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On certain paranormed spaces using modulus function

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Abstract. The sequence spaces of paranormed nature have been given various structures by using modulus function as it were first given in [26]. The goal of this paper is to present some new spaces $\mathfrak{S}^t_\infty(\Delta, s, \mathcal{G}, p, \eta)$, $\mathfrak{S}^t(\Delta, s, \mathcal{G}, p, \eta)$ and $\mathfrak{S}^t_0(\Delta, s, \mathcal{G}, p, \eta)$. Also, their complete linear paranormed structure will be given. Further, some inclusion relations will be computed corresponding to these new spaces.

1 Preliminaries, introduction and notation

Let the spaces of all real valued sequences be abbreviated by Ω. By sequence space, we mean a vector subspace of Ω. By ℓ_∞, c and c_0 we shall abbreviate for the spaces of all bounded, convergent and null sequences, respectively, as can be seen in [1, 2, 18, 20, 29, 31, 32].

As in [26], we call function $\daleth$ to be modulus function $\daleth : [0, \infty) \to [0, \infty)$ satisfying (i) $\daleth(u) = 0$ if and only if $u = 0$, (ii) $\daleth(u + v) = \daleth(u) + \daleth(v)$ for all $u = 0$, $v = 0$, (iii) $\daleth$ is increasing, and (iv) $\daleth$ is continuous from the right at zero. Consequently, $\daleth$ should be continuous everywhere on $[0, \infty)$. It was studied by various authors as can be seen in [8, 9, 10, 11, 17, 34, 37], and many others.

The notion of paranormed sequence space was studied at the initial stage by Nakano [26]. Let $\mathcal{U}$ be a linear space. A function $\mathfrak{H} : \mathcal{U} \to \mathbb{R}$ is a paranorm, if
(i) $\mathfrak{H}(\xi) \geq 0$ for each $\xi \in \mathcal{U}$,
(ii) $\mathfrak{H}(-\xi) = \mathfrak{H}(\xi)$ for each $\xi \in \mathcal{U}$,
(iii) $\mathfrak{H}(\xi + \rho) \leq \mathfrak{H}(\xi) + \mathfrak{H}(\rho)$, for each $\xi, \rho \in \mathcal{U}$,
(iv) if (ξ_j) is a sequence of vectors with $\lim_{j\to\infty} \mathfrak{H}(\xi_j - \xi) = 0$ and (μ_j) is a sequence of scalars with $\lim_{j\to\infty} \mu_j = \mu$, then $\lim_{j\to\infty} \mathfrak{H}(\mu_j \xi_j - \mu\xi) = 0$ (continuity of multiplication by scalars). A paranormed space $(\mathcal{U}, \mathfrak{H})$ is a linear space endowed with a paranorm $\mathfrak{H}$.

Later on, it was further investigated by many others as Candan [3], Erdem [6], Fadime [7], Hamid et al. [13, 14], Maddox [16], Murat et al. [21], Sheikh et al. [33] and many others.

Let $p = (p_l)$ be a sequence of positive real numbers with $0 < p_l \leq \sup_l p_l = H < \infty$ and $\mathcal{M} = \max(1, \sup_l p_l = H)$. Then, for any $a_l, b_l \in \mathbb{C}$ with $l \in \mathbb{N}$, we have

$$|a_l + b_l|^{p_l} \leq \mathcal{M}\{|a_l|^{p_l} + | + |b_l|^{p_l}\}.$$

Kizmaz [15] defined the following sequence spaces:

$$\ell_\infty(\Delta) = \{\xi = (\xi_j) : \Delta\xi \in \ell_\infty\},$$

$$c(\Delta) = \{\xi = (\xi_j) : \Delta\xi \in c\},$$

$$c_0(\Delta) = \{\xi = (\xi_j) : \Delta\xi \in c_0\}.$$

He showed that these spaces are Banach with the norm given by

$$\|\xi\| = |\xi_1| + \|\Delta\xi\|_\infty.$$

Further, it was studied by Et at al. [4, 5], Gaur [11], Hamid et al. [12, 13], Mohiuddine et al. [19], Mursaleen et al. [22, 23, 24, 25], Rahman et al. [28], Sheikh et al. [33], Zeren et al. [36] and many others. As in [15], the generalized Δ_η-difference sequence spaces were setup as follows:

$$\Delta_\eta u = (\Delta_\eta u_j) = u_j - u_{j+\eta}, \text{ for all } \eta \in \mathbb{N} = \{0, 1, 2, \cdots\}.$$

