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Uncertainty inequalities for the Gabor transform on certain Lie groups

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Abstract In this paper, we establish Pitt's uncertainty inequality for the Gabor transform on certain classes of Lie groups such as Heisenberg motion group and the class of diamond Lie groups. These inequalities can be used to prove some important uncertainty inequalities like logarithmic uncertainty inequality and Heisenberg uncertainty inequality for the Gabor transform on these classes of Lie groups. Hölder's inequality, Fubini's theorem, Plancherel formula and the representation theory are the important tools used to prove the main results.

1 Introduction

The uncertainty principle refers to a fundamental concept in harmonic analysis. At its core, it describes limits on how precisely certain pairs of physical quantities (conjugate variables) can be known or measured simultaneously. The idea is that there are inherent limitations to the precision with which certain properties of a system can be observed or determined at the same time. Mathematically, a non-zero function g and its Fourier transform $\widehat{g}$ cannot both be concentrated simultaneously. For various mathematical formulations of uncertainty principles, refer to [1, 2]. Pitt's uncertainty inequality (PUI) is one of the most important uncertainty principles. This inequality operates in the domain of harmonic analysis, relating the norms of functions and their Fourier transforms, which is essential for understanding the broader mathematical framework that supports other two important uncertainty inequalities, viz., logarithmic uncertainty inequality (LUI) and Heisenberg uncertainty inequality (HUI).
Let $\mathbb{S}(\mathbb{R}^n)$ denote the Schwartz space, i.e., the space of rapidly decreasing functions on $\mathbb{R}^n$ and the Fourier transform of $g \in L^1(\mathbb{R}^n)$ be defined by

$$\widehat{g}(\omega) = \int_{\mathbb{R}^n} g(y)\, e^{-2\pi i \omega\cdot y}\, dy \text{ for all } \omega \in \mathbb{R}^n.$$

PUI [3] for the Fourier transform on $\mathbb{R}^n$ can be stated as follows:

Theorem 1.1. *For $g \in \mathbb{S}(\mathbb{R}^n)$ and $0 \le m < n$,*

$$\int_{\mathbb{R}^n} \|\omega\|^{-m}\, |\widehat{g}(\omega)|^2\, d\omega \le E_m \int_{\mathbb{R}^n} \|y\|^m\, |g(y)|^2\, dy,$$

where $E_m = \pi^m \left[\Gamma\left(\frac{n-m}{4}\right) \Big/ \Gamma\left(\frac{n+m}{4}\right)\right]^2$, $\|\cdot\|$ being the Euclidean norm on $\mathbb{R}^n$ and Γ denotes the gamma function.

The LUI, introduced in [3], is another important quantitative uncertainty principle. It refines the uncertainty principles by introducing a logarithmic term, providing a stronger bound in certain situations, but still tied to the quantum mechanical setting and the Fourier relationship between the position and momentum. We state the generalised form of LUI [4, Theorem 1.4] for the Fourier transform on $\mathbb{R}^n$.

Theorem 1.2. *For $g \in \mathbb{S}(\mathbb{R}^n)$ and $a, b \geq 1$, we have*

$$D\,\|g\|_2^{\frac{2}{a}+\frac{2}{b}} \leq \|g\|_2^{\frac{2}{b}} \left(\int_{\mathbb{R}^n} (\ln\|y\|)^a\,|g(y)|^2\,dy\right)^{\frac{1}{a}} + \|g\|_2^{\frac{2}{a}} \left(\int_{\mathbb{R}^n} (\ln\|\omega\|)^b\,|\widehat{g}(\omega)|^2\,d\omega\right)^{\frac{1}{b}}, \quad (1.1)$$

where $D = \dfrac{\Gamma'(n/4)}{\Gamma(n/4)} - \ln\pi$, $\|\cdot\|_2$ and $\|\cdot\|$ denote the L^2-norm in $L^2(\mathbb{R}^n)$ and the Euclidean norm on $\mathbb{R}^n$ respectively.

The HUI is one of the precise quantitative formulations of the uncertainty principles. It establishes the minimum uncertainty between conjugate variables (like position and momentum), with a specific focus on the product of the uncertainties being bounded below. The following theorem gives the generalised HUI [5] for functions on $\mathbb{R}^n$.

Theorem 1.3. *For $g \in L^2(\mathbb{R}^n)$ and $a, b \geq 1$, we have*

$$\frac{n\|g\|_2^{\left(\frac{1}{a}+\frac{1}{b}\right)}}{4\pi} \leq \left(\int_{\mathbb{R}^n} \|y\|^{2a}\,|g(y)|^2\,dy\right)^{\frac{1}{2a}} \left(\int_{\mathbb{R}^n} \|\omega\|^{2b}\,|\widehat{g}(\omega)|^2\,d\omega\right)^{\frac{1}{2b}}.$$

For more details related to uncertainty principles, see [1], [5] and [6].
The Gabor transform is a mathematical tool used in signal processing and time-frequency analysis. It is a type of the Fourier transform that allows a signal to be analysed in both the time and frequency domains simultaneously. This is especially useful when analysing signals whose frequency content changes over time, a common occurrence in many real-world signals, such as speech, music, and biomedical signals. The Gabor transform provides a way to represent a signal as a collection of localised frequency components. Let $g, \varphi \in L^2(\mathbb{R}^n)$ with φ a fixed non-zero window function, then the Gabor transform $G_\varphi g$ of g with respect to φ is

$$G_\varphi g(t,\omega) = \int_{\mathbb{R}^n} g(y)\,\overline{\varphi}(y-t)\,e^{-2\pi i\omega\cdot y}\,dy$$

for all $(t,\omega) \in \mathbb{R}^{2n}$.
In Section 2, we shall briefly discuss some basic notations and results related to the Gabor transform. The alternate versions of PUI, LUI and HUI for the Gabor transform on $\mathbb{R}^n$ are proved in the next section. The Section 4 incorporates the proofs of PUI, LUI and HUI for the Gabor transform on the Heisenberg motion group. Finally, in Section 5, we establish the three inequalities for the Gabor transform on the class of diamond Lie groups.

2 Gabor Transform

In this section, we briefly introduce the Gabor transform. Assume G to be a unimodular locally compact group of type I which is second-countable and separable. The unitary dual of G is denoted by $\widehat{G}$ and it is the set of all equivalence classes of unitary representations of G that are irreducible. $\widehat{G}$ is endowed with the Mackey-Borel structure. Let the fixed left Haar measure on G be μ_G that uniquely determines the Plancherel measure $\mu_{\widehat{G}}$ on $\widehat{G}$. For $\zeta \in \widehat{G}$, let $\mathrm{HS}(\mathscr{H}_\zeta)$ denote the space of all Hilbert-Schmidt operators on the representation space $\mathscr{H}_\zeta$ of ζ.
For each $(x,\zeta) \in G \times \widehat{G}$, define $\mathscr{H}_{(x,\zeta)} = \zeta(x)\mathrm{HS}(\mathscr{H}_\zeta)$, where $\zeta(x)\mathrm{HS}(\mathscr{H}_\zeta) = \{\zeta(x)T : T \in \mathrm{HS}(\mathscr{H}_\zeta)\}$. The space $\mathscr{H}_{(x,\zeta)}$ forms a Hilbert space with the inner product

$$\langle \zeta(x)T, \zeta(x)S\rangle_{\mathscr{H}_{(x,\zeta)}} = \mathrm{tr}\,(S^*T) = \langle T, S\rangle_{\mathrm{HS}(\mathscr{H}_\zeta)}.$$

It is clear that $\mathscr{H}_{(x,\zeta)} = \mathrm{HS}(\mathscr{H}_\zeta)$ for all $(x,\zeta) \in G \times \widehat{G}$. Also, the family $\{\mathscr{H}_{(x,\zeta)}\}_{(x,\zeta)\in G\times\widehat{G}}$ of Hilbert spaces indexed by $G \times \widehat{G}$ is a field of Hilbert spaces over $G \times \widehat{G}$.

