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Random Langevin fractional system driven by two tempered (k, ϕ)-Caputo derivatives: A high-order study

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Abstract. A Langevin fractional system involving two tempered (k, ϕ)-Caputo derivatives with random effects within a generalized separable Banach space framework is investigated. Firstly, the system at hand is rigorously converted into an integral form with the help of some newly proved properties. Secondly, a new Gronwall-type inequality with tempered (k, ϕ)-Riemann-Liouville integral kernel is generalized. Thirdly, a random version of Perov's fixed-point theorem associated with the Bielecki-type vector-valued norm is employed to derive a new uniqueness criterion. Additionally, the existence result is established under growth and compactness-type conditions on the nonlinear forcing terms. Fourthly, an analysis of Ulam–Hyers type stabilities is conducted. Finally, the theoretical results are justified by providing some illustrative examples.

1 Introduction

Nowadays, the concept of fractional derivative (shortly, FD) has emerged as the most suitable mathematical tool for describing various models in numerous engineering and scientific disciplines. Many scholars have addressed different theoretical aspects of the field, see [37, 7, 4, 29, 21]. The tempered fractional derivative (shortly, TFD) is one generalized version of FD, introduced by multiplying the kernel of the usual fractional integral by an exponential factor and depends on a parameter which reduces the FD as a particular case. This kind of derivative has been found to have applications in geophysics [26], finance [12], and random walk models [13]. The authors [27] develop the theory of TFD with respect to another function. For some interesting studies on differential systems driven by TFD, we refer to [3, 43].

The classical mathematical Langevin model is highly significant for illustrating how particles interact within their surrounding medium and the stochastic forces or fluctuations that cause their erratic movements. Nevertheless, the reliance on the specific relationship between a particle's position and velocity has prompted the development of the fractional Langevin model, aimed at describing anomalous diffusion phenomena [22]. Also, it's important to highlight that certain phenomena are more accurately described by coupled random systems. For example, in epidemiology, the migration of birds from various regions worldwide can introduce infectious diseases. Accordingly, the transmission rate of these diseases increases as migratory birds flock together. Moreover, this scenario warrants consideration of the presence of random disturbances. While the above-mentioned motivational models have a great advantage, the difficulty of the corresponding mathematical model may significantly increase, complicating the study of the existence of solutions. Furthermore, Ulam type stability, initiated in [40], has emerged recently as a powerful tool in the qualitative study of in nonlinear fractional differential systems [16, 31]. Therefore, investigating the qualitative aspect for the tempered (k, φ)-Caputo Langevin coupled systems with random effects became important.

Recently, the authors in [1, 45] studied theoretically some quantitative aspects for the following problem:

$$\begin{cases} \left({}^{c}\mathcal{D}_{a^+}^{\vartheta+1;\phi} + \varpi\, {}^{c}\mathcal{D}_{a^+}^{\vartheta;\phi}\right)\mathfrak{s}(\varsigma) = f(\varsigma, \mathfrak{s}(\varsigma)), \quad \varsigma \in [a,b], \\ \mathfrak{s}(a) = \mathfrak{s}'(a) = 0, \end{cases}$$

where $0 < \vartheta < 1$, ${}^{c}\mathcal{D}_{a^+}^{\theta;\phi}$ is the Caputo FD with respect to ϕ of order $\theta \in \{\vartheta+1, \vartheta\}$, $f : [a,b] \times \mathbb{X} \to \mathbb{X}$ is a given function and $\mathbb{X}$ is a Banach space.

The authors in [11] have investigated some existence and stability results for the following problem:

$$\begin{cases} {}^{c}\mathcal{D}_{a^+}^{\vartheta;\phi}\left({}^{c}\mathcal{D}_{a^+}^{\mu;\phi} - \varpi\right)\mathfrak{s}(\varsigma) = f(\varsigma, \mathfrak{u}(\varsigma)), & a < \varsigma < b, \\ D^{[\ell];\phi}\mathfrak{s}(a) = c_\ell, & 0 \le \ell < J, \\ {}^{c}\mathcal{D}_{a^+}^{\mu+\ell;\phi}\mathfrak{s}(a) = d_\ell, & 0 \le \ell < n, \end{cases}$$

where $n-1 < \mu \le n$, $m-1 < \vartheta \le m$, $J = \max\{n,m\}$, $m,n \in \mathbb{N}^*$, $\varpi, c_\ell, d_\ell \in \mathbb{R}$, ${}^{c}\mathcal{D}_{a^+}^{\theta;\phi}$ is the Caputo FD with respect to ϕ of order $\theta \in \{\vartheta, \mu, \mu+\ell\}$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $D^{[\ell];\phi}\mathfrak{s}(\varsigma) = \left(\frac{1}{\phi'(\varsigma)}\frac{d}{d\varsigma}\right)^{\ell}\mathfrak{s}(\varsigma)$.

O. Zentar *et al.* in [41] discussed the existence of solutions for the following problem:

$$\begin{cases} \mathcal{D}_{0^+}^{\vartheta_1}\mathfrak{s}_1(\varsigma,\omega) = f_1(\varsigma, \mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega), \quad \varsigma \in (0,b], \\ \mathcal{D}_{0^+}^{\vartheta_2}\mathfrak{s}_2(\varsigma,\omega) = f_2(\varsigma, \mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega), \quad \varsigma \in (0,b], \\ \lim_{\varsigma\to 0^+} \varsigma^{1-\vartheta_1}\mathfrak{s}_1(\varsigma,\omega) = \mathfrak{z}_3(\omega), \quad \omega \in \Omega, \\ \lim_{\varsigma\to 0^+} \varsigma^{1-\vartheta_2}\mathfrak{s}_2(\varsigma,\omega) = \mathfrak{z}_4(\omega), \quad \omega \in \Omega, \end{cases}$$

where $\mathfrak{z}_3, \mathfrak{z}_4 : \Omega \to \mathbb{O}$ are random variables, $\mathcal{D}_{0^+}^{\vartheta_i}$ represents the standard Riemann-Liouville FD of order $\vartheta_i \in (0,1]$ for each $i = 1,2$ and $f_i : [0,b] \times \mathbb{O} \times \mathbb{O} \times \Omega \to \mathbb{O}$ satisfy certain compactness and growth-type conditions and $\mathbb{O}$ is a real separable Banach space.