Later in [5, 35], for a sequence of positive real number $v = (v_j)$, we have the following:

$$\mathfrak{Z}(\Delta_v^\eta) = \left\{u = (u_j) \in \Omega : \ \Delta_v^\eta u \in \mathfrak{Z}\right\},$$

for $\eta \in \mathbb{N}$, $\Delta_v^0 u = (v_j u_j)$, $\Delta_v u = (v_j u_j - v_{j+1} u_{j+1})$, $\Delta_v^\eta u = \left(\Delta_v^{\eta-1} u_j - \Delta_v^{\eta-1} u_{j+1}\right)$, and thus,

$$\Delta_v^\eta u_j = \sum_{r=0}^{\eta} (-1)^r \binom{\eta}{r} v_{j+r} u_{j+r}.$$

Now, we consider $q = (q_j)$ as a sequence of positive numbers and set, $\mathfrak{S}_r = \sum_{j=0}^{r} q_j$ for $r \in \mathbb{N}$. Then, $\mathfrak{S}^q = (s_{rj}^q)$ represents the matrix of the Riesz mean $(\mathfrak{S}, q_r)$ as follows:

$$s_{rj}^q = \begin{cases} \frac{q_j}{\mathfrak{S}_r}, & \text{if } 0 \le j \le r, \\ 0, & \text{if } j > r. \end{cases}$$

This means $(\mathfrak{S}, q_r)$ is regular if and only if $\mathfrak{S}_r \to \infty$ as $r \to \infty$ as in [27, 30].

It is also important to note that as in [8], for some $s \ge 0$, various transformations on the spaces like $l_\infty(p, s)$ have been computed. Its further properties have been computed in [14, 28, 30, 37].

2 Main Results

This portion deals with the determination of new spaces given by a sequence of modulus functions. Also, some topological properties and inclusion relations between these spaces will be computed.

Let $p = (p_l)$ be bounded sequence with $0 < p_l \le \sup_l p_l < \infty$ and $\mathcal{G}$ be a modulus function. We now define

$$\mathfrak{S}_\infty^q(\Delta, s, \mathcal{G}, p, \eta) = \left\{u = (u_j) : \ \sup_l \left| l^{-s} \frac{1}{Q_l} \sum_{j=0}^{l} q_j \mathcal{G}(\Delta_\eta u_j) \right|^{p_l} < \infty, \ s \ge 0\right\};$$

$$\mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta) = \left\{u = (u_j) : \ \sup_l \left| l^{-s} \frac{1}{Q_l} \sum_{j=0}^{l} q_j \mathcal{G}(\Delta_\eta u_j) - \alpha \right|^{p_l} = 0, \ s \ge 0\right\};$$

for $\alpha \in \mathbb{R}$ and

$$\mathfrak{S}_0^q(\Delta, s, \mathcal{G}, p, \eta) = \left\{u = (u_j) : \ \sup_l \left| l^{-s} \frac{1}{Q_l} \sum_{j=0}^{l} q_j \mathcal{G}(\Delta_\eta u_j) \right|^{p_l} = 0, \ s \ge 0\right\}.$$

Theorem 2.1. *The spaces $\mathfrak{S}_\infty^q(\Delta, s, \mathcal{G}, p, \eta), \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta)$ and $\mathfrak{S}_0^q(\Delta, s, \mathcal{G}, p, \eta)$ are complete linear spaces paranormed by $\mathfrak{G}$, where*

$$\mathfrak{G}(\xi) = \sum_{r=1}^{\eta} |\xi_r| + \sup_l \left| l^{-s} \frac{1}{\mathcal{Q}_l} \sum_{i=0}^{l} q_i \mathcal{G}(\Delta_\eta \xi_i) \right|^{\frac{p_l}{\mathcal{M}}},$$

with $\mathcal{M} = (1, \sup_l p_l)$.

Proof. To prove the result, we only consider the case of $\mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, p, \eta)$ and then others can be proved on similar lines.