Let $\mathscr{H}^2(G \times \widehat{G})$ denote the direct integral of $\{\mathscr{H}_{(x,\zeta)}\}_{(x,\zeta)\in G\times\widehat{G}}$ with respect to the product measure $dx\, d\zeta$, i.e., the space of all vector fields U on $G \times \widehat{G}$ that are measurable and

$$\|U\|^2_{\mathscr{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \|U(x,\zeta)\|^2_{(x,\zeta)}\, dx\, d\zeta < \infty.$$

The space $\mathscr{H}^2(G \times \widehat{G})$ forms a Hilbert space under the inner product

$$\langle U, V\rangle_{\mathscr{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \text{tr}\,[U(x,\zeta)V(x,\zeta)^*]\, dx\, d\zeta.$$

Let $\varphi \in L^2(G)$ be a window function and $g \in C_c(G)$, the set of all complex-valued functions on G that are continuous and have compact supports. For $(x, \zeta) \in G \times \widehat{G}$, the *Gabor Transform* of g with respect to φ is defined as a field of operators on $G \times \widehat{G}$ that are measurable and

$$G_\varphi g(x,\zeta) := \int_G g(y)\, \overline{\varphi}(x^{-1}y)\, \zeta(y)^*\, dy.$$

For each $x \in G$, define $g^\varphi_x : G \to \mathbb{C}$ by $g^\varphi_x(y) := g(y)\, \overline{\varphi}(x^{-1}y)$. Since $g \in C_c(G)$ and $\varphi \in L^2(G)$, we have $g^\varphi_x \in L^1(G) \cap L^2(G)$ for all $x \in G$. Thus,

$$\mathscr{F}(g^\varphi_x)(\zeta) = \int_G g^\varphi_x(y)\, \zeta(y)^*\, dy = \int_G g(y)\, \overline{\varphi}(x^{-1}y)\, \zeta(y)^*\, dy = G_\varphi g(x,\zeta).$$

For almost all $\zeta \in \widehat{G}$, the operator $\mathscr{F}(g^\varphi_x)(\zeta)$ is a Hilbert-Schmidt operator (by Plancherel theorem [7, Theorem 7.44]) and hence $G_\varphi g(x,\zeta)$ is a Hilbert-Schmidt operator for all $x \in G$ and for almost all $\zeta \in \widehat{G}$. In [8], it is shown that $\|G_\varphi g\|_{\mathscr{H}^2(G\times\widehat{G})} = \|\varphi\|_2\, \|g\|_2$.

The map $G_\varphi : C_c(G) \to \mathscr{H}^2(G \times \widehat{G})$ given by $g \mapsto G_\varphi g$, being a multiple of an isometry, can be extended uniquely to a linear operator from $L^2(G)$ into a closed subspace of $\mathscr{H}^2(G \times \widehat{G})$ and the extension is still denoted by G_φ. The extension operator G_φ is bounded and it satisfies

$$\|G_\varphi g\|_{\mathscr{H}^2(G\times\widehat{G})} = \|\varphi\|_2\, \|g\|_2. \tag{2.1}$$

for all $g \in L^2(G)$. For $g, \varphi \in L^2(G)$, we have $G_\varphi g(x,\zeta) = \mathscr{F}(g^\varphi_x)(\zeta)$ (see [6]).

3 Euclidean Group $\mathbb{R}^n$

In this section, we shall discuss PUI, LUI and HUI for the Gabor transform on $\mathbb{R}^n$. We now prove the PUI for the Gabor transform on $\mathbb{R}^n$.

Theorem 3.1. *For any $g, \varphi \in \mathbb{S}(\mathbb{R}^n)$ with φ a window function and $0 \le m < n$, we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^{-m}\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega \le E_m\, \|\varphi\|_2^2 \int_{\mathbb{R}^n} \|y\|^m\, |g(y)|^2\, dy,$$

where E_m is as in Theorem 1.1.

Proof. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function. For $x \in \mathbb{R}^n$, we define $g^\varphi_x : \mathbb{R}^n \to \mathbb{C}$ by $g^\varphi_x(y) = g(y)\, \overline{\varphi}(y - x)$ for all $y \in \mathbb{R}^n$. Then $g^\varphi_x \in \mathbb{S}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and

$$G_\varphi g(x,\omega) = \mathscr{F}(g^\varphi_x)(\omega) \text{ for all } \omega \in \mathbb{R}^n.$$

Using Theorem 1.1, we have for all $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \|\omega\|^{-m}\, |\mathscr{F}(g^\varphi_x)(\omega)|^2\, d\omega \le E_m \int_{\mathbb{R}^n} \|y\|^m\, |g^\varphi_x(y)|^2\, dy$$

$$\Rightarrow \quad \int_{\mathbb{R}^n} \|\omega\|^{-m}\, |G_\varphi g(x,\omega)|^2\, d\omega \le E_m \int_{\mathbb{R}^n} \|y\|^m\, |g(y)|^2\, |\varphi(y-x)|^2\, dy.$$

On integrating both sides w.r.t. dx and using Fubini's theorem, we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^{-m}\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega \le E_m \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|y\|^m\, |g(y)|^2\, |\varphi(y-x)|^2\, dx\, dy$$
$$= E_m\, \|\varphi\|_2^2 \int_{\mathbb{R}^n} \|y\|^m\, |g(y)|^2\, dy. \qquad \square$$

We now prove LUI for the Gabor transform on $\mathbb{R}^n$.

Theorem 3.2. *For any $g, \varphi \in \mathbb{S}(\mathbb{R}^n)$ with φ a window function and $a, b \ge 1$, we have*

$$D\, \|\varphi\|_2^{\frac{2}{b}}\, \|g\|_2^{\frac{2}{a}+\frac{2}{b}}$$
$$\le \|\varphi\|_2^{\frac{2}{b}}\, \|g\|_2^{\frac{2}{b}} \left(\int_{\mathbb{R}^n} (\ln\|y\|)^a\, |g(y)|^2\, dy\right)^{\frac{1}{a}} + \|g\|_2^{\frac{2}{a}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\ln\|\omega\|)^b\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega\right)^{\frac{1}{b}}, \tag{3.1}$$

where D is as in Theorem 1.2.

Proof. Let $g, \varphi \in \mathbb{S}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, define $g_x^\varphi : \mathbb{R}^n \to \mathbb{C}$ by $g_x^\varphi(y) = g(y)\, \overline{\varphi}(y-x)$ for all $y \in \mathbb{R}^n$. Then $g_x^\varphi \in \mathbb{S}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and

$$G_\varphi g(x,\omega) = \mathscr{F}(g_x^\varphi)(\omega).$$

Using Theorem 1.2 for $a = b = 1$, we have

$$D\, \|g_x^\varphi\|_2^2 \le \int_{\mathbb{R}^n} \ln\|y\|\, |g_x^\varphi(y)|^2\, dy + \int_{\mathbb{R}^n} \ln\|\omega\|\, |\mathscr{F}(g_x^\varphi)(\omega)|^2\, d\omega$$
$$\Rightarrow \qquad D \int_{\mathbb{R}^n} |g_x^\varphi(y)|^2\, dy \le \int_{\mathbb{R}^n} \ln\|y\|\, |g_x^\varphi(y)|^2\, dy + \int_{\mathbb{R}^n} \ln\|\omega\|\, |G_\varphi g(x,\omega)|^2\, d\omega$$

for all $x \in \mathbb{R}^n$. On integrating both sides w.r.t. dx and using Fubini's theorem, we get

$$D \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |g(y)|^2\, |\varphi(y-x)|^2\, dx\, dy$$
$$\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln\|y\|\, |g(y)|^2\, |\varphi(y-x)|^2\, dx\, dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln\|\omega\|\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega$$

which implies

$$D\, \|\varphi\|_2^2\, \|g\|_2^2 \le \|\varphi\|_2^2 \int_{\mathbb{R}^n} \ln\|y\|\, |g(y)|^2\, dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln\|\omega\|\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega. \tag{3.2}$$