Motivated by the preceding discussions, this paper presents new qualitative results for the following random coupled Langevin system involving tempered (k,ϕ)-Caputo FD:

$$\begin{cases} {}^{Tc}_{k}\mathcal{D}_{a^+}^{\vartheta_1,\varrho;\phi}\left({}^{Tc}_{k}\mathcal{D}_{a^+}^{\mu_1,\varrho;\phi} - \varpi_1\right)\mathfrak{s}_1(\varsigma,\omega) = \mathfrak{f}_1(\varsigma, \mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega), \;\; a \leqslant \varsigma \leqslant b, \\ {}^{Tc}_{k}\mathcal{D}_{a^+}^{\vartheta_2,\varrho;\phi}\left({}^{Tc}_{k}\mathcal{D}_{a^+}^{\mu_2,\varrho;\phi} - \varpi_2\right)\mathfrak{s}_2(\varsigma,\omega) = \mathfrak{f}_2(\varsigma, \mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega), \;\; a \leqslant \varsigma \leqslant b, \\ \left(D^{[\ell],\varrho;\phi}\mathfrak{s}_1(a,\omega), D^{[\ell],\varrho;\phi}\mathfrak{s}_2(a,\omega)\right) = \left(z_{1,\ell}(\omega), z_{2,\ell}(\omega)\right), \;\; \omega \in \Omega, \;\; 0 \le \ell < J, \\ \left({}^{Tc}_{k}\mathcal{D}_{a^+}^{\mu_1+k\ell,\varrho;\phi}\mathfrak{s}_1(a,\omega), {}^{Tc}_{k}\mathcal{D}_{a^+}^{\mu_2+k\ell,\varrho;\phi}\mathfrak{s}_2(a,\omega)\right) = \left(w_{1,\ell}(\omega), w_{2,\ell}(\omega)\right), \;\; \omega \in \Omega, \; 0 \le \ell < n, \end{cases} \tag{1.1}$$

where $(\Omega, \mathcal{G})$ is a measurable space, $\mathbb{O}$ is a separable Banach space. Moreover, $k > 0$, $(n-1)k < \mu_i \le nk$, $(m-1)k < \vartheta_i \le mk$, $J = \max\{n,m\}$, $m,n \in \mathbb{N}^*$, $z_{i,\ell}, w_{i,\ell} : \Omega \to \mathbb{O}$ are random variables, $\varpi_i \in \mathbb{R}$, ${}^{T}_{k}\mathcal{D}_{a^+}^{\theta_i,\varrho;\phi}$ denotes the tempered (k,ϕ)-Caputo FD of order $\theta_i \in \{\vartheta_i, \mu_i, \mu_i + k\ell\}$, $i = 1,2$, and index $\varrho \in [0,\infty)$, and $\mathfrak{f}_i : [a,b] \times \mathbb{O} \times \mathbb{O} \times \Omega \to \mathbb{O}$ are a given appropriate functions specified later and $D^{[\ell],\varrho;\phi}\mathfrak{s}(\varsigma) = \left(\frac{1}{\phi'(\varsigma)}\frac{d}{d\varsigma}\right)^{\ell} e^{\varrho\chi(\varsigma,a)}\mathfrak{s}(\varsigma)$.

As a distinctive aspect of our investigation, we can highlight the following:

(1) Gronwall's inequality with the weakly singular tempered (k,ϕ) Riemann-Liouville integral kernel is generalized, which is concerned with priori bounds of the solutions, allowing the global solvability of the considered system.

(2) The employment of Perov's fixed-point theorem associated with the Bielecki-type vector-valued norm allows us to provide a less restrictive hypothesis under which system (1.1) is uniquely solvable.

(3) A new existence is established, addressing an unresolved question concerning high-order coupled random Langevin fractional systems in general setting, namely, when the nonlinear forcing terms $\mathfrak{f}_i$, $i = 1, 2$, acts on an infinite dimensional separable Banach space. This is achieved through a combination of Sadovskii's fixed-point principle in a random setting, along with the noncompactness measure (shortly, MNC) procedure and the a priori estimate technique.

(4) Established some Ulam-Hyers type stabilities for the system at hand.

(5) Our findings generalize, improve and extend the results demonstrated in [14, 15, 28, 1, 45, 11, 8, 44].

The rest of the paper is organized as follows: In Section 2, we gather some necessary background required for the subsequent sections. Section 3 presents a new Gronwall-type inequality. In Section 4, we will address the aforementioned points (2)-(3). In Section 5 we treat the point (4). Finally, to validate our abstract result, some illustrative examples are provided in Section 6.

2 Preliminaries

Throughout the paper, let $(\mathbb{O}, \|\cdot\|)$ be a separable Banach space, we endow the space $C(\mathfrak{I}, \mathbb{O})$ of $\mathbb{O}$-valued continuous functions on $\mathfrak{I}$ with the supnorm

$$\|\mathfrak{u}\|_\infty = \sup_{\varsigma \in \mathfrak{I}} \|\mathfrak{u}(\varsigma)\|. \tag{2.1}$$

$L^1(\mathfrak{I}, \mathbb{O})$ denotes the space of Bochner integrable functions $\mathfrak{u} : \mathfrak{I} \to \mathbb{O}$ normed by

$$\|\mathfrak{u}\|_{L^1} = \int_a^b \|\mathfrak{u}(\varsigma)\| d\varsigma, \quad \text{for all } \mathfrak{u} \in L^1(\mathfrak{I}, \mathbb{O}).$$

$L^\infty(\mathfrak{I}, \mathbb{R}_+)$ stands for the space all essentially bounded functions normed by

$$\|\mathfrak{u}\|_{L^\infty} = \operatorname*{ess\,sup}_{\varsigma \in \mathfrak{I}} \|\mathfrak{u}(\varsigma)\| = \inf\{M > 0;\ \|\mathfrak{u}(\varsigma)\| \le M \text{ for almost every } \varsigma \in \mathfrak{I}\}.$$

For our convenience, define the set

$$\mathbb{H}^1_+(\mathfrak{I}, \mathbb{R}) = \{\phi \ :\ \phi \in C^1(\mathfrak{I}, \mathbb{R}) \text{ and } \phi'(\varsigma) > 0 \text{ for all } \varsigma \in \mathfrak{I}\}.$$

For $\phi \in \mathbb{H}^1_+(\mathfrak{I}, \mathbb{R})$ and $\varsigma, s \in \mathfrak{I}$, $(\varsigma > s)$, we pose

$$\chi(\varsigma, s) = \phi(\varsigma) - \phi(s) \text{ and } \chi(\varsigma, s)^\alpha = (\phi(\varsigma) - \phi(s))^\alpha, \quad \alpha \in \mathbb{R}.$$

If, $y, v \in \mathbb{R}^n$, $y = (y_1, \dots, y_n)$, $v = (v_1, \dots, v_n)$, by $y \le v$ we mean $y_i \le v_i$ for all $i = 1, \dots, n$. Also $|y| = (|y_1|, \dots, |y_n|)$, $\max(y, v) = (\max(y_1, v_1), \dots, \max(y_n, v_n))$ and $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : \ y_i > 0\}$. If $c \in \mathbb{R}$, then $y \le c$ means $y_i \le c$ for each $i = 1, \dots, n$.

Definition 2.1. Let $\mathbb{F}$ be a nonempty set. By a vector-valued metric on $\mathbb{F}$ we mean a map $d : \mathbb{F} \times \mathbb{F} \to \mathbb{R}^n_+$ with the following properties:

(i) $d(y, v) \ge 0$ for all $y, v \in \mathbb{F}$; if $d(y, v) = 0$ then $y = v$;

(ii) $d(y, v) = d(v, u)$ for all $y, v \in \mathbb{F}$;

(iii) $d(y, v) \le d(y, u) + d(u, v)$ for all $u, v, y \in \mathbb{F}$.

For d_i, $i = 1, \dots, n$ are metrics on $\mathbb{F}$, the pair $(\mathbb{F}, d)$ is called a generalized metric space (shortly, GMS) (or a vector-valued metric space) with $d(y, v) := \begin{pmatrix} d_1(y, v) \\ \vdots \\ d_n(y, v) \end{pmatrix}$.

Definition 2.2. We call a matrix $\mathcal{M} \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers convergent to zero if its spectral radius $\rho(\mathcal{M}) < 1$. In other words, this means that all the eigenvalues of $\mathcal{M}$ are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(\mathcal{M} - \lambda I) = 0$, where I denote the unit matrix.

Proposition 2.3. [33] *Let $\mathcal{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) $\mathcal{M}^r \to 0$ *when* $r \to \infty$.

(ii) $\mathcal{M}$ *is convergent to zero.*

(iii) The matrix $(I - \mathcal{M})$ *is nonsingular and*

$$(I - \mathcal{M})^{-1} = I + \mathcal{M} + \mathcal{M}^2 + \ldots + \mathcal{M}^k + \ldots.$$

(iv) $(I - \mathcal{M})$ *is nonsingular matrix and* $(I - \mathcal{M})^{-1}$ *has positive elements.*

Let $\mathbb{O}$ be a separable GMS and $(\Omega, \mathcal{F})$ be a measurable space. We denote $\mathcal{B}(\mathbb{O})$ the Borel σ-algebra on $\Omega \times \mathbb{O}$. Therefore, $\mathcal{F} \times \mathcal{B}(\mathbb{O})$ is the smallest σ-algebra on $\Omega \times \mathbb{O}$ which contains all the sets $F \times S$, where $F \in \mathcal{F}$ and $S \in \mathcal{B}(\mathbb{O})$.

Definition 2.4. Given two separable GMSs $\mathbb{O}$ and $\mathbb{X}$, a mapping $\mathcal{Q} : \Omega \times \mathbb{O} \to \mathbb{X}$ is called a random operator if $\omega \longmapsto \mathcal{Q}(\omega, \mathfrak{u})$ is measurable for all $\mathfrak{u} \in \mathbb{O}$. The random operator $\mathcal{L}$ on $\mathbb{O}$ will be denoted by

$$\mathcal{Q}(\mathfrak{u})(\omega) = \mathcal{Q}(\omega, \mathfrak{u}), \ \omega \in \Omega, \ \ \mathfrak{u} \in \mathbb{O}.$$

Definition 2.5. The fixed point of a random operator $\mathcal{Q}$ is a measurable function $\mathfrak{u} : \Omega \to \mathbb{O}$ such that

$$\mathfrak{u}(\omega) = \mathcal{Q}(\omega, \mathfrak{u}(\omega)) \quad \text{for all} \quad \omega \in \Omega.$$

Definition 2.6. Let $f : \mathfrak{I} \times \mathbb{O} \times \Omega \to \mathbb{X}$ is called random Carathéodory if the following statements are verified:

(i) The map $\mathfrak{u} \longmapsto f(\varsigma, \mathfrak{u}, \omega)$ is continuous for all $\varsigma \in \mathfrak{I}$ and $\omega \in \Omega$.

(ii) The map $(\varsigma, \omega) \longmapsto f(\varsigma, \mathfrak{u}, \omega)$ is jointly measurable for all $\mathfrak{u} \in \mathbb{O}$.

Lemma 2.7. [36] *Let $\mathbb{O}$ be a separable metric space and $\mathcal{Q} : \Omega \times \mathbb{O} \to \mathbb{O}$ be a mapping such that $\mathcal{Q}(\omega, \cdot)$ is continuous for all $\omega \in \Omega$ and $\mathcal{Q}(\cdot, \mathfrak{u})$ is measurable for all $\mathfrak{u} \in \mathbb{O}$. Then the map $(\omega, \mathfrak{u}) \to \mathcal{Q}(\omega, \mathfrak{u})$ is jointly measurable.*

Definition 2.8. [19] Let $\mathbb{O}$ be a generalized Banach space and $(\mathcal{O}, \leq)$ be a partially ordered set. A map $\Lambda : \mathcal{P}(\mathbb{O}) \to \mathcal{O} \times \mathcal{O} \times \ldots \times \mathcal{O}$ is called a generalized MNC on $\mathbb{O}$, if

$$\Lambda(\overline{co}\, \mathcal{O}) = \Lambda(\mathcal{O}) \text{ for every } \mathcal{O} \in \mathcal{P}(\mathbb{O}),$$

where $\Lambda(\mathcal{O}) := \begin{pmatrix} \Lambda_1(\mathcal{O}) \\ \vdots \\ \Lambda_n(\mathcal{O}) \end{pmatrix}$, $\mathcal{P}(\mathbb{O})$ denotes the family of all bounded subsets of $\mathbb{O}$ and $\overline{co}\mathcal{O}$ is the closed convex hull of $\mathcal{O}$.

Definition 2.9. The application Λ is called:

(i) Monotone if $\mathcal{O}_0, \mathcal{O}_1 \in \mathcal{P}(\mathbb{O}), \mathcal{O}_0 \subset \mathcal{O}_1$ implies $\Lambda(\mathcal{O}_0) \leq \Lambda(\mathcal{O}_1)$.

(ii) Nonsingular if $\Lambda(\{a\} \cup \mathcal{O}) = \Lambda(\mathcal{O})$ for every $a \in \mathbb{O}$ and $\mathcal{O} \in \mathcal{P}(\mathbb{O})$.

If $\mathcal{O}$ is a cone in a normed space, we say that the MNC is

(iii) Regular if the condition $\Lambda(\mathcal{O}) = 0$ is equivalent to the compactness of $\overline{\mathcal{O}}$.

The most well-known example of a MNC possessing all previous properties is the Hausdorff MNC defined by:

$$\eta(\mathcal{O}) = \inf\{\epsilon > 0 : \mathcal{O} \text{ has a finite } \epsilon - \text{net}\}.$$

Definition 2.10. [19] Let $\mathbb{X}, \mathbb{Y}$ be two generalized normed spaces. A continuous map $G : \mathbb{X} \to \mathbb{Y}$ is called a $\mathcal{M}$-contraction (with respect to the generalized MNC Λ) if there exists a matrix $\mathcal{M} \in \mathcal{M}_{n\times n}(\mathbb{R})$ converges to zero such that for every $D \in \mathcal{P}(\mathbb{X})$, one has

$$\Lambda(G(D)) \leq \mathcal{M}\Lambda(D).$$

Lemma 2.11. [20] *If $\{x_n\}_{n=1}^{+\infty} \subset L^1(\mathfrak{I}, \mathbb{O})$ satisfies $\|x_n(\varsigma)\| \leq \iota(\varsigma)$ a.e. on $\mathfrak{I}$ for all $n \geq 1$ with some $\iota \in L^1(\mathfrak{I}, \mathbb{R}_+)$. Then, the function $\eta(\{x_n(\varsigma)\}_{n=1}^{+\infty})$ is integrable and*

$$\eta\left(\left\{\int_0^\varsigma x_n(s)ds \ : \ n \geq 1\right\}\right) \leq \int_0^\varsigma \eta(x_n(s) \ : \ n \geq 1)ds. \tag{2.2}$$

Definition 2.12. [17] For $\vartheta, \mu, k > 0$, the k-gamma function is given by

$$\Gamma_k(\vartheta) = \int_0^\infty \varsigma^{\vartheta-1} e^{-\frac{\varsigma^k}{k}} d\varsigma,$$

We have also the following relations

$$\Gamma_k(\vartheta) = k^{\frac{\vartheta}{k}-1}\Gamma\left(\frac{\vartheta}{k}\right), \ \ \Gamma_k(\vartheta + k) = \vartheta\Gamma_k(\vartheta), \ \ \Gamma_k(k) = \Gamma(1) = 1.$$

Definition 2.13. [17] The k-Mittag-Leffler function is defined as follows:

$$\mathbb{E}^k_{\vartheta,\mu}(\mathfrak{u}) = \sum_{j=0}^{\infty} \frac{\mathfrak{u}^j}{\Gamma_k(j\vartheta + \mu)}, \qquad \vartheta, \mu > 0.$$

In particular, $\mathbb{E}^k_{\vartheta,1}(\mathfrak{u}) = \mathbb{E}^k_\vartheta(\mathfrak{u})$.

Definition 2.14. [35] Let $\alpha > 0$, $\varrho \in [0, \infty)$, $k > 0$ and $\phi \in \mathbb{H}^1_+(\mathfrak{I}, \mathbb{R})$. The tempered (k, ϕ)-fractional integral of a function $\mathfrak{u} \in C(\mathfrak{I}, \mathbb{R})$ is defined by

$${}^T_k\mathcal{I}^{\alpha,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = e^{-\varrho\chi(\varsigma,a)}{}_k\mathcal{I}^{\alpha;\phi}_{a^+}\left(\mathfrak{u}(\varsigma)e^{\lambda\chi(\varsigma,a)}\right) = \frac{1}{k\Gamma_k(\alpha)}\int_a^\varsigma \phi'(s)\mathfrak{T}^{\frac{\alpha}{k}-1}_{\varrho,\chi}(\varsigma, s)\mathfrak{u}(s)ds,$$

where $\mathfrak{T}^{\frac{\alpha}{k}-1}_{\varrho,\chi}(\varsigma, s) = e^{-\varrho\chi(\varsigma,s)}\chi(\varsigma, s)^{\frac{\alpha}{k}-1}$ and ${}_k\mathcal{I}^{\alpha;\phi}_{a^+}$ the (k, ϕ)-fractional integral [32], defined by

$${}_k\mathcal{I}^{\alpha;\phi}_{a^+}\mathfrak{u}(\varsigma) = \frac{1}{k\Gamma_k(\alpha)}\int_a^\varsigma \phi'(s)\chi(\varsigma, s)^{\frac{\alpha}{k}-1}\mathfrak{u}(s)ds.$$

Lemma 2.15. [35] *Let $\vartheta_1, \vartheta_2, k > 0$ and $\varrho \in [0, \infty)$, then we have*

$${}^T_k\mathcal{I}^{\vartheta_1,\varrho;\phi}_{a^+}\mathfrak{T}^{\frac{\vartheta_2}{k}-1}_{\varrho,\chi}(\varsigma, a) = \frac{\Gamma_k(\vartheta_2)}{\Gamma_k(\vartheta_1 + \vartheta_2)}\mathfrak{T}^{\frac{\vartheta_1+\vartheta_2}{k}-1}_{\varrho,\chi}(\varsigma, a).$$

Lemma 2.16. *Let $\vartheta, k > 0$ and $\varrho \in [0, \infty)$. Then,*

(i) ${}^T_k\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = k^{-\frac{\vartheta}{k}}\,{}^T\mathcal{I}^{\frac{\vartheta}{k},\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma)$

(ii) ${}^T_k\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}(1) = \mathfrak{T}^{\frac{\vartheta}{k}}_{\varrho,\chi}(\varsigma, a)\mathbb{E}^k_{k,k+\vartheta}(k\varrho\chi(\varsigma, a))$.

where ${}^T\mathcal{I}^{\frac{\vartheta}{k},\varrho;\phi}_{a^+}$ the tempered ϕ-fractional integral [24].

Proof. Using the definition of ${}^T_k\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}(\cdot)$ and [23, Theorem 4.9], we have

$$\begin{aligned}
{}^T_k\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) &= e^{-\varrho\chi(\varsigma,a)}{}_k\mathcal{I}^{\vartheta;\phi}_{a^+}\left(\mathfrak{u}(\varsigma)e^{\varrho\chi(\varsigma,a)}\right) \\
&= e^{-\varrho\chi(\varsigma,a)}k^{-\frac{\vartheta}{k}}\mathcal{I}^{\frac{\vartheta}{k};\phi}_{a^+}\left(\mathfrak{u}(\varsigma)e^{\varrho\chi(\varsigma,a)}\right) \\
&= k^{-\frac{\vartheta}{k}}\,{}^T\mathcal{I}^{\frac{\vartheta}{k},\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma)
\end{aligned} \tag{2.3}$$

Next, by [24, Theorem 7.5] and Definition 2.12, it follows that

$$\begin{aligned} {}^{T}_{k}\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}(1) &= k^{-\frac{\vartheta}{k}}\mathfrak{T}^{\frac{\vartheta}{k}}_{\varrho,\chi}(\varsigma,a)\mathbb{E}_{1,1+\frac{\varrho}{k}}(\varrho\chi(\varsigma,a)) \\ &= \mathfrak{T}^{\frac{\vartheta}{k}}_{\varrho,\chi}(\varsigma,a)\mathbb{E}^{k}_{k,k+\vartheta}(k\varrho\chi(\varsigma,a)). \end{aligned}$$

□

Definition 2.17. [35] Let $k > 0$ and $\phi \in \mathbb{H}^1_+(\mathfrak{I},\mathbb{R})$. The (k,ϕ)-Caputo tempered FD of $\mathfrak{u} \in C^n(\mathfrak{I},\mathbb{R})$ of order $(n-1)k < \vartheta < nk$, and index $\varrho \in [0,\infty)$ is defined by

$${}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = e^{-\varrho\chi(\varsigma,a)}\,{}^{c}_{k}\mathcal{D}^{\vartheta;\phi}_{a^+}\left(\mathfrak{u}(\varsigma)e^{\varrho\chi(\varsigma,a)}\right),$$

where ${}^{c}_{k}\mathcal{D}^{\vartheta;\phi}_{a^+}(\cdot)$ is the (k,ϕ)-Caputo fractional derivatives, defined in [35, 23] and $n \in \mathbb{N}$.

Lemma 2.18. *Let $(n-1)k < \vartheta < nk$ and $\varrho \in [0,\infty)$ where $k > 0$ and $n \in \mathbb{N}$, then for $\mathfrak{u} \in C^n(\mathfrak{I},\mathbb{R})$*

$${}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = k^{\frac{\vartheta}{k}}\,{}^{Tc}\mathcal{D}^{\frac{\vartheta}{k},\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma).$$

Proof. Proof follows by using the definition 2.17 and [23, Theorem 6.1]. □

Theorem 2.19. [35] *Let $\phi \in \mathbb{H}^1_+(\mathfrak{I},\mathbb{R})$, $(n-1)k < \vartheta \le nk$ and $\varrho \in [0,\infty)$ where $k > 0$ and $n \in \mathbb{N}$. Then, for $\mathfrak{u} \in C^n(\mathfrak{I},\mathbb{O})$, we have*

$${}^{T}_{k}\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}\,{}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = \mathfrak{u}(\varsigma) - e^{-\varrho\chi(\varsigma,a)}\sum_{j=0}^{n-1}\frac{\left[D^{[j],\varrho;\phi}\mathfrak{u}(\varsigma)\right]_{\varsigma=a}}{k^{-j}\Gamma_k(k(j+1))}\chi(\varsigma,a)^j,$$

where $D^{[j],\varrho;\phi}\mathfrak{u}(\varsigma) = \left(\frac{1}{\phi'(\varsigma)}\frac{d}{d\zeta}\right)^j e^{\varrho\chi(\varsigma,a)}\mathfrak{u}(\varsigma)$.

Lemma 2.20. [35] *Let $\vartheta, k > 0$, $\varrho \ge 0$, and $\mathfrak{u} \in C(\mathfrak{I},\mathbb{O})$, then for $\varsigma \in \mathfrak{I}'$, we have*

$${}^{T}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\,{}^{T}_{k}\mathcal{I}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = \mathfrak{u}(\varsigma).$$

Theorem 2.21. *Let $(n-1)k < \vartheta \le nk$, $m,n \in \mathbb{N}$, $y \in C^{m+n}(\mathfrak{I},\mathbb{R})$ and $\varrho \in [0,\infty)$. Then,*

$$\mathfrak{D}^{m,\varrho;\phi}\,{}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = \frac{{}^{Tc}_{k}\mathcal{D}^{\vartheta+mk,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma)}{k^m} + e^{-\varrho\chi(\varsigma,a)}\sum_{j=0}^{m-1}\frac{\chi(\varsigma,a)^{j+n-m-\frac{\vartheta}{k}}\left[D^{[j+n],\varrho;\phi}\mathfrak{u}(\varsigma)\right]_{\varsigma=a}}{k^{m-j-n+}\Gamma_k(k(j+n-m+1)-\vartheta)}, \tag{2.4}$$

and

$$D^{[m],\varrho;\phi}\,{}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) = \frac{e^{\varrho\chi(\varsigma,a)}}{k^m}\,{}^{Tc}_{k}\mathcal{D}^{\vartheta+mk,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) + \sum_{j=0}^{m-1}\frac{\chi(\varsigma,a)^{j+n-m-\frac{\vartheta}{k}}\left[D^{[j+n],\varrho;\phi}\mathfrak{u}(\varsigma)\right]_{\varsigma=a}}{k^{m-j-n+}\Gamma_k(k(j+n-m+1)-\vartheta)}, \tag{2.5}$$

where $\mathfrak{D}^{m,\varrho;\phi}\mathfrak{u}(\varsigma) = \left(\frac{1}{\phi'(\varsigma)}\frac{d}{d\varsigma}+\varrho\right)^m\mathfrak{u}(\varsigma)$ and $C^{m+n}(\mathfrak{I},\mathbb{R})$ be the space of $m+n$-times continuously differentiable functions.

Proof. Using the Lemma 2.18 and the [24, Theorem 6.12], we obtain

$$\begin{aligned} &\mathfrak{D}^{m,\varrho;\phi}\,{}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) \\ &= k^{\frac{\vartheta}{k}}\mathfrak{D}^{m,\varrho;\phi}\,{}^{Tc}\mathcal{D}^{\frac{\vartheta}{k},\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) \\ &= k^{\frac{\vartheta}{k}}\left({}^{Tc}\mathcal{D}^{\frac{\vartheta}{k}+m,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) + e^{-\varrho\chi(\varsigma,a)}\sum_{j=0}^{m-1}\frac{\chi(\varsigma,a)^{j+n-m-\frac{\vartheta}{k}}\left[\mathfrak{D}^{[j+n],\varrho;\phi}\mathfrak{u}(\varsigma)\right]_{\varsigma=a}}{\Gamma(j+n-m+1-\frac{\vartheta}{k})}\right). \end{aligned}$$

Using the properties of k-gamma function and Lemma 2.18, we have

$$\begin{aligned} &\mathfrak{D}^{m,\varrho;\phi}\,{}^{Tc}_{k}\mathcal{D}^{\vartheta,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma) \\ &= \frac{{}^{Tc}_{k}\mathcal{D}^{\vartheta+mk,\varrho;\phi}_{a^+}\mathfrak{u}(\varsigma)}{k^m} + e^{-\varrho\chi(\varsigma,a)}\sum_{j=0}^{m-1}\frac{\chi(\varsigma,a)^{j+n-m-\frac{\vartheta}{k}}\left[\mathfrak{D}^{[j+n],\varrho;\phi}\mathfrak{u}(\varsigma)\right]_{\varsigma=a}}{k^{m-j-n}\Gamma_k(k(j+n-m+1)-\vartheta)}. \end{aligned}$$

Next, by [24, Theorem 5.1] and Eq (2.4), we get Eq (2.5). □

Remark 2.22. Observe that from Equation (2.5), if $D^{[j],\varrho;\phi}\mathfrak{u}(a)=0$, for all $j=n,n+1,\cdots,n+m-1$ we can get the following relation

$$D^{[m],\varrho;\phi}\ {}_{k}^{Tc}\mathcal{D}_{a^+}^{\vartheta,\varrho;\phi}\mathfrak{u}(\varsigma)=\frac{e^{\varrho\chi(\varsigma,a)}}{k^m}{}_{k}^{Tc}\mathcal{D}_{a^+}^{\vartheta+mk,\varrho;\phi}\mathfrak{u}(\varsigma).$$

Lemma 2.23. *Let $\vartheta,\xi>0$, $\varrho\geq 0$. Then for all $\varsigma\in\mathfrak{I}$ we have*

$${}_{k}^{T}\mathcal{I}_{a}^{\vartheta,\varrho;\phi}e^{\xi\chi(\varsigma,a)}\leq(k\varrho+k\xi)^{-\frac{\vartheta}{k}}e^{\xi\chi(\varsigma,a)}.$$

Proof. From (2.3) and using [11, Lemma 2.6], we have

$$\begin{aligned}{}_{k}^{T}\mathcal{I}_{a}^{\vartheta,\varrho;\phi}e^{\xi\chi(\varsigma,a)}&=k^{-\frac{\vartheta}{k}}e^{-\varrho\chi(\varsigma,a)}\mathcal{I}_{a^+}^{\frac{\vartheta}{k};\phi}\left(e^{(\xi+\varrho)\chi(\varsigma,a)}\right)\\&\leq(\xi+\varrho)^{-\frac{\vartheta}{k}}k^{-\frac{\vartheta}{k}}e^{-\varrho\chi(\varsigma,a)}e^{(\xi+\varrho)\chi(\varsigma,a)}\\&\leq(k\varrho+k\xi)^{-\frac{\vartheta}{k}}e^{\xi\chi(\varsigma,a)}.\end{aligned}$$

□

Remark 2.24. [38] On the space $C(\mathfrak{I},\mathbb{O})$ we define a Bielecki type norm $\|\cdot\|_{\mathfrak{B}}$ as below

$$\|\mathfrak{u}\|_{\mathfrak{B}}:=\sup_{\varsigma\in\mathfrak{I}}e^{-\xi\chi(\varsigma,a)}\|\mathfrak{u}(\varsigma)\|,\quad \xi>0.$$

Consequently, the norms $\|\cdot\|_{\mathfrak{B}}$ and $\|\cdot\|_{\infty}$ are equivalent on $C(\mathfrak{I},\mathbb{O})$, i.e;

$$\|\cdot\|_{\mathfrak{B}}\leq\|\cdot\|_{\infty}\leq e^{\xi\chi(\varsigma,a)}\|\cdot\|_{\mathfrak{B}}.$$

Theorem 2.25. [9, 36] *Let $\mathbb{X}$ be a real separable generalized Banach space and $(\Omega,\mathcal{G})$ be a measurable space and $\mathcal{Q}:\Omega\times\mathbb{X}\to\mathbb{X}$ a continuous random operator, and let $\mathcal{M}(\omega)\in\mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that for every $\omega\in\Omega$, the matrix $\mathcal{M}(\omega)$ converges to zero and:*

$$d(\mathcal{Q}(\omega,\mathfrak{s}_1),\mathcal{Q}(\omega,\mathfrak{s}_2))\leq\mathcal{M}(\omega)d(\mathfrak{s}_1,\mathfrak{s}_2),\quad \textit{for each } \mathfrak{s}_1,\mathfrak{s}_2\in\mathbb{X} \textit{ and } \omega\in\Omega.$$

Then, there exists a unique random fixed point of $\mathcal{Q}$.

Theorem 2.26. [9, 36] *Let $\mathbb{O}$ be a separable generalized Banach space, and let $\mathcal{Q}:\Omega\times\mathbb{O}\to\mathbb{O}$ be a condensing continuous random operator. Then either of the following holds:*

(i) The random equation $\mathcal{Q}(\omega,\mathfrak{s})=\mathfrak{s}$ has a random solution, i.e., there is a measurable function $\mathfrak{s}:\Omega\to\mathbb{O}$ such that $\mathcal{Q}(\omega,\mathfrak{s}(\omega))=\mathfrak{s}(\omega)$ for all $\omega\in\Omega$,
or

(ii) The set

$$\mathcal{W}=\{\mathfrak{s}:\Omega\to\mathbb{O}\ \textit{is measurable}\ \kappa(\omega)\mathcal{Q}(\omega,\mathfrak{s})=\mathfrak{s}\}$$

is unbounded for some measurable function $\kappa:\Omega\to\mathbb{O}$ with $\mu(\omega)\in(0,1)$ on Ω.

Lemma 2.27. *Let $0<\vartheta_i,\mu_i<k$, $i=1,2$, $k>0$, $\varrho>0$ and $\phi\in\mathbb{H}_+^1(\mathfrak{I},\mathbb{R})$. Then*

$$\mathfrak{h}_{i,r}(\gamma,\omega):=\sup_{\varsigma\in\mathfrak{I}}\frac{2\mathbb{G}_{i,r}(\omega)}{k\Gamma_k(\vartheta_i+\mu_i)}\int_a^{\varsigma}\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma,s)e^{-\gamma(\varsigma-s)}\phi'(s)ds\xrightarrow[\gamma\to+\infty]{}0,\ i,r=1,2,\tag{2.6}$$

and

$$\mathfrak{g}_i(\gamma,\omega):=\sup_{\varsigma\in\mathfrak{I}}\frac{2}{k\Gamma_k(\mu_i)}\int_a^{\varsigma}\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i}{k}-1}(\varsigma,s)e^{-\gamma(\varsigma-s)}\phi'(s)ds\xrightarrow[\gamma\to+\infty]{}0,\ i=1,2.\tag{2.7}$$

Proof. From Lemma 2.16, we get

$$\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma,\cdot)\phi'(\cdot)\in L^1(\mathfrak{I},\mathbb{R}),\quad i=1,2.$$

So, there exists $\widehat{\mathfrak{g}} \in C(\mathfrak{I}, \mathbb{R})$ such that

$$\int_a^{\varsigma} \left| \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)\phi'(s) - \widehat{\mathfrak{g}}(s) \right| ds < \frac{1}{2}\epsilon.$$

Hence

$$\begin{aligned} &\left| \int_a^{\varsigma} \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)\phi'(s)e^{-\gamma(\varsigma-s)}ds \right| \\ &\quad \le \int_a^{\varsigma} \left| \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)\phi'(s) - \widehat{\mathfrak{g}}(s) \right| e^{-\gamma(\varsigma-s)}ds + \int_a^{\varsigma} |\widehat{\mathfrak{g}}(s)|e^{-\gamma(\varsigma-s)}ds \\ &\quad \le \frac{\epsilon}{2} + \frac{1-e^{-\gamma(b-a)}}{\gamma}\widehat{\mathfrak{g}}^*, \quad i = 1, 2, \end{aligned}$$

where $\widehat{\mathfrak{g}}^* := \|\widehat{\mathfrak{g}}\|_\infty$. Consequently,

$$\int_a^{\varsigma} \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)e^{-\gamma(\varsigma-s)}\phi'(s)ds \longrightarrow 0 \text{ as } \gamma \longrightarrow +\infty, \quad i = 1, 2.$$

Similar to the above process, we have

$$\int_a^{\varsigma} \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma, s)e^{-\gamma(\varsigma-s)}\phi'(s)ds \longrightarrow 0 \text{ as } \gamma \longrightarrow +\infty, \quad i = 1, 2.$$

□

3 Gronwall-type inequality via tempered (k, ϕ)-RL integral kernel

In this section, a new Gronwall's inequality within tempered (k, ϕ)-RL integral kernel is established.

Theorem 3.1. [6] *Let $\alpha_l > 0$, $l = \overline{1, n}$, $n \in \mathbb{N}$ and $\phi \in \mathbb{H}_+^1(\mathcal{J}, \mathbb{R})$. Assume that*

(i) The functions g_l are the bounded and monotonic increasing functions on $[a, b)$,

(ii) $\mathfrak{z}$ and $\mathfrak{u}$ are nonnegative functions locally integrable on $[a, b)$.

If

$$\mathfrak{z}(\varsigma) \le \mathfrak{u}(\varsigma) + \sum_{l=1}^{n} g_l(\varsigma) \int_a^{\varsigma} \chi(\varsigma, s)^{\alpha_l-1}\mathfrak{z}(s)\phi'(s)ds, \tag{3.1}$$

then,

$$\mathfrak{z}(\varsigma) \le \mathfrak{u}(\varsigma) + \sum_{j=1}^{\infty} \left(\sum_{1',2',3',\ldots,j'=1}^{n} \frac{\prod_{l=1}^{j}(g_{l'}(\varsigma)\Gamma(\alpha_{l'}))}{\Gamma(\sum_{l=1}^{j}\alpha_{l'})} \int_a^{\varsigma} \chi(\varsigma, s)^{\sum_{l=1}^{j}\alpha_{l'}-1}\mathfrak{u}(s)\phi'(s)ds \right). \tag{3.2}$$

Theorem 3.2. *Let $\varrho \in \mathbb{R}$, $\vartheta_l, k > 0$, $l = \overline{1, n}$, $n \in \mathbb{N}$ and $\phi \in \mathbb{H}_+^1(\mathcal{J}, \mathbb{R})$. Assume that*

(i) The functions h_l are the bounded and monotonic increasing functions on $[a, b)$,

(ii) $\mathfrak{z}$ and $\mathfrak{u}$ are nonnegative functions locally integrable on $[a, b)$.

If

$$\mathfrak{z}(\varsigma) \le \mathfrak{u}(\varsigma) + \sum_{l=1}^{n} \frac{h_l(\varsigma)}{k} \int_a^{\varsigma} \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_l}{k}-1}(\varsigma, s)\mathfrak{z}(s)\phi'(s)ds, \quad \varsigma \in \mathfrak{I}, \tag{3.3}$$

then,

$$\mathfrak{z}(\varsigma) \le \mathfrak{u}(\varsigma) + \sum_{j=1}^{\infty} \left(\int_a^{\varsigma} \sum_{1',2',3',\ldots,j'=1}^{n} \frac{\left[\prod_{l=1}^{j}\Theta h_{l'}(\varsigma)\Gamma_k(\vartheta_{l'})\right]}{k\Gamma_k(\sum_{l=1}^{l}\vartheta_{l'})} \chi(\varsigma, s)^{\sum_{l=1}^{j}\frac{\vartheta_{l'}}{k}-1}\mathfrak{u}(s)\phi'(s)ds \right), \tag{3.4}$$

where

$$\Theta = \max_{(\varsigma,s)\in\mathfrak{I}\times[a,\varsigma]} e^{-\varrho\chi(\varsigma,s)} = \begin{cases} 1 & \text{if} \quad \varrho \ge 0 \\ e^{-\varrho\chi(b,a)} & \text{if} \quad \varrho < 0. \end{cases}$$

Proof. Using the definition of ${}_{k}^{T}\mathcal{I}_{a^+}^{\vartheta_l,\varrho;\phi}(\cdot)$ and [23, Theorem 4.9], we have

$$ {}_{k}^{T}\mathcal{I}_{a^+}^{\vartheta_l,\varrho;\phi}\mathfrak{z}(\varsigma) \quad \leq \Theta\left({}_{k}\mathcal{I}_{a^+}^{\vartheta_l;\phi}\mathfrak{z}(s)\right)(\varsigma) = \Theta k^{-\frac{\vartheta_l}{k}}\left(\mathcal{I}_{a^+}^{\frac{\vartheta_l}{k};\phi}\mathfrak{z}(s)\right)(\varsigma). \tag{3.5}$$

where $\mathcal{I}_{a^+}^{\frac{\vartheta_l}{k};\phi}(\cdot)$ the ϕ-Riemann–Liouville FI [21]. Then, by (3.3) and (3.5), we get

$$\begin{aligned}\mathfrak{z}(\varsigma) \quad &\leq \mathfrak{u}(\varsigma) + \textstyle\sum_{l=1}^{n}\Gamma_k(\vartheta_l)h_l(\varsigma)\left({}_{k}^{T}\mathcal{I}_{a^+}^{\vartheta_l,\varrho;\phi}\mathfrak{z}(s)\right)(\varsigma)\\ &\leq \mathfrak{u}(\varsigma) + \textstyle\sum_{l=1}^{n}\Gamma_k(\vartheta_l)h_l(\varsigma)\Theta k^{-\frac{\vartheta_l}{k}}\left(\mathcal{I}_{a^+}^{\frac{\vartheta_l}{k};\phi}\mathfrak{z}(s)\right)(\varsigma).\end{aligned}$$

Put $kg_l(\varsigma) = h_l(\varsigma)\Theta$ and $k\alpha_l = \vartheta_l$, $l = 1, \cdots, n$, by using the properties of k-gamma function, we have

$$\begin{aligned}\mathfrak{z}(\varsigma) \quad &\leq \mathfrak{u}(\varsigma) + \textstyle\sum_{l=1}^{n}k^{\alpha_l-1}\Gamma(\alpha_l)h_l(\varsigma)\Theta k^{-\alpha_l}\left(\mathcal{I}_{a^+}^{\alpha_l;\phi}\mathfrak{z}(s)\right)(\varsigma)\\ &\leq \mathfrak{u}(\varsigma) + \textstyle\sum_{l=1}^{n}\Gamma(\alpha_l)g_l(\varsigma)\left(\mathcal{I}_{a^+}^{\alpha_l;\phi}\mathfrak{z}(s)\right)(\varsigma)\\ &\leq \mathfrak{u}(\varsigma) + \textstyle\sum_{l=1}^{n}g_l(\varsigma)\int_a^\varsigma\chi(\varsigma,s)^{\alpha_l-1}\phi'(s)\mathfrak{z}(s)ds.\end{aligned}$$

Applying Theorem 3.1, one obtains

$$\mathfrak{z}(\varsigma) \quad \leq \mathfrak{u}(\varsigma) + \textstyle\sum_{j=1}^{\infty}\left(\sum_{1',2',3',\ldots,j'=1}^{n}\frac{\left[\prod_{l=1}^{j}g_{l'}(\varsigma)\Gamma(\alpha_{l'})\right]}{\Gamma(\sum_{l=1}^{j}\alpha_{l'})}\int_a^\varsigma\chi(\varsigma,s)^{\sum_{l=1}^{j}\alpha_{l'}-1}\phi'(s)\mathfrak{u}(s)ds\right)$$

Using the properties of k-gamma function, one more time, one can get

$$\begin{aligned}\mathfrak{z}(\varsigma) \quad &\leq \mathfrak{u}(\varsigma) + e^{-\varrho\chi(\varsigma,a)}\textstyle\sum_{j=1}^{\infty}\left(\sum_{1',2',3',\ldots,j'=1}^{n}\frac{\left[\prod_{l=1}^{j}g_{l'}(\varsigma)k^{1-\frac{\vartheta_{l'}}{k}}\Gamma_k(\vartheta_{l'})\right]}{k^{1-\sum_{l=1}^{j}\vartheta_{l'}}\Gamma(\sum_{l=1}^{j}\vartheta_{l'})}\times\right.\\ &\qquad\left.\int_a^\varsigma\chi(\varsigma,s)^{\sum_{l=1}^{j}\frac{\vartheta_{l'}}{k}-1}\phi'(s)\mathfrak{u}(s)ds\right)\\ &\leq \mathfrak{u}(\varsigma) + \textstyle\sum_{j=1}^{\infty}\left(\sum_{1',2',3',\ldots,j'=1}^{n}\frac{\left[\prod_{l=1}^{j}\Theta h_{l'}(\varsigma)\Gamma_k(\vartheta_{l'})\right]}{k\Gamma(\sum_{l=1}^{j}\vartheta_{l'})}\int_a^\varsigma\chi(\varsigma,s)^{\sum_{l=1}^{j}\frac{\vartheta_{l'}}{k}-1}\phi'(s)\mathfrak{u}(s)ds\right).\end{aligned}$$

Hence

$$\mathfrak{z}(\varsigma) \leq \mathfrak{u}(\varsigma) + \sum_{j=1}^{\infty}\left(\int_a^\varsigma\sum_{1',2',3',\ldots,j'=1}^{n}\frac{\left[\prod_{l=1}^{j}\Theta h_{l'}(\varsigma)\Gamma_k(\vartheta_{l'})\right]}{k\Gamma_k(\sum_{l=1}^{l}\vartheta_{l'})}\chi(\varsigma,s)^{\sum_{l=1}^{j}\frac{\vartheta_{l'}}{k}-1}\mathfrak{u}(s)\phi'(s)ds\right).$$

□

Corollary 3.3. *Under the hypotheses of Theorem 3.2, assume further that* $\mathfrak{u}(\varsigma)$ *is a nondecreasing function for* $\varsigma \in [a, b)$*, then*

$$\mathfrak{z}(\varsigma) \quad \leq \mathfrak{u}(\varsigma)\textstyle\sum_{l=0}^{n}\mathbb{E}_{\vartheta_l}^{k}\left(\Theta h_l(\varsigma)\Gamma_k(\vartheta_l)\chi(\varsigma,a)^{\frac{\vartheta_l}{k}}\right). \tag{3.6}$$

Proof. From (3.4) and $\mathfrak{u}$ is a nondecreasing function for $\varsigma \in [a, b)$, we can write

$$\mathfrak{z}(\varsigma) \leq \mathfrak{u}(\varsigma)\left[1 + \sum_{j=1}^{\infty}\left(\sum_{1',2',3',\ldots,j'=1}^{n}\frac{\left[\prod_{l=1}^{j}\Theta h_{l'}(\varsigma)\Gamma_k(\vartheta_{l'})\right]}{k\Gamma_k(\sum_{l=1}^{l}\vartheta_{l'})}\int_a^\varsigma\chi(\varsigma,s)^{\sum_{l=1}^{j}\frac{\vartheta_{l'}}{k}-1}\phi'(s)ds\right)\right].$$

Then,

$$\begin{aligned}\mathfrak{z}(\varsigma) \quad &\leq \mathfrak{u}(\varsigma)\left[1 + \textstyle\sum_{j=1}^{\infty}\left(\sum_{1',2',3',\ldots,j'=1}^{n}\left[\prod_{l=1}^{j}\Theta h_{l'}(\varsigma)\Gamma_k(\vartheta_{l'})\right]\frac{\chi(\varsigma,a)^{\sum_{l=1}^{j}\frac{\vartheta_{l'}}{k}}}{\Gamma_k(\sum_{l=1}^{l}\vartheta_{l'}+k)}\right)\right]\\ &\leq \mathfrak{u}(\varsigma)\textstyle\sum_{l=0}^{n}\mathbb{E}_{\vartheta_l}^{k}\left(\Theta h_l(\varsigma)\Gamma_k(\vartheta_l)\chi(\varsigma,a)^{\frac{\vartheta_l}{k}}\right).\end{aligned}$$

□

Remark 3.4. From Theorem 3.2 and Corollary 3.3, we have the following particular cases:

- If $\varrho = 0$ and $k = 1$, then Theorem 3.2 and Corollary 3.3 reduces to the inequality given by Theorem 3.1 and [6, Corollary 2.1].
- If $n = 1$, then Theorem 3.2 and Corollary 3.3 reduces to the inequalities given by [35, Theorem 3.1].

4 Existence and Uniqueness of Solutions

In this section, we establish the existence and uniqueness results for the system (1.1).

Lemma 4.1. *For $\widehat{c}_\ell, \widehat{d}_\ell \in \mathbb{O}$ and $f : C(\mathfrak{I}, \mathbb{R})$, the problem*

$$\begin{cases} {}^{Tc}_{k}\mathcal{D}^{\vartheta_1,\varrho;\phi}_{a^+}\left({}^{Tc}_{k}\mathcal{D}^{\mu_1,\varrho;\phi}_{a^+} - \varpi_1\right)\mathfrak{s}_1(\varsigma) = f(\varsigma), & a \leqslant \varsigma \leqslant b, \\ D^{[\ell],\varrho;\phi}\mathfrak{s}_1(a) = \widehat{c}_\ell, & 0 \leq \ell < J, \\ {}^{Tc}_{k}\mathcal{D}^{\mu_1+k\ell,\varrho;\phi}_{a^+}\mathfrak{s}_1(a) = \widehat{d}_\ell, & 0 \leq \ell < n, \end{cases} \tag{4.1}$$

has a solution given by

$$\begin{aligned} \mathfrak{s}_1(\varsigma) &= {}^{T}_{k}\mathcal{I}^{\vartheta_1+\mu_1,\varrho;\phi}_{a^+} f(\varsigma) + \varpi_1 {}^{T}_{k}\mathcal{I}^{\mu_1,\varrho;\phi}_{a^+}\mathfrak{s}_1(\varsigma) + \sum_{j=0}^{n-1} \frac{\widehat{d}_j - \varpi_1 k^j \widehat{c}_j}{\Gamma_k(kj+k+\mu 1)} \mathfrak{T}^{\frac{\mu_1}{k}+j}_{\varrho,\chi}(\varsigma, a) \\ &+ \sum_{l=0}^{m-1} \frac{k^l \widehat{c}_l}{\Gamma_k(kl+k)} \mathfrak{T}^{l}_{\varrho,\chi}(\varsigma, a). \end{aligned} \tag{4.2}$$

Proof. Let $\mathfrak{s}_1(\cdot)$ be the solution of (4.1). Applying the ${}^{T}_{k}\mathcal{I}^{\vartheta_1,\varrho;\phi}_{a^+}(\cdot)$ on both sides of the Eq (4.1), and by using Theorem 2.19 and Remark 2.22, we obtain .

$$\begin{aligned} {}^{Tc}_{k}\mathcal{D}^{\mu 1,\varrho;\phi}_{a^+}\mathfrak{s}_1(\varsigma) &= {}^{T}_{k}\mathcal{I}^{\vartheta_1,\varrho;\phi}_{a^+} f(\varsigma) + \varpi_1\mathfrak{s}_1(\varsigma) \ + \\ &\quad e^{-\varrho\chi(\varsigma,a)} \sum_{j=0}^{n-1} \frac{\left[D^{[j],\varrho;\phi}\left({}^{Tc}_{k}\mathcal{D}^{\mu_1,\varrho;\phi}_{a^+}\mathfrak{s}_1(\varsigma) - \varpi_1\mathfrak{s}_1(\varsigma)\right)\right]_{\varsigma=a}}{k^{-j}\Gamma_k(k(j+1))}\chi(\varsigma, a)^j \\ &= {}^{T}_{k}\mathcal{I}^{\vartheta_1,\varrho;\phi}_{a^+} f(\varsigma) + \varpi_1\mathfrak{s}_1(\varsigma) \ + \\ &\quad e^{-\varrho\chi(\varsigma,a)} \sum_{j=0}^{n-1} \frac{{}^{Tc}_{k}\mathcal{D}^{\mu_1+jk,\varrho;\phi}_{a^+}\mathfrak{s}_1(a) - k^j\varpi_1 D^{[j],\varrho;\phi}\mathfrak{s}_1(a)}{\Gamma_k(k(j+1))}\chi(\varsigma, a)^j. \end{aligned}$$

From the initial condition (4.1), it follows that

$${}^{Tc}_{k}\mathcal{D}^{\mu_1,\varrho;\phi}_{a^+}\mathfrak{s}_1(\varsigma) \quad = {}^{T}_{k}\mathcal{I}^{\vartheta_1,\varrho;\phi}_{a^+} f(\varsigma) + