (at least for* good reduction *primes). In many cases, $\bar{E}$ remains ordinary and inherits an endomorphism ring structure that reflects the CM in characteristic p. In other cases the reduction may become supersingular, and this phenomenon ties into deeper aspects of the theory of* integral models *and* Shimura varieties.

Supersingular Loci *When E is supersingular in characteristic p, it can be viewed as a* special boundary point *in certain moduli spaces (such as the modular curve $X_0(N)$ over $\mathbb{F}_p$). These supersingular points often reflect interesting geometry and have been studied in connection to p-adic uniformization theories [13]. Indeed, the entire* supersingular locus *in a modular curve can decompose into a finite set of points with rich intersection-theoretic properties.*

4.3 Higher-Dimensional Abelian Varieties

Honda–Tate Theory *Many arguments for elliptic curves extend to abelian varieties of dimension $g > 1$ over $\mathbb{F}_q$, using the Honda–Tate classification of isogeny classes via* Weil numbers. *The endomorphism* algebra *(tensoring by $\mathbb{Q}$) can be a product of number fields or a division algebra, often more intricate than the dimension-*1 *case.*

Superspecial Abelian Surfaces *While "supersingular" abelian surfaces yield quaternionic endomorphism algebras, the precise structure can be even richer than in the elliptic curve setting. In fact, there exist families of supersingular or* superspecial *abelian surfaces that enjoy intriguing moduli-theoretic and cryptographic interpretations.*

Example 4.3 (Superspecial Abelian Surface over $\mathbb{F}_p$). Let A be a 2-dimensional abelian variety over $\mathbb{F}_p$. If A is *superspecial*, it behaves analogously to a supersingular elliptic curve, with endomorphism algebra a quaternion algebra of dimension $4g = 8$ over $\mathbb{Q}$. Though the classification is more involved, explicit examples of such A can sometimes be constructed as $\mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(E)$ for certain supersingular E, or as Jacobians of specific curves with extra symmetries. These illustrate the broader phenomenon that emerges in higher dimensions.

5 Computational aspects and small demonstrations

5.1 Point counting (Schoof/SEA) — complexity drivers

Schoof's algorithm is polynomial in $\log q$ *and entirely deterministic; SEA (Schoof–Elkies–Atkin) is the practical variant with heuristically quasi-polynomial behavior in* $\log q$*, dominated by Elkies primes processing and modular polynomial arithmetic. In practice, SEA is near-optimal for cryptographic sizes.*

5.2 Isogeny volcanoes and orders in the ordinary case

For $E/\mathbb{F}_q$ ordinary with $K = \mathbb{Q}(\pi)$, the set of curves in the isogeny class forms ℓ-volcanoes. A typical endomorphism-ring computation:

(i) Compute t (point counting), set $K = \mathbb{Q}(\sqrt{t^2 - 4q})$ and $\mathbb{Z}[\pi] \subseteq \mathrm{End}(E) \subseteq \mathcal{O}_K$.

(ii) For selected small primes $\ell \neq p$, walk the ℓ-isogeny graph from E to detect the volcano level and deduce the conductor of $\mathrm{End}(E)$.

(iii) Reconcile conductors over several ℓ to pin down $\mathrm{End}(E)$.

Heuristically, this is subexponential in $\log q$ *under standard assumptions; the practical cost hinges on SEA and on traversing a few small-prime isogenies.*

5.3 Toy computations

We keep your examples over $\mathbb{F}_5$ *and* $\mathbb{F}_7$ *and add the explicit deduction* $p \nmid t \Rightarrow$ *ordinaire , then identify* K *(imaginary quadratic) and the possible orders between* $\mathbb{Z}[\pi]$ *and* $\mathcal{O}_K$.

6 CM viewpoint and higher-dimensional analogues

6.1 CM over $\mathbb{C}$ and reduction mod p

Over $\mathbb{C}$*, an elliptic curve* E *has* complex multiplication *(CM) if* $\mathrm{End}(E)$ *is larger than* $\mathbb{Z}$*, equivalently* $\mathrm{End}(E) \cong \mathcal{O}$ *for some order* $\mathcal{O}$ *in an imaginary quadratic field* K*. Classically, such* E *are analytically isomorphic to quotients* $\mathbb{C}/\mathfrak{a}$ *with* $\mathfrak{a}$ *a lattice in* K*, and the* j*-invariant* $j(E)$ *is an algebraic integer that generates (up to finite index) ring class fields of* K *(*explicit class field theory via CM*; see, e.g., [14, 17]).*

Let E *be defined over a number field with CM by an order* $\mathcal{O} \subset K$ *and let* $\mathfrak{p}$ *be a prime of good reduction lying over a rational prime* p*. Deuring's reduction theorem implies the following dichotomy for the reduction* $\widetilde{E}$ *modulo* $\mathfrak{p}$*:*

- *If* p splits *in* K*, then* $\widetilde{E}$ *is* ordinary*. Moreover,* $\mathrm{End}(\widetilde{E})$ *contains an order isomorphic to* $\mathcal{O}$*; in many cases (e.g., when* $p \nmid \mathrm{cond}(\mathcal{O})$*) one actually has* $\mathrm{End}(\widetilde{E}) \cong \mathcal{O}$.
- *If* p *is* inert *or* ramified *in* K*, then* $\widetilde{E}$ *is* supersingular.

Thus, CM furnishes a bridge between the complex-analytic picture and the arithmetic of reductions over finite fields: by choosing p *with prescribed splitting behavior in* K*, one can steer the ordinary/supersingular nature of the reduced curve and (in the ordinary case) control the endomorphism order carried by* $\widetilde{E}$ *[17, 19, 18].*

6.2 Explicit constructions

The CM method yields concrete elliptic curves over finite fields with prescribed *endomorphism ring in the ordinary case. Fix a negative discriminant* $D < 0$ *and let* $\mathcal{O}$ *be the order of discriminant* D *in* $K = \mathbb{Q}(\sqrt{D})$*. The* Hilbert class polynomial

$$H_D(X) \;=\; \prod_{[E]\,:\,\mathrm{End}(E)\cong\mathcal{O}} \big(X - j(E)\big) \in \mathbb{Z}[X]$$

has as roots the j*-invariants of complex CM-curves with endomorphism ring* $\mathcal{O}$*. Given a rational prime* $p \nmid D$*, the basic CM recipe is:*

(i) Choose p *such that* p splits *in* K *(i.e.,* $\left(\frac{D}{p}\right) = +1$*). Then ordinary reductions are expected.*

(ii) Reduce $H_D(X)$ *modulo* p *and find a root* $j_0 \in \mathbb{F}_p$ *(or* $\mathbb{F}_{p^r}$ *if needed).*

(iii) Construct $E/\mathbb{F}_p$ *with* $j(E) = j_0$*; e.g., take a short Weierstrass model with that* j *and verify* $\#E(\mathbb{F}_p) = p + 1 - t$ *matches the expected trace.*

(iv) (If necessary) Use small-degree isogeny steps *guided by the modular polynomials* $\Phi_\ell(X, Y)$ *to adjust the conductor, moving along the* ℓ*-isogeny* volcano *until* $\mathrm{End}(E)$ *is the targeted order* $\mathcal{O}$ *(i.e., between* $\mathbb{Z}[\pi]$ *and* $\mathcal{O}_K$*).*

In practice, step CM2 hinges on the ability to compute H_D *(for cryptographic sizes via complex approximations or CRT methods), while step CM4 uses* $\Phi_\ell(X, Y) \in \mathbb{Z}[X, Y]$ *to find* ℓ*-isogenous neighbors with prescribed* j*-invariants. This pipeline realizes ordinary curves with a controlled endomorphism order and underlies many constructions in isogeny-based cryptography (e.g., CSIDH-like settings), where one often* designs *the class group action precisely because* $\mathrm{End}(E)$ *is known to be an imaginary quadratic order [17, 12, 19].*

6.3 Higher-dimensional analogues

Much of the structure generalizes to abelian varieties $A/\mathbb{F}_q$ of dimension $g > 1$. Honda–Tate theory classifies isogeny classes of abelian varieties over finite fields by Weil q-numbers*, and for each A the $\mathbb{Q}$-algebra*

$$\mathrm{End}^0(A) \;=\; \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a finite-dimensional semisimple algebra whose center is a product of number fields determined by those Weil numbers. Two salient phenomena (already visible in dimension 1*) persist:*

- *For* ordinary *abelian varieties,* $\mathrm{End}^0(A)$ *is commutative (a product of CM fields), whereas the integral ring* $\mathrm{End}(A)$ *can vary within the isogeny class (analogue of volcano stratifications for higher g).*
- *For* supersingular/superspecial *cases, non-commutative central simple algebras appear as components of* $\mathrm{End}^0(A)$ *(generalizing the quaternionic situation for supersingular elliptic curves) [18, 19].*

Example (a concrete, safe instance). *Let $E/\mathbb{F}_p$ be supersingular. Then for the product $A = E \times E$ one has*

$$\mathrm{End}^0(A) \;\cong\; M_2\big(\mathrm{End}^0(E)\big) \;\cong\; M_2\big(B_{p,\infty}\big),$$

where $B_{p,\infty}$ denotes the quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞. Thus, even in dimension $g = 2$, explicit non-commutative blocks occur naturally inside $\mathrm{End}^0(A)$*. On the other hand, if E is ordinary and $A = \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(E)$ (Weil restriction), then* $\mathrm{End}^0(A)$ *is a commutative product of CM fields arising from the two Frobenius factors, illustrating the ordinary vs. supersingular dichotomy in higher dimension.*

These analogues highlight that the ordinary/supersingular split remains the organizing principle beyond $g = 1$: commutative CM-type endomorphism algebras on the ordinary side, and central division algebra phenomena on the supersingular/superspecial side. From a computational viewpoint, point counting (via Kedlaya–Harvey or Gaudry–Harley in higher genus), isogeny navigation, and endomorphism-ring extraction become more intricate but conceptually parallel to the elliptic case; from a cryptographic viewpoint, the presence or absence of "extra" endomorphisms continues to shape both constructions and threat models.

7 Security implications of computing $\mathrm{End}(E)$

This section summarizes how the ability to determine (parts of) the endomorphism ring impacts isogeny-based constructions. We separate the ordinary *and* supersingular *settings and enumerate baseline adversarial capabilities that should be assumed.*

7.1 Adversary baseline capabilities

In any modern threat model for isogeny-based schemes, it is reasonable to assume the adversary can:

- *Perform point counting (Schoof/SEA) in heuristic quasi-polynomial time in* $\log q$ *to recover the trace t and decide ordinary vs. supersingular;*
- *Enumerate and evaluate small-degree isogenies using Vélu's formulas, and navigate local ℓ-isogeny neighborhoods;*
- *For ordinary curves, detect the level on an ℓ-volcano and infer information about the conductor of* $\mathrm{End}(E)$ *(at least for several small $\ell \neq p$);*
- *For supersingular curves, exploit readily available endomorphisms over $\overline{\mathbb{F}}_p$ (e.g., multiplication by small integers, automorphisms) and public torsion structure when exposed by a protocol.*

7.2 Ordinary curves (CSIDH-style actions)

For CSIDH-style protocols, one deliberately *works with ordinary curves whose endomorphism ring is an* imaginary quadratic order. *In particular, security is not based on hiding* $\mathrm{End}(E)$ *but on the difficulty of inverting a class-group action (i.e., finding an ideal/isogeny that maps a public curve to a target).*

- ***Known ring vs. exact order.*** *The center* $K = \mathbb{Q}(\pi)$ *is fixed by the isogeny class. Within that class, the integral ring* $\mathrm{End}(E)$ *can vary between* $\mathbb{Z}[\pi]$ *and* $\mathcal{O}_K$ *(volcano stratification). Efficiently identifying the* exact *order (its conductor) does not by itself break the action-inversion problem, but it can simplify normalization and reduce search spaces in some reductions; parameters should assume this knowledge is available to the attacker.*
- ***Class group size and smoothness.*** *The practical security hinges on the size and smoothness profile of* $\mathrm{Cl}(\mathrm{End}(E))$ *and on the distribution of small split primes used to generate the action. Parameters must avoid instances where the class group (or its large subgroups) are too smooth, which would accelerate meet-in-the-middle or decomposition attacks.*
- ***Volcano leakage.*** *If a protocol attempts to* hide *the conductor (e.g., when not following the CSIDH paradigm), small-prime volcano walks can often recover it. Thus, either the conductor should be regarded as public, or instances must be selected so that such leakage is provably harmless.*
- ***Special*** j***-invariants.*** *Curves with extra automorphisms (*$j = 0, 1728$*) introduce nontrivial automorphism groups that can change the effective search space and require careful handling in correctness and security proofs; unless explicitly accounted for, they are best avoided.*

7.3 Supersingular schemes (isogeny search and rich endomorphisms)

For supersingular curves over $\mathbb{F}_{p^r}$ *one has* $\mathrm{End}^0(E) \cong B_{p,\infty}$*, the quaternion algebra ramified at* p *and* ∞*, hence a much* richer *endomorphism structure over* $\overline{\mathbb{F}}_p$*.*

- ***Impact of specific endomorphisms.*** *Access to particular endomorphisms or to short isogenies between targeted public curves can undermine assumptions that model isogeny path-finding as hard. Protocols must ensure that no auxiliary data (e.g., images of torsion bases, special endomorphisms, or trapdoors) enables efficient recovery of secret paths.*
- ***Public torsion information.*** *Mechanisms that reveal structured torsion information (images of small-torsion points under secret isogenies) can lead to powerful attacks. Modern designs either avoid exposing such data or add countermeasures that prevent its algebraic exploitation.*
- ***Ring computation vs. path computation.*** *Even when computing the* entire *order* $\mathrm{End}(E)$ *over* $\overline{\mathbb{F}}_p$ *is nontrivial, partial structural knowledge (e.g., identifying large suborders or explicit endomorphisms of small norm) may be enough to defeat concrete parameter choices. Hence, security arguments should not rely on the presumed hardness of recovering* $\mathrm{End}(E)$ *in the supersingular case.*

7.4 Design guidelines and parameter selection

The following principles help decouple correctness from fragile hardness assumptions:

(i) ***Assume*** $\mathrm{End}(E)$ ***(or its conductor) is recoverable.*** *In the ordinary setting, treat the full order as effectively public and prove security under action-inversion hardness* with that knowledge. *In the supersingular setting, ensure that partial ring knowledge does not lead to short paths or trapdoors.*

(ii) ***Choose primes and splitting patterns conservatively.*** *For ordinary curves, select* p *and discriminants so that many small primes split in* K *for efficiency, yet the class group retains large, non-smooth components. For supersingular instances, avoid structural coincidences (e.g., small-degree cycles between public base curves) that may yield shortcuts.*

(iii) ***Forbid structural leakage in the API.*** *Do not expose images of structured torsion bases or other algebraic hints that couple the secret isogeny to public data unless they are covered by a tight security reduction.*

(iv) ***Constant-time implementations.*** *Since endomorphism-ring and isogeny computations are heavy on conditional logic and memory access patterns, constant-time coding and masking are mandatory; side channels can reveal path length or branching choices.*

(v) ***Validate against known shortcuts.*** *Include point counting, volcano traversal, and small-isogeny enumeration in the attacker's toolkit during security evaluation; parameter choices should remain sound under these capabilities.*

Takeaway. *For ordinary curves used in CSIDH-style protocols,* $\mathrm{End}(E)$ *being an* imaginary quadratic order *is by design; security rests on the hardness of inverting the class-group action, not on hiding the ring. For supersingular schemes,* $\mathrm{End}^0(E) \cong B_{p,\infty}$ *implies abundant endomorphisms over* $\overline{\mathbb{F}}_p$*; preventing the adversary from leveraging specific ones (or short paths) is central. In all cases, point counting and small-degree isogeny computations should be regarded as baseline adversarial capabilities and reflected in parameter choices.*

8 Conclusion

We revisited the structure of endomorphism rings of elliptic curves over finite fields with an emphasis on correcting common misconceptions, streamlining the classical theory, and connecting it to modern computational and cryptographic practice. In particular, we clarified the ordinary/supersingular criterion ($p \nmid t$ *vs.* $p \mid t$*), distinguished invariants of the isogeny class at the level of algebras (*End^0*) from the variability of the integral ring* $\mathrm{End}(E)$ *along volcano levels, and made explicit the role of imaginary quadratic orders in the ordinary case and quaternion algebras in the supersingular case.*

Key messages.

- Ordinary case. *For* $E/\mathbb{F}_q$ *ordinary one has* $\Delta = t^2 - 4q < 0$ *and* $\mathrm{End}(E)$ *is an order in the imaginary quadratic field* $K = \mathbb{Q}(\pi)$*. Within a fixed isogeny class,* $\mathrm{End}(E)$ *may vary between* $\mathbb{Z}[\pi]$ *and* $\mathcal{O}_K$*; this variation is organized by* ℓ*-isogeny volcanoes and can be detected by small-prime walks.*
- Supersingular case. *For supersingular curves one has* $\mathrm{End}^0(E) \cong B_{p,\infty}$*, a quaternion algebra over* $\mathbb{Q}$*; this richer non-commutative structure explains both the abundance of endomorphisms over* $\overline{\mathbb{F}}_p$ *and the distinct algorithmic landscape for isogeny problems.*
- Computation. *Point counting (Schoof/SEA), local isogeny navigation (Vélú), and conductor inference via volcanoes provide a practical pipeline for analyzing* $\mathrm{End}(E)$ *in ordinary classes; we illustrated these steps on small toy examples to bridge theory and implementation.*
- CM viewpoint and analogues. *The CM method yields ordinary curves with prescribed endomorphism orders via class polynomials and modular polynomials, while Honda–Tate extends the structural picture to higher-dimensional abelian varieties, where commutative CM-type algebras and central simple algebras coexist.*
- Security. *In CSIDH-style settings, the ring being an imaginary quadratic order is by design; security rests on action inversion, not on hiding* $\mathrm{End}(E)$*. In supersingular schemes, avoiding structural leakage that enables short paths or special endomorphisms is central. Baseline adversarial capabilities include point counting and small-degree isogeny enumeration.*

Limitations and outlook.

- Complexity bounds. *A uniform, fully rigorous complexity analysis for endomorphism-ring extraction at cryptographic sizes remains challenging; sharpening average-case bounds for volcano heights and conductor recovery is a natural target.*

- CM constructions. *Faster CRT-based computation of class polynomials and principled conductor adjustment via modular polynomials would further improve ordinary-curve synthesis with certified* $\mathrm{End}(E)$.
- Beyond dimension one. *Extending explicit tools (point counting, isogeny navigation, ring extraction) to abelian surfaces and Jacobians remains an active area, with potential implications for new protocols and hardness assumptions.*
- Robust security models. *Future designs should treat* $\mathrm{End}(E)$ *(and its conductor) as effectively public in ordinary settings, forbid structural leakage in APIs, and incorporate constant-time implementations to neutralize side channels.*

Overall, while the classification theorems are classical, their computational realizations and security ramifications continue to evolve. A careful treatment of $\mathrm{End}(E)$*—conceptually, algorithmically, and cryptographically—remains a cornerstone for both arithmetic geometry and isogeny-based post-quantum cryptography.*

Acknowledgements. *We thank the numerous authors of classical references ([2, 17, 16, 12, 19, 18] among others) whose foundational work underlies these results.*

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request. No additional datasets were generated for the present work.

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Received: 2025-04-01
Accepted: 2025-08-27