Now applying Hölder's inequality with exponents a and $\dfrac{a}{a-1}$ when $a \ne 1$, we have

$$\int_{\mathbb{R}^n} \ln\|y\|\, |g(y)|^2\, dy = \int_{\mathbb{R}^n} \ln\|y\|\, |g(y)|^{\frac{2}{a}}\, |g(y)|^{2\left(1-\frac{1}{a}\right)}\, dy$$
$$\le \left(\int_{\mathbb{R}^n} (\ln\|y\|)^a\, |g(y)|^2\, dy\right)^{\frac{1}{a}} \left(\int_{\mathbb{R}^n} |g(y)|^2\, dy\right)^{1-\frac{1}{a}}$$
$$= \left(\int_{\mathbb{R}^n} (\ln\|y\|)^a\, |g(y)|^2\, dy\right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}}.$$

Similarly applying Hölder's inequality with exponents b and $\dfrac{b}{b-1}$ when $b \ne 1$ and then using (2.1), we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln\|\omega\|\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega \le \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\ln\|\omega\|)^b\, |G_\varphi g(x,\omega)|^2\, dx\, d\omega\right)^{\frac{1}{b}} \left(\|g\|_2^2\, \|\varphi\|_2^2\right)^{1-\frac{1}{b}}.$$

Using the above two inequalities, we can express inequality (3.2) as

$$D\,\|\varphi\|_2^2\,\|g\|_2^2 \le \|\varphi\|_2^2 \left(\int_{\mathbb{R}^n} (\ln\|y\|)^a\,|g(y)|^2\,dy\right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\ln\|\omega\|)^b\,|G_\varphi g(x,\omega)|^2\,dx\,d\omega\right)^{\frac{1}{b}} \left(\|g\|_2^2\,\|\varphi\|_2^2\right)^{1-\frac{1}{b}}$$

which implies

$$\begin{aligned} &D\,\|\varphi\|_2^{\frac{2}{b}}\,\|g\|_2^{\frac{2}{a}+\frac{2}{b}} \\ &\le \|\varphi\|_2^{\frac{2}{b}}\,\|g\|_2^{\frac{2}{b}} \left(\int_{\mathbb{R}^n} (\ln\|y\|)^a\,|g(y)|^2\,dy\right)^{\frac{1}{a}} \\ &\quad + \|g\|_2^{\frac{2}{a}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\ln\|\omega\|)^b\,|G_\varphi g(x,\omega)|^2\,dx\,d\omega\right)^{\frac{1}{b}}. \end{aligned}$$ □

Remark 3.3. The above inequality can also be deduced from Theorem 3.1. For every $0 \le m < 1$, consider

$$P(m) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^{-m}\,|G_\varphi g(x,\omega)|^2\,dx\,d\omega - E_m\,\|\varphi\|_2^2 \int_{\mathbb{R}^n} \|y\|^m\,|g(y)|^2\,dy,$$

where E_m is as in Theorem 1.1. On applying Plancherel formula for $\mathbb{R}^n$, Hölder's inequality and using the same mechanism as in [4, Theorem 2.2], we get the required inequality (3.1).

The following HUI for the Gabor transform on $\mathbb{R}^n$ has been proved in [6]. Here, we shall give an alternative proof of the inequality using the LUI.

Theorem 3.4. *For any $g, \varphi \in L^2(\mathbb{R}^n)$ with φ a window function and $a, b \ge 1$, we have*

$$\|\varphi\|_2^{\frac{1}{b}}\,\|g\|_2^{\frac{1}{a}+\frac{1}{b}} \le C \left(\int_{\mathbb{R}^n} \|y\|^{2a}\,|g(y)|^2\,dy\right)^{\frac{1}{2a}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^{2b}\,|G_\varphi g(x,\omega)|^2\,dx\,d\omega\right)^{\frac{1}{2b}}. \tag{3.3}$$

Proof. Proceeding as in [5, Theorem 3.2], it is sufficient to prove the required inequality for functions in $\mathbb{S}(\mathbb{R}^n)$. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function, then taking $a = b = 1$ in (3.1), we have

$$D\,\|\varphi\|_2^2\,\|g\|_2^2 \le \|\varphi\|_2^2 \int_{\mathbb{R}^n} \ln\|y\|\,|g(y)|^2\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln\|\omega\|\,|G_\varphi g(x,\omega)|^2\,dx\,d\omega. \tag{3.4}$$

For every $x > 0$, we have $\ln x - \dfrac{1}{x} \le \psi(x)$, where $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. So we can write

$$\ln\left(\frac{ne^{-4}}{4\pi}\right) \le \ln\left(\frac{n}{4}\right) - \frac{4}{n} - \ln\pi \le \psi\left(\frac{n}{4}\right) - \ln\pi = D.$$

Using (3.4) and then applying Jensen's inequality for ln which is a concave function, we have

$$\begin{aligned} \ln\left(\frac{ne^{-4}}{4\pi}\right) &\le \int_{\mathbb{R}^n} \ln\|y\|\,\frac{|g(y)|^2}{\|g\|_2^2}\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln\|\omega\|\,\frac{|G_\varphi g(x,\omega)|^2}{\|\varphi\|_2^2\,\|g\|_2^2}\,dx\,d\omega \\ &\le \frac{1}{2}\ln\left[\int_{\mathbb{R}^n} \|y\|^2\,\frac{|g(y)|^2}{\|g\|_2^2}\,dy\right] + \frac{1}{2}\ln\left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^2\,\frac{|G_\varphi g(x,\omega)|^2}{\|\varphi\|_2^2\,\|g\|_2^2}\,dx\,d\omega\right] \\ &= \ln\left[\left(\int_{\mathbb{R}^n} \|y\|^2\,\frac{|g(y)|^2}{\|g\|_2^2}\,dy\right)^{1/2} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^2\,\frac{|G_\varphi g(x,\omega)|^2}{\|\varphi\|_2^2\,\|g\|_2^2}\,dx\,d\omega\right)^{1/2}\right]. \end{aligned}$$

Using the fact that ln is an increasing function, we obtain

$$\frac{ne^{-4}}{4\pi} \leq \left(\int_{\mathbb{R}^n} \|y\|^2 \, \frac{|g(y)|^2}{\|g\|_2^2} \, dy\right)^{1/2} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^2 \, \frac{|G_\varphi g(x,\omega)|^2}{\|\varphi\|_2^2 \, \|g\|_2^2} \, dx \, d\omega\right)^{1/2}$$

$$\Rightarrow \quad \|\varphi\|_2 \, \|g\|_2^2 \leq C \left(\int_{\mathbb{R}^n} \|y\|^2 \, |g(y)|^2 \, dy\right)^{1/2} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^2 \, |G_\varphi g(x,\omega)|^2 \, dx \, d\omega\right)^{1/2}. \tag{3.5}$$

Applying Hölder's inequality and then using (2.1) as in Theorem 3.2, we obtain

$$\int_{\mathbb{R}^n} \|y\|^2 \, |g(y)|^2 \, dy \leq \left(\int_{\mathbb{R}^n} \|y\|^{2a} \, |g(y)|^2 \, dy\right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}}$$

and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^2 \, |G_\varphi g(x,\omega)|^2 \, dx \, d\omega \leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \|\omega\|^{2b} \, |G_\varphi g(x,\omega)|^2 \, dx \, d\omega\right)^{\frac{1}{b}} \left(\|\varphi\|_2^2 \, \|g\|_2^2\right)^{1-\frac{1}{b}}.$$

Combining the above inequalities with (3.5), we obtain the required inequality (3.3). □

4 Heisenberg Motion Group

In this section, we establish the three inequalities PUI, LUI and HUI for the Gabor transform on Heisenberg motion group. Consider the Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ having the group law

$$(v,s)\cdot(v',s') = \left(v+v', s+s' + \frac{1}{2}\,\mathrm{Im}\,(v\cdot\overline{v'})\right),$$

where $v,\ v' \in \mathbb{C}^n$ and $s,\ s' \in \mathbb{R}$. The unitary group $U(n)$ consists of automorphisms of $\mathbb{H}_n$ and its action on $\mathbb{H}_n$ is given by $(k,(v,s)) \mapsto (kv,s)$, where $k \in K$ and $(v,s) \in \mathbb{H}_n$. Consider a compact, connected Lie subgroup K of $U(n)$ such that $(K,\mathbb{H}_n)$ forms a Gelfand pair. Then $G = \mathbb{H}_n \rtimes K$ is the Heisenberg motion group whose group law is $(v,s,k)\cdot(v',s',k') = ((v,s)\cdot(kv',s'),kk')$, where $(v,s),\ (v',s') \in \mathbb{H}_n$ and $k,\ k' \in K$. For more details, refer to [9]. The following theorem gives the PUI [4] for the Fourier transform on the Heisenberg motion group.

Theorem 4.1. *Let $G = \mathbb{H}_n \rtimes K$ be the Heisenberg motion group. Then for any $g \in \mathbb{S}(G)$ and $0 \leq m < 1$, we have*

$$\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, |\delta|^{-m} \, \|\widehat{g}(\delta,\rho)\|_{HS}^2 \, |\delta|^n \, d\delta \leq E_m \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m \, |g(u,t,k)|^2 \, dt \, du \, dk,$$

where E_m is as in Theorem 1.1.

We now prove the PUI for the Gabor transform on Heisenberg motion group.

Theorem 4.2. *Let $G = \mathbb{H}_n \rtimes K$ be the Heisenberg motion group. Then for any $g \in \mathbb{S}(G)$, window function $\varphi \in \mathbb{S}(G)$ and $0 \leq m < 1$, we have*

$$\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho \, |\delta|^{-m} \, \|G_\varphi g(v,s,h,\delta,\rho)\|_{HS}^2 \, |\delta|^n \, dv \, ds \, dh \, d\delta$$
$$\leq E_m \, \|\varphi\|_2^2 \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m \, |g(u,t,k)|^2 \, du \, dt \, dk,$$

where E_m is as in Theorem 1.1.

Proof. Let $g,\varphi \in \mathbb{S}(G)$ with φ a window function. For $(v,s,h) \in G$, define $g^\varphi_{(v,s,h)} : G \to \mathbb{C}$ by

$$g^\varphi_{(v,s,h)}(u,t,k) = g(u,t,k)\,\overline{\varphi}((v,s,h)^{-1}(u,t,k)) \text{ for all } (u,t,k) \in G.$$

Then $g^{\varphi}_{(v,s,h)} \in \mathbb{S}(G)$ for all $(v,s,h) \in G$ and

$$G_{\varphi}g(v,s,h,\delta,\rho) = \mathscr{F}(g^{\varphi}_{(v,s,h)})(\delta,\rho).$$

Using Theorem 4.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_{\rho}\, |\delta|^{-m}\, \|\mathscr{F}(g^{\varphi}_{(v,s,h)})(\delta,\rho)\|^2_{\mathrm{HS}}\, |\delta|^n\, d\delta \\
&\leq E_m \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m\, |g^{\varphi}_{(v,s,h)}(u,t,k)|^2\, du\, dt\, dk
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_{\rho}\, |\delta|^{-m}\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|^2_{\mathrm{HS}}\, |\delta|^n\, d\delta \\
&\leq E_m \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m\, |g(u,t,k)|^2\, |\varphi((v,s,h)^{-1}(u,t,k))|^2\, du\, dt\, dk
\end{aligned}$$

for all $(v,s,h) \in G$. Integrating both sides with respect to $dv\, ds\, dh$, we get

$$\begin{aligned}
&\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_{\rho}\, |\delta|^{-m}\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|^2_{\mathrm{HS}}\, |\delta|^n\, dv\, ds\, dh\, d\delta \\
&\leq E_m \int_K \int_{\mathbb{H}_n} \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m\, |g(u,t,k)|^2\, |\varphi((v,s,h)^{-1}(u,t,k))|^2\, dv\, ds\, dh\, du\, dt\, dk \\
&= E_m\, \|\varphi\|_2^2 \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m\, |g(u,t,k)|^2\, du\, dt\, dk. \qquad \square
\end{aligned}$$

The following theorem gives the LUI [4] for the Fourier transform on the Heisenberg motion group.

Theorem 4.3. *Let $G = \mathbb{H}_n \rtimes K$ be the Heisenberg motion group. Then for any $g \in \mathbb{S}(G)$ and $a, b \geq 1$, we have*

$$\begin{aligned}
D\, \|g\|_2^{\frac{2}{a}+\frac{2}{b}} \leq{}& \|g\|_2^{\frac{2}{b}} \left(\int_K \int_{\mathbb{H}_n} (\ln \|(u,t)\|)^a\, |g(u,t,k)|^2\, dt\, du\, dk\right)^{\frac{1}{a}} \\
&+ \|g\|_2^{\frac{2}{a}} \left(\int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_{\rho}\, (\ln|\delta|)^b\, \|\widehat{g}(\delta,\rho)\|^2_{HS}\, |\delta|^n\, d\delta\right)^{\frac{1}{b}}, \qquad (4.1)
\end{aligned}$$

where D is as in Theorem 1.2.

We now prove the LUI for the Gabor transform on Heisenberg motion group.

Theorem 4.4. *Let $G = \mathbb{H}_n \rtimes K$ be the Heisenberg motion group. Then for any $g \in \mathbb{S}(G)$, window function $\varphi \in \mathbb{S}(G)$ and $a, b \geq 1$, we have*

$$\begin{aligned}
&D\, \|\varphi\|_2^{\frac{2}{b}}\, \|g\|_2^{\frac{2}{a}+\frac{2}{b}} \\
&\leq \|\varphi\|_2^{\frac{2}{b}}\, \|g\|_2^{\frac{2}{b}} \left(\int_K \int_{\mathbb{H}_n} (\ln \|(u,t)\|)^a\, |g(u,t,k)|^2\, du\, dt\, dk\right)^{\frac{1}{a}} \\
&\quad + \|g\|_2^{\frac{2}{a}} \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_{\rho}\, (\ln|\delta|)^b\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|^2_{HS}\, |\delta|^n\, dv\, ds\, dh\, d\delta\right)^{\frac{1}{b}}, \qquad (4.2)
\end{aligned}$$

where D is as in Theorem 1.2.

Proof. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function. For $(v,s,h) \in G$, we define $g^{\varphi}_{(v,s,h)} : G \to \mathbb{C}$ by $g^{\varphi}_{(v,s,h)}(u,t,k) = g(u,t,k)\, \overline{\varphi}((v,s,h)^{-1}(u,t,k))$ for all $(u,t,k) \in G$.
Then $g^{\varphi}_{(v,s,h)} \in \mathbb{S}(G)$ for all $(v,s,h) \in G$ and $G_{\varphi}g(v,s,h,\delta,\rho) = \mathscr{F}(g^{\varphi}_{(v,s,h)})(\delta,\rho)$.
Using Theorem 4.3 for $a = b = 1$, we have

$$D\, \|g^{\varphi}_{(v,s,h)}\|_2^2 \le \int_K \int_{\mathbb{H}_n} \ln \|(u,t)\|\, |g^{\varphi}_{(v,s,h)}(u,t,k)|^2 \, du\, dt\, dk + \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho\, \ln|\delta|\, \|\mathscr{F}(g^{\varphi}_{(v,s,h)})(\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, d\delta$$

which implies

$$\begin{aligned} &D \int_K \int_{\mathbb{H}_n} |g^{\varphi}_{(v,s,h)}(u,t,k)|^2 \, du\, dt\, dk \\ &\le \int_K \int_{\mathbb{H}_n} \ln \|(u,t)\|\, |g^{\varphi}_{(v,s,h)}(u,t,k)|^2 \, du\, dt\, dk \\ &\qquad + \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho\, \ln|\delta|\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, d\delta \end{aligned}$$

for all $(v,s,h) \in G$. Integrating both sides with respect to $dv\, ds\, dh$, we get

$$\begin{aligned} &D \int_K \int_{\mathbb{H}_n} \int_K \int_{\mathbb{H}_n} |g(u,t,k)|^2\, |\varphi((v,s,h)^{-1}(u,t,k))|^2\, dv\, ds\, dh\, du\, dt\, dk \\ &\le \int_K \int_{\mathbb{H}_n} \int_K \int_{\mathbb{H}_n} \ln\|(u,t)\|\, |g(u,t,k)|^2\, |\varphi((v,s,h)^{-1}(u,t,k))|^2\, dv\, ds\, dh\, du\, dt\, dk \\ &\quad + \int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho\, \ln|\delta|\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, dv\, ds\, dh\, d\delta. \end{aligned}$$

Thus,

$$\begin{aligned} D\, \|\varphi\|_2^2\, \|g\|_2^2 \le \|\varphi\|_2^2 &\int_K \int_{\mathbb{H}_n} \ln\|(u,t)\|\, |g(u,t,k)|^2\, du\, dt\, dk \\ &+ \int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho\, \ln|\delta|\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, dv\, ds\, dh\, d\delta. \end{aligned} \tag{4.3}$$

Now applying Hölder's inequality with exponents a and $\dfrac{a}{a-1}$ when $a \neq 1$, we have

$$\begin{aligned} &\int_K \int_{\mathbb{H}_n} \ln\|(u,t)\|\, |g(u,t,k)|^2\, du\, dt\, dk \\ &\le \left(\int_K \int_{\mathbb{H}_n} (\ln\|(u,t)\|)^a\, |g(u,t,k)|^2\, du\, dt\, dk \right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}}. \end{aligned}$$

Similarly applying Hölder's inequality with exponents b and $\dfrac{b}{b-1}$ when $b \neq 1$ and then using (2.1), we obtain

$$\begin{aligned} &\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho\, \ln|\delta|\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, dv\, ds\, dh\, d\delta \\ &\le \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho\, (\ln|\delta|)^b\, \|G_{\varphi}g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, dv\, ds\, dh\, d\delta \right)^{\frac{1}{b}} \left(\|g\|_2^2\, \|\varphi\|_2^2\right)^{1-\frac{1}{b}}. \end{aligned}$$

Using the above two inequalities, we can express inequality (4.3) as

$$
\begin{aligned}
&D\,\|\varphi\|_2^2\,\|g\|_2^2 \\
&\leq \|\varphi\|_2^2 \left(\int_K \int_{\mathbb{H}_n} (\ln\|(u,t)\|)^a\,|g(u,t,k)|^2\,du\,dt\,dk\right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}} \\
&\quad + \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\,(\ln|\delta|)^b\,\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\,|\delta|^n\,dv\,ds\,dh\,d\delta\right)^{\frac{1}{b}} \left(\|g\|_2^2\,\|\varphi\|_2^2\right)^{1-\frac{1}{b}}
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
&D\,\|\varphi\|_2^{\frac{2}{b}}\,\|g\|_2^{\frac{2}{a}+\frac{2}{b}} \\
&\leq\ \|\varphi\|_2^{\frac{2}{b}}\,\|g\|_2^{\frac{2}{b}} \left(\int_K \int_{\mathbb{H}_n} (\ln\|(u,t)\|)^a\,|g(u,t,k)|^2\,du\,dt\,dk\right)^{\frac{1}{a}} \\
&\quad + \|g\|_2^{\frac{2}{a}} \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\,(\ln|\delta|)^b\,\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\,|\delta|^n\,dv\,ds\,dh\,d\delta\right)^{\frac{1}{b}}. \qquad \square
\end{aligned}
$$

Remark 4.5. The LUI can also be deduced from PUI. For every $0 \leq m < 1$, consider

$$
\begin{aligned}
P(m) = &\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\,|\delta|^{-m}\,\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\,|\delta|^n\,dv\,ds\,dh\,d\delta \\
&- E_m\,\|\varphi\|_2^2 \int_K \int_{\mathbb{H}_n} \|(u,t)\|^m\,|g(u,t,k)|^2\,du\,dt\,dk,
\end{aligned}
$$

where E_m is as in Theorem 1.1. As discussed in Remark 3.3, we can obtain the required LUI (4.2).

The following HUI for the Gabor transform on Heisenberg motion group has been proved in [10]. We provide an alternative proof of the HUI using the LUI.

Theorem 4.6. *Let $G = \mathbb{H}_n \rtimes K$ be the Heisenberg motion group. Then for any $g \in L^2(G)$, window function $\varphi \in L^2(G)$ and $a, b \geq 1$, we have*

$$
\begin{aligned}
&\|\varphi\|_2^{\frac{1}{b}}\,\|g\|_2^{\frac{1}{a}+\frac{1}{b}} \\
&\leq C \left(\int_K \int_{\mathbb{H}_n} \|(u,t)\|^{2a}\,|g(u,t,k)|^2\,du\,dt\,dk\right)^{\frac{1}{2a}} \\
&\quad \times \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\,|\delta|^{2b}\,\|G_\varphi g(v,s,h,\delta,\rho)\|_{HS}^2\,|\delta|^n\,dv\,ds\,dh\,d\delta\right)^{\frac{1}{2b}}. \qquad (4.4)
\end{aligned}
$$

Proof. Proceeding as in [5, Theorem 3.2], it is sufficient to prove the required inequality for functions in $\mathbb{S}(\mathbb{R}^n)$. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function, then taking $a = b = 1$ in (4.2), we have

$$
\begin{aligned}
&D\,\|\varphi\|_2^2\,\|g\|_2^2 \\
&\leq \|\varphi\|_2^2 \int_K \int_{\mathbb{H}_n} \ln\|(u,t)\|\,|g(u,t,k)|^2\,du\,dt\,dk \\
&\quad + \int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\,\ln|\delta|\,\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\,|\delta|^n\,dv\,ds\,dh\,d\delta. \qquad (4.5)
\end{aligned}
$$

For every $x > 0$, we have $\ln x - \frac{1}{x} \leq \psi(x)$, where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. So we can write

$$\begin{aligned}\ln\left(\frac{ne^{-4}}{4\pi}\right) &\leq \ln\left(\frac{n}{4}\right) - \frac{4}{n} - \ln \pi \\ &\leq \psi\left(\frac{n}{4}\right) - \ln \pi \\ &= D.\end{aligned}$$

Using (4.5) and then applying Jensen's inequality for ln which is a concave function, we have

$$\begin{aligned}&\ln\left(\frac{ne^{-4}}{4\pi}\right) \\ &\leq \int_K \int_{\mathbb{H}_n} \ln \|(u,t)\| \, \frac{|g(u,t,k)|^2}{\|g\|_2^2} \, du \, dt \, dk \\ &\quad + \int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho \, \ln|\delta| \, \frac{\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2 \, \|g\|_2^2} \, |\delta|^n \, dv \, ds \, dh \, d\delta \\ &\leq \frac{1}{2} \ln\left[\int_K \int_{\mathbb{H}_n} \|(u,t)\|^2 \, \frac{|g(u,t,k)|^2}{\|g\|_2^2} \, du \, dt \, dk\right] \\ &\quad + \frac{1}{2} \ln\left[\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho \, |\delta|^2 \, \frac{\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2 \, \|g\|_2^2} \, |\delta|^n \, dv \, ds \, dh \, d\delta\right] \\ &= \ln\Bigg[\left(\int_K \int_{\mathbb{H}_n} \|(u,t)\|^2 \, \frac{|g(u,t,k)|^2}{\|g\|_2^2} \, du \, dt \, dk\right)^{1/2} \\ &\qquad \times \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho \, |\delta|^2 \, \frac{\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2 \, \|g\|_2^2} \, |\delta|^n \, dv \, ds \, dh \, d\delta\right)^{1/2}\Bigg].\end{aligned}$$

Using the fact that ln is an increasing function, we obtain

$$\begin{aligned}\frac{ne^{-4}}{4\pi} \leq &\left(\int_K \int_{\mathbb{H}_n} \|(u,t)\|^2 \, \frac{|g(u,t,k)|^2}{\|g\|_2^2} \, du \, dt \, dk\right)^{1/2} \\ &\times \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho \, |\delta|^2 \, \frac{\|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2 \, \|g\|_2^2} \, |\delta|^n \, dv \, ds \, dh \, d\delta\right)^{1/2}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\varphi\|_2 \, \|g\|_2^2 \leq C &\left(\int_K \int_{\mathbb{H}_n} \|(u,t)\|^2 \, |g(u,t,k)|^2 \, du \, dt \, dk\right)^{1/2} \\ &\times \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho \in \widehat{K}} d_\rho \, |\delta|^2 \, \|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2 \, |\delta|^n \, dv \, ds \, dh \, d\delta\right)^{1/2}.\end{aligned} \tag{4.6}$$

On applying Hölder's inequality and then using (2.1) as in Theorem 4.4, we obtain

$$\begin{aligned}&\int_K \int_{\mathbb{H}_n} \|(u,t)\|^2 \, |g(u,t,k)|^2 \, du \, dt \, dk \\ &\leq \left(\int_K \int_{\mathbb{H}_n} \|(u,t)\|^{2a} \, |g(u,t,k)|^2 \, du \, dt \, dk\right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}}\end{aligned}$$

and

$$\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\, |\delta|^2\, \|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, dv\, ds\, dh\, d\delta$$

$$\leq \left(\int_K \int_{\mathbb{H}_n} \int_{\mathbb{R}^*} \sum_{\rho\in\widehat{K}} d_\rho\, |\delta|^{2b}\, \|G_\varphi g(v,s,h,\delta,\rho)\|_{\mathrm{HS}}^2\, |\delta|^n\, dv\, ds\, dh\, d\delta \right)^{\frac{1}{b}} \left(\|\varphi\|_2^2\, \|g\|_2^2 \right)^{1-\frac{1}{b}}.$$

Combining the above inequalities with (4.6), we obtain the required inequality (4.4). □

5 Diamond Lie Groups

In this section, we shall establish the three inequalities PUI, LUI and HUI for the Gabor transform on the class of diamond Lie groups. Let $\mathfrak{h}_{2n+1}$ be the Heisenberg Lie algebra having a basis $\{X_1, Y_1, \ldots, X_n, Y_n, Z\}$ such that the non-trivial brackets are given by $[X_i, Y_i] = Z$ for all $i = 1, 2, \ldots, n$. Let $\mathbb{H}$ be the simply connected Lie group associated to the $\mathfrak{h}_{2n+1}$. A diamond Lie group G is the semi-direct product $\mathbb{R}^n \ltimes \mathbb{H}$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ acts on $d \in \mathbb{H}$ by

$$(\gamma_1, \ldots, \gamma_n) \cdot d = ((r(\gamma_1)d_1)^T, \ldots, (r(\gamma_n)d_n)^T, z),$$

where $d = (d_1, d_2, \ldots, d_n, z)^T$, $d_i = (a_i, b_i)^T \in \mathbb{R}^2$, $z \in \mathbb{R}$ and for $\phi \in \mathbb{R}$ we have

$$r(\phi) = \begin{bmatrix} \cos 2\pi\phi & -\sin 2\pi\phi \\ \sin 2\pi\phi & \cos 2\pi\phi \end{bmatrix}.$$

For more details, refer to [10] and [11]. The elements of $\widehat{G}$ are the representations of the form $\sigma_{w,z} \otimes \chi$, where χ is an arbitrary character of $\mathbb{T}^n$. For $\eta \in \mathbb{Z}^n$, let χ_η denote the associated character of $\mathbb{T}^n$. Then

$$\widehat{G} \setminus \widehat{G/Z(\mathbb{H})} = \{\sigma_{w,z} \otimes \chi_\eta : \eta \in \mathbb{Z}^n, w \in \mathbb{R}^*, z \in \mathbb{T}^n\}.$$

Consider a cross-section $\Sigma = \{(r, (x, y, 0)) : r, x, y \in \mathbb{R}^n\}$ for the cosets of $\mathbb{R} = Z(\mathbb{H})$ in G. The Haar measures on G and $G/\mathbb{R}$ are normalised in such a way that Weil's formula holds for G, $\mathbb{R}$ and $G/\mathbb{R}$. The space Σ is endowed with the image of the Haar measure on $G/\mathbb{R}$ under the homeomorphism between $G/\mathbb{R}$ and Σ given by $\mathbb{R}(r, (x, y, 0)) \mapsto (r, (x, y, 0))$. The following theorem gives the PUI [4] for the Fourier transform on diamond Lie groups.

Theorem 5.1. *Let $G = \mathbb{R}^n \ltimes \mathbb{H}$ be a diamond Lie group. For any $g \in \mathbb{S}(G)$ and $0 \leq m < 1$, we have*

$$\int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta\in\mathbb{Z}^n} (|w|^2 + \|\eta\|^2)^{-m/2}\, \|(\sigma_{w,z} \otimes \chi_\eta)(g)\|_{HS}^2\, |w|^n\, dz\, dw$$

$$\leq E_m \int_\Sigma \int_{\mathbb{R}} \|(s,t)\|^m\, |g(s,t)|^2\, ds\, dt,$$

where E_m is as in Theorem 1.1.

We now prove the PUI for the Gabor transform on diamond Lie group G.

Theorem 5.2. *Let $G = \mathbb{R}^n \ltimes \mathbb{H}$ be a diamond Lie group. Then for any $g \in \mathbb{S}(G)$, window function $\varphi \in \mathbb{S}(G)$ and $0 \leq m < 1$, we have*

$$\int_\Sigma \int_{\mathbb{R}} \int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta\in\mathbb{Z}^n} (|w|^2 + \|\eta\|^2)^{-m/2}\, \|G_\varphi g(u, v, \sigma_{w,z} \otimes \chi_\eta)\|_{HS}^2\, |w|^n\, du\, dv\, dz\, dw$$

$$\leq E_m\, \|\varphi\|_2^2 \int_\Sigma \int_{\mathbb{R}} \|(s,t)\|^m\, |g(s,t)|^2\, ds\, dt,$$

where E_m is as in Theorem 1.1.

Proof. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function. For $(s,t) \in G$, we define $g^{\varphi}_{(u,v)} : G \to \mathbb{C}$ by $g^{\varphi}_{(u,v)}(s,t) = g(s,t)\, \overline{\varphi}((u,v)^{-1}(s,t))$ for all $(s,t) \in G$. Then $g^{\varphi}_{(u,v)} \in \mathbb{S}(G)$ for all $(u,v) \in G$ and

$$G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta}) = (\sigma_{w,z} \otimes \chi_{\eta})(g^{\varphi}_{(u,v)}).$$

Using Theorem 5.1, we have

$$\int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} (|w|^2 + \|\eta\|^2)^{-m/2} \, \|(\sigma_{w,z} \otimes \chi_{\eta})(g^{\varphi}_{(u,v)})\|^2_{\text{HS}} \, |w|^n \, dz \, dw$$
$$\leq E_m \int_{\Sigma} \int_{\mathbb{R}} \|(s,t)\|^m \, |g^{\varphi}_{(u,v)}(s,t)|^2 \, ds \, dt$$

which can be written as

$$\int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} (|w|^2 + \|\eta\|^2)^{-m/2} \, \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|^2_{\text{HS}} \, |w|^n \, dz \, dw$$
$$\leq E_m \int_{\Sigma} \int_{\mathbb{R}} \|(s,t)\|^m \, |g(s,t)|^2 \, |\varphi((u,v)^{-1}(s,t))|^2 \, ds \, dt$$

for all $(u,v) \in G$. Integrating both sides with respect to $du\, dv$, we get

$$\int_{\Sigma} \int_{\mathbb{R}} \int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} (|w|^2 + \|\eta\|^2)^{-m/2} \, \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|^2_{\text{HS}} \, |w|^n \, du \, dv \, dz \, dw$$
$$\leq E_m \int_{\Sigma} \int_{\mathbb{R}} \int_{\Sigma} \int_{\mathbb{R}} \|(s,t)\|^m \, |g(s,t)|^2 \, |\varphi((u,v)^{-1}(s,t))|^2 \, du \, dv \, ds \, dt$$
$$= E_m \, \|\varphi\|_2^2 \int_{\Sigma} \int_{\mathbb{R}} \|(s,t)\|^m \, |g(s,t)|^2 \, ds \, dt. \qquad \square$$

Following theorem gives the LUI [4] for the Fourier transform on the class of diamond Lie groups.

Theorem 5.3. *Let $G = \mathbb{R}^n \ltimes \mathbb{H}$ be a diamond Lie group. For any $g \in \mathbb{S}(G)$ and $a, b \geq 1$, we have*

$$D \, \|g\|_2^{\frac{2}{a}+\frac{2}{b}} \leq \|g\|_2^{\frac{2}{b}} \left(\int_{\Sigma} \int_{\mathbb{R}} (\ln \|(s,t)\|)^a \, |g(s,t)|^2 \, ds \, dt \right)^{\frac{1}{a}}$$
$$+ \|g\|_2^{\frac{2}{a}} \left(\int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \left[\ln \left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \right]^b \|(\sigma_{w,z} \otimes \chi_{\eta})(g)\|^2_{HS} \, |w|^n \, dz \, dw \right)^{\frac{1}{b}}, \tag{5.1}$$

where D is as in Theorem 1.2.

We now prove the LUI for the Gabor transform on the class of diamond Lie groups.

Theorem 5.4. *Let $G = \mathbb{R}^n \ltimes \mathbb{H}$ be a diamond Lie group. Then for any $g \in \mathbb{S}(G)$, window function $\varphi \in \mathbb{S}(G)$ and $a, b \geq 1$, we have*

$$D \, \|\varphi\|_2^{\frac{2}{b}} \, \|g\|_2^{\frac{2}{a}+\frac{2}{b}}$$
$$\leq \; \|\varphi\|_2^{\frac{2}{b}} \, \|g\|_2^{\frac{2}{b}} \left(\int_{\Sigma} \int_{\mathbb{R}} (\ln \|(s,t)\|)^a \, |g(s,t)|^2 \, ds \, dt \right)^{\frac{1}{a}}$$
$$+ \|g\|_2^{\frac{2}{a}} \left(\int_{\Sigma} \int_{\mathbb{R}} \int_{\mathbb{R}^*} \int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \left[\ln \left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \right]^b \, \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|^2_{HS} \, |w|^n \, du \, dv \, dz \, dw \right)^{\frac{1}{b}}, \tag{5.2}$$

where D is as in Theorem 1.2.

Proof. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function. For $(s,t) \in G$, we define $g^{\varphi}_{(u,v)} : G \to \mathbb{C}$ by $g^{\varphi}_{(u,v)}(s,t) = g(s,t)\, \overline{\varphi}((u,v)^{-1}(s,t))$ for all $(s,t) \in G$. Then $g^{\varphi}_{(u,v)} \in \mathbb{S}(G)$ for all $(u,v) \in G$ and $G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta}) = (\sigma_{w,z} \otimes \chi_{\eta})(g^{\varphi}_{(u,v)})$. Using Theorem 5.3 for $a = b = 1$, we have

$$D\, \|g^{\varphi}_{(u,v)}\|_2^2 \leq \int_{\Sigma}\int_{\mathbb{R}} \ln \|(s,t)\|\, |g^{\varphi}_{(u,v)}(s,t)|^2\, ds\, dt + \int_{\mathbb{R}^*}\int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \ln\left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \|(\sigma_{w,z} \otimes \chi_{\eta})(g^{\varphi}_{(u,v)})\|_{\text{HS}}^2\, |w|^n\, dz\, dw$$

which can be written as

$$\begin{aligned} &D \int_{\Sigma}\int_{\mathbb{R}} |g^{\varphi}_{(u,v)}(s,t)|^2\, ds\, dt \\ &\leq \int_{\Sigma}\int_{\mathbb{R}} \ln \|(s,t)\|\, |g^{\varphi}_{(u,v)}(s,t)|^2\, ds\, dt \\ &\quad + \int_{\mathbb{R}^*}\int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \ln\left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|_{\text{HS}}^2\, |w|^n\, dz\, dw \end{aligned}$$

for all $(u,v) \in G$. Integrating both sides with respect to $du\, dv$, we get

$$\begin{aligned} &D \int_{\Sigma}\int_{\mathbb{R}} |g(s,t)|^2\, |\varphi((u,v)^{-1}(s,t))|^2\, du\, dv\, ds\, dt \\ &\leq \int_{\Sigma}\int_{\mathbb{R}}\int_{\Sigma}\int_{\mathbb{R}} \ln \|(s,t)\|\, |g(s,t)|^2\, |\varphi((u,v)^{-1}(s,t))|^2\, du\, dv\, ds\, dt \\ &\quad + \int_{\Sigma}\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \ln\left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|_{\text{HS}}^2\, |w|^n\, du\, dv\, dz\, dw. \end{aligned}$$

Thus,

$$\begin{aligned} &D\, \|\varphi\|_2^2\, \|g\|_2^2 \\ &\leq \|\varphi\|_2^2 \int_{\Sigma}\int_{\mathbb{R}} \ln \|(s,t)\|\, |g(s,t)|^2\, ds\, dt \\ &\quad + \int_{\Sigma}\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \ln\left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|_{\text{HS}}^2\, |w|^n\, du\, dv\, dz\, dw. \end{aligned} \tag{5.3}$$

On applying Hölder's inequality with exponents a and $\dfrac{a}{a-1}$ when $a \neq 1$, we have

$$\int_{\Sigma}\int_{\mathbb{R}} \ln \|(s,t)\|\, |g(s,t)|^2\, ds\, dt \leq \left(\int_{\Sigma}\int_{\mathbb{R}} (\ln \|(s,t)\|)^a\, |g(s,t)|^2\, ds\, dt\right)^{\frac{1}{a}} \left(\|g\|_2^2\right)^{1-\frac{1}{a}}.$$

On applying Hölder's inequality with exponents b and $\dfrac{b}{b-1}$ when $b \neq 1$ and then using (2.1), we obtain

$$\begin{aligned} &\int_{\Sigma}\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \ln\left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}} \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|_{\text{HS}}^2\, |w|^n\, du\, dv\, dz\, dw \\ &\leq \left(\int_{\Sigma}\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n} \sum_{\eta \in \mathbb{Z}^n} \left[\ln\left(|w|^2 + \|\eta\|^2\right)^{\frac{1}{2}}\right]^b \|G_{\varphi}g(u,v,\sigma_{w,z} \otimes \chi_{\eta})\|_{\text{HS}}^2\, |w|^n\, du\, dv\, dz\, dw\right)^{\frac{1}{b}} \\ &\quad \times \left(\|g\|_2^2\, \|\varphi\|_2^2\right)^{1-\frac{1}{b}}. \end{aligned}$$

Using the above two inequalities, we can express inequality (5.3) as

$$\begin{aligned}
&D\,\|\varphi\|_2^2\,\|g\|_2^2\\
&\leq \|\varphi\|_2^2\left(\int_\Sigma\int_{\mathbb{R}}(\ln\|(s,t)\|)^a\,|g(s,t)|^2\,ds\,dt\right)^{\frac{1}{a}}\left(\|g\|_2^2\right)^{1-\frac{1}{a}}\\
&\quad+\left(\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n}\left[\ln(|w|^2+\|\eta\|^2)^{\frac{1}{2}}\right]^b\,\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2\,|w|^n\,du\,dv\,dz\,dw\right)^{\frac{1}{b}}\\
&\qquad\times\left(\|g\|_2^2\,\|\varphi\|_2^2\right)^{1-\frac{1}{b}}
\end{aligned}$$

which implies

$$\begin{aligned}
&D\,\|\varphi\|_2^{\frac{2}{b}}\,\|g\|_2^{\frac{2}{a}+\frac{2}{b}}\\
&\leq \|\varphi\|_2^{\frac{2}{b}}\,\|g\|_2^{\frac{2}{b}}\left(\int_\Sigma\int_{\mathbb{R}}(\ln\|(s,t)\|)^a\,|g(s,t)|^2\,ds\,dt\right)^{\frac{1}{a}}\\
&+\|g\|_2^{\frac{2}{a}}\left(\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n}\left[\ln(|w|^2+\|\eta\|^2)^{\frac{1}{2}}\right]^b\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2\,|w|^n\,du\,dv\,dz\,dw\right)^{\frac{1}{b}}.
\end{aligned}$$

□

Remark 5.5. The LUI can also be deduced from PUI. For every $0 \leq m < 1$, consider

$$\begin{aligned}
P(m) = &\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n}(|w|^2+\|\eta\|^2)^{-m/2}\,\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2\,|w|^n\,du\,dv\,dz\,dw\\
&- E_m\,\|\varphi\|_2^2\int_{\mathbb{R}}\|(s,t)\|^m\,|g(s,t)|^2\,ds\,dt,
\end{aligned}$$

where E_m is as in Theorem 1.1. As discussed in Remark 3.3, we obtain the required LUI (5.2).

Following HUI for the Gabor transform on the class of diamond Lie groups has been proved in [10]. We provide an alternative proof of the HUI using the LUI.

Theorem 5.6. *Let $G = \mathbb{R}^n \ltimes \mathbb{H}$ be a diamond Lie group. Then for any $g \in L^2(G)$, window function $\varphi \in L^2(G)$ and $a, b \geq 1$, we have*

$$\begin{aligned}
\|\varphi\|_2^{\frac{1}{b}}\,\|g\|_2^{\frac{1}{a}+\frac{1}{b}} \leq C&\left(\int_\Sigma\int_{\mathbb{R}}\|(s,t)\|^{2a}\,|g(s,t)|^2\,ds\,dt\right)^{\frac{1}{2a}}\\
&\times\left(\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n}(|w|^2+\|\eta\|^2)^b\,\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{HS}^2\,|w|^n\,du\,dv\,dz\,dw\right)^{\frac{1}{2b}}.
\end{aligned}\tag{5.4}$$

Proof. Proceeding as in [5, Theorem 3.2], it is sufficient to prove the required inequality for functions in $\mathbb{S}(\mathbb{R}^n)$. Let $g, \varphi \in \mathbb{S}(G)$ with φ a window function, then taking $a = b = 1$ in (5.2), we have

$$\begin{aligned}
&D\,\|\varphi\|_2^2\,\|g\|_2^2\\
&\leq \|\varphi\|_2^2\int_\Sigma\int_{\mathbb{R}}\ln\|(s,t)\|\,|g(s,t)|^2\,ds\,dt\\
&\quad+\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n}\ln(|w|^2+\|\eta\|^2)^{\frac{1}{2}}\,\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2\,|w|^n\,du\,dv\,dz\,dw.
\end{aligned}\tag{5.5}$$

For every $x > 0$, we have $\ln x - \frac{1}{x} \leq \psi(x)$, where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. So we can write

$$\begin{aligned}\ln\left(\frac{ne^{-4}}{4\pi}\right) &\leq \ln\left(\frac{n}{4}\right) - \frac{4}{n} - \ln\pi \\ &\leq \psi\left(\frac{n}{4}\right) - \ln\pi \\ &= D.\end{aligned}$$

Using (5.5) and then applying Jensen's inequality for ln which is a concave function, we have

$$\begin{aligned}&\ln\left(\frac{ne^{-4}}{4\pi}\right) \\ &\leq \int_\Sigma\int_{\mathbb{R}} \ln\|(s,t)\| \frac{|g(s,t)|^2}{\|g\|_2^2}\, ds\, dt \\ &\quad + \int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} \ln\left(|w|^2+\|\eta\|^2\right)^{\frac{1}{2}} \frac{\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2\,\|g\|_2^2}\, |w|^n\, du\, dv\, dz\, dw \\ &\leq \frac{1}{2}\ln\left[\int_\Sigma\int_{\mathbb{R}} \|(s,t)\|^2 \frac{|g(s,t)|^2}{\|g\|_2^2}\, ds\, dt\right] \\ &\quad + \frac{1}{2}\ln\left[\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} \left(|w|^2+\|\eta\|^2\right) \frac{\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2\,\|g\|_2^2}\, |w|^n\, du\, dv\, dz\, dw\right] \\ &= \ln\left[\left(\int_\Sigma\int_{\mathbb{R}} \|(s,t)\|^2 \frac{|g(s,t)|^2}{\|g\|_2^2}\, ds\, dt\right)^{1/2}\right. \\ &\quad \left.\left(\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} \left(|w|^2+\|\eta\|^2\right) \frac{\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2\,\|g\|_2^2}\, |w|^n\, du\, dv\, dz\, dw\right)^{1/2}\right].\end{aligned}$$

Using the fact that ln is an increasing function, we obtain

$$\begin{aligned}&\frac{ne^{-4}}{4\pi} \\ &\leq \left(\int_\Sigma\int_{\mathbb{R}} \ln\|(s,t)\| \frac{|g(s,t)|^2}{\|g\|_2^2}\, ds\, dt\right)^{1/2} \\ &\quad \times\left(\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} \left(|w|^2+\|\eta\|^2\right) \frac{\|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2}{\|\varphi\|_2^2\,\|g\|_2^2}\, |w|^n\, du\, dv\, dz\, dw\right)^{1/2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\varphi\|_2\,\|g\|_2^2 \\ &\leq C\left(\int_\Sigma\int_{\mathbb{R}} \|(s,t)\|^2\, |g(s,t)|^2\, ds\, dt\right)^{1/2} \\ &\quad \times\left(\int_\Sigma\int_{\mathbb{R}}\int_{\mathbb{R}^*}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} \left(|w|^2+\|\eta\|^2\right) \|G_\varphi g(u,v,\sigma_{w,z}\otimes\chi_\eta)\|_{\mathrm{HS}}^2\, |w|^n\, du\, dv\, dz\, dw\right)^{1/2}.\end{aligned}$$

On applying Hölder's inequality and then using (2.1) as in Theorem 5.4, we obtain the required inequality. □

6 Conclusion remarks

The study of uncertainty inequalities is very important as they provide important insights and tools in various fields of mathematics, economics, physics, and statistics. These inequalities are used to quantify the amount of uncertainty or variability in a system and provide bounds on unknown values, which is crucial in decision-making, risk management, and scientific predictions. Obtaining explicit expressions of various uncertainty inequalities involving integral transforms on different locally compact groups helps in strengthening this branch of mathematics.

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