The linearity follows from the following inequality:

$$\begin{aligned}\mathfrak{G}(\mho+\xi) &= \sum_{r=1}^{\eta}|\mho_r+\xi_r| + \sup_l \left| l^{-s}\frac{1}{\mathcal{Q}_l}\sum_{i=0}^{l} q_i\mathcal{G}\left(\Delta^v_\eta(\mho_j+\xi_j)\right)\right|^{\frac{p_l}{\mathcal{M}}}\\ &\le \sum_{r=1}^{\eta}|\mho_r| + \sup_l \left| l^{-s}\frac{1}{\mathcal{Q}_l}\sum_{i=0}^{l} q_i\mathcal{G}\left(\Delta^v_\eta\mho_j)\right)\right|^{\frac{p_l}{\mathcal{M}}}\\ &\qquad + \sum_{r=1}^{\eta}|\xi_r| + \sup_l \left| \frac{1}{\mathcal{Q}_l}\sum_{i=0}^{l} q_i\mathcal{G}\left(\Delta^v_\eta\xi_j\right)\right|^{\frac{p_l}{\mathcal{M}}}\\ &= \mathfrak{G}(\mho)+\mathfrak{G}(\xi).\end{aligned}$$

Also, as in [4] for $\sigma \in \mathbb{R}$, the scalar multiplication follows by employing the inequality

$$|\sigma|^{p_j} \le \max\{1, |\sigma|^{\mathcal{M}}\}.$$

Obviously, $\mathfrak{G}(0) = 0$ and $\mathfrak{G}(-\xi) = -\mathfrak{G}(\xi)$ for each $\xi \in \mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, p, \eta)$. Further, the subadditivity of $\mathfrak{G}$ follows from the following inequality:

$$\mathfrak{G}(\sigma\xi) \le \max(1, |\sigma|)\mathfrak{G}(\xi).$$

Now to prove it is complete, let (ξ^n) be any Cauchy sequence of $\mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, p, \eta)$, where $(\xi^n) = (\xi^1_n, \xi^2_n, \cdots)$. Thus, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$\mathfrak{G}(\xi^\varsigma - \xi^t) < \varepsilon,$$

for each $\varsigma, t \ge n_0(\varepsilon)$. Now for a fixed $r \in \mathbb{N}$ and by definition of $\mathfrak{G}$, we have

$$\sum_{r=1}^{\eta}|\xi^\varsigma_r - \xi^t_r| < \varepsilon \quad and \quad \sup_l \left| l^{-s}\frac{1}{\mathcal{Q}_l}\sum_{i=0}^{l} q_i\mathcal{G}\left(\Delta_\eta\xi^\varsigma_i - \xi^t_i\right)\right|^{\frac{p_l}{\mathcal{M}}} < \varepsilon, \tag{2.1}$$

for each $\varsigma, t \ge n_0(\varepsilon)$. This yields that

$$\mathfrak{G}(\xi^\varsigma_r - \xi^t_r) < \varepsilon,$$

for each $\varsigma, t \ge n_0(\varepsilon)$. Consequently, (ξ^ς_r) is a Cauchy sequence in $\mathbb{R}$, so let $\lim_{\varsigma\to\infty}\xi^\varsigma_r = \xi_r$. Using (2.1), for $\varepsilon > 0$, we can find a positive integer $n_0(\varepsilon)$ such that

$$\left| \frac{l^{-s}}{\mathcal{Q}_l}\sum_{i=0}^{l} q_i\mathcal{G}\left(\Delta_\eta\xi^\varsigma_i - \xi^t_i\right)\right|^{p_l} < \varepsilon^{\mathcal{M}}, \tag{2.2}$$

for each $\varsigma, t \ge n_0(\varepsilon)$. Now, by choosing $t \to \infty$ in (2.2), we see

$$\left| \frac{l^{-s}}{\mathcal{Q}_l}\sum_{i=0}^{l} q_i\mathcal{G}\left(\Delta_\eta\xi^\varsigma_i - \xi_i\right)\right|^{p_l} < \varepsilon^{\mathcal{M}},$$

for each $\varsigma \ge n_0(\varepsilon)$. Hence, $\mathfrak{G}(\xi^\varsigma - \xi) \le \varepsilon$ for each $\varsigma \ge n_0(\varepsilon)$. Lastly, by Minkowski's inequality and by taking $\varepsilon = 1$, we have

$$\mathfrak{G}(\xi^\varsigma) \le \mathfrak{G}(\xi) + \mathfrak{G}(\xi^\varsigma - \xi) \le 1 + \mathfrak{G}(\xi),$$

for each $\varsigma \ge n_0(1)$, showing thereby $\xi \in \mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, p, \eta)$. Since, $\mathfrak{G}(\xi^\varsigma - \xi) \le \varepsilon$ for each $\varsigma \ge n_0(\varepsilon)$, consequently, we see $\xi^\varsigma \to \xi$ as $\varsigma \to \infty$, showing thereby that $\mathfrak{S}^q_\infty(\Delta, \mathcal{G}, p, \eta)$ is complete. □

Theorem 2.2. *If for a bounded sequences $p = (p_k)$ and $t = (t_k)$ of positive numbers with $0 < p_k \leq t_k < \infty$ for all $k \in \mathbb{N}$, then for any modulus function $\mathcal{G}$, we have*

(i) $\mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, p, \eta) \subset \mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, t, \eta)$.

(ii) $\mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta) \subset \mathfrak{S}^q(\Delta, s, \mathcal{G}, t, \eta)$.

(iii) $\mathfrak{S}^q_0(\Delta, s, \mathcal{G}, p, \eta) \subset \mathfrak{S}^q_0(\Delta, s, \mathcal{G}, t, \eta)$.

Proof. To prove the result, we only prove (i). Let $\xi \in \mathfrak{S}^q_\infty(\Delta, s, \mathcal{G}, p, \eta)$, then

$$\sup_l \left| \frac{l^{-s}}{Q_l} \sum_{i=0}^{l} q_i \mathcal{G}\left(\Delta_\eta \xi_i\right) \right|^{p_l} < \infty.$$

Now, for l large, say $l \geq l_0$ for some fixed l_0, we see

$$\left| \frac{l^{-s}}{Q_l} \sum_{i=0}^{l} q_i \mathcal{G}\left(\Delta_\eta \xi_i\right) \right|^{p_l} \leq \left| \frac{l^{-s}}{Q_l} \sum_{i=0}^{l} q_i \mathcal{G}\left(\Delta_\eta \xi_i\right) \right|^{t_l} < \infty,$$

and consequently the result follows. □

Theorem 2.3. *Let the bounded sequences $p = (p_k)$ and $t = (t_k)$ of positive numbers with $0 < p_k, t_k < \infty$ for each $k \in \mathbb{N}$, then for a modulus function $\mathcal{G}$, we have*

$$\mathfrak{S}^q(\Delta, s, \mathcal{G}, r, \eta) = \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta) \cap \mathfrak{S}^q(\Delta, s, \mathcal{G}, t, \eta),$$

where $r_k = \min(p_k, t_k)$.

Proof. Clearly from Theorem **??** (ii), we have

$$\mathfrak{S}^q(\Delta, s, \mathcal{G}, r, \eta) \subset \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta) \cap \mathfrak{S}^q(\Delta, s, \mathcal{G}, t, \eta).$$

Now for any complex number δ, we see

$$|\delta|^{r_j} \leq \max\{|\delta|^{p_j}, |\delta|^{t_j}\},$$

hence it follows that

$$\mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta) \cap \mathfrak{S}^q(\Delta, s, \mathcal{G}, t, \eta) \subset \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta).$$

□

Theorem 2.4. *For any modulus function $\mathcal{G}$, we have*

(i) $\mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta) \subset \mathfrak{S}^q(\Delta, s, p, \eta)$, *provided* $\lim_{t\to\infty} \frac{\mathcal{G}(t)}{t}$ *exists.*

(ii) $\mathfrak{S}^q(\Delta, s, p, \eta) \subset \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta)$, *if there exists a positive constant λ such that $\mathcal{G}(t) \leq \lambda t$ for all $t \geq 0$.*

Proof. To prove (i), using proposition 1 of Maddox [16], we see

$$\beta = \inf\left\{\frac{\mathcal{G}(t)}{t} : t > 0\right\},$$

so that $0 \leq \beta \leq \mathcal{G}(1)$. Let $\xi = (\xi_j) \in \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta)$ and $\beta > 0$, then by definition of β, we have $\beta t \leq f(t)$ for all $t \geq 0$ and

$$\left| l^{-s} \frac{1}{Q_l} \sum_{i=0}^{l} q_i \left(\Delta_\eta \xi_i - L\right) \right|^{p_l} \leq \max\left(1, \beta^{-H}\right) \left| l^{-s} \frac{1}{Q_l} \sum_{i=0}^{l} q_i \mathcal{G}\left(\Delta_\eta \xi_i - L\right) \right|^{p_l}.$$

Hence, it follows that $\xi = (\xi_j) \in \mathfrak{S}^q(\Delta, s, p, \eta)$.

Now to prove (ii), let $\mathcal{G}(t) \leq \lambda t$ for each $t \geq 0$ and $\lambda > 0$. Thus, for $\xi = (\xi_j) \in \mathfrak{S}^q(\Delta, s, p, \eta)$, we see $\beta t \leq f(t)$ for all $t \geq 0$ and

$$\left| l^{-s} \frac{1}{Q_l} \sum_{i=0}^{l} q_i \mathcal{G}\left(\Delta_\eta \xi_i - L\right) \right|^{p_l} \leq \max\left(1, \lambda^{-H}\right) \left| l^{-s} \frac{1}{Q_l} \sum_{i=0}^{l} q_i \left(\Delta_\eta \xi_i - L\right) \right|^{p_l}.$$

Hence, it follows that $\xi = (\xi_j) \in \mathfrak{S}^q(\Delta, s, \mathcal{G}, p, \eta)$.

□

Theorem 2.5. *If $\mathcal{G}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are modulus function, then for $s \geq 0$, we have*
(i) $\mathfrak{S}^q(\Delta, s, \mathcal{G}_1, p, \eta) \subset \mathfrak{S}^q(\Delta, s, \mathcal{G}o\mathcal{G}_1, p, \eta)$.
(ii) $\mathfrak{S}^q(\Delta, s, \mathcal{G}_1, p, \eta) \cap \mathfrak{S}^q(\Delta, s, \mathcal{G}_2, p, \eta) \subset \mathfrak{S}^q(\Delta, s, \mathcal{G}_1 + \mathcal{G}_2, p, \eta)$.

Proof. To prove (i), let $\xi \in \mathfrak{S}^q(\Delta, s, \mathcal{G}_1, p, \eta)$. Then, for $\epsilon > 0$, choose $\delta > 0$ such that $\mathcal{G}(t) < \epsilon$ for each $0 < t < \delta$.

Therefore, for $y_l = \left|l^{-s}\frac{1}{Q_l}\sum_{y_l>\delta} q_j\mathcal{G}_1(\Delta_\eta u_j)\right|^{p_l}$, we have

$$\begin{aligned}
&\left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j(\mathcal{G}o\mathcal{G}_1)(\Delta_\eta\xi_j)\ \right|^{p_l} \\
&= \left|l^{-s}\frac{1}{Q_l}\sum_{y_l\leq\delta} q_j(\mathcal{G}o\mathcal{G}_1)(\Delta_\eta\xi_j)\right|^{p_l} + \left|l^{-s}\frac{1}{Q_l}\sum_{y_l>\delta} q_j(\mathcal{G}o\mathcal{G}_1)(\Delta_\eta\xi_j)\right|^{p_l} \\
&\leq (\epsilon)^{\mathcal{H}} + \max\left(1, \left(2\frac{\mathcal{G}(1)}{\delta}\right)^{\mathcal{H}}\right)\left|l^{-s}\frac{1}{Q_l}\sum_{y_l\leq\delta} q_j\mathcal{G}_1(\Delta_\eta\xi_j)\right|^{p_l}.
\end{aligned}$$

where $\sup_l p_l = \mathcal{H}$. Consequently, $\xi \in \mathfrak{S}^q(\Delta, s, \mathcal{G}_1 o\mathcal{G}_2, p, \eta)$ and the proof of the theorem is complete.

(ii) Let $\xi \in \mathfrak{S}^q(\Delta, s, \mathcal{G}_1, p, \eta) \cap \mathfrak{S}^q(\Delta, s, \mathcal{G}_2, p, \eta)$. Therefore, we have

$$\sup_l\left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j\mathcal{G}_1(\Delta_\eta\xi_j)\right|^{p_l} < \infty$$

and

$$\sup_l\left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j\mathcal{G}_2(\Delta_\eta\xi_j)\right|^{p_l} < \infty.$$

The rest of the proof follows from the equality

$$\begin{aligned}
&\left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j(\mathcal{G}_1 + \mathcal{G}_2)(\Delta_\eta\xi_j)\right|^{p_l} \\
&\quad = \left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j\big(\mathcal{G}_1(\Delta_\eta\xi_j) + \mathcal{G}_2(\Delta_\eta\xi_j)\big)\right|^{p_l} \\
&\quad \leq \mathcal{M}\left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j\big(\mathcal{G}_1(\Delta_\eta\xi_j)\big)\right|^{p_l} + \mathcal{M}\left|l^{-s}\frac{1}{Q_l}\sum_{j=0}^{l} q_j\big(\mathcal{G}_2(\Delta_\eta\xi_j)\big)\right|^{p_l}.
\end{aligned}$$

Therefore, we conclude that, $\xi \in \mathfrak{S}^q(\Delta, s, \mathcal{G}_1 + \mathcal{G}_2, p, \eta)$ and the proof of the theorem is complete. □

3 Conclusion remarks

In this study, some new difference sequence spaces have been constructed using modulus function and establish some properties corresponding to these spaces. Moreover, the inclusion properties have been given. The given spaces in this study can be converted into new types of spaces by changing the difference operator.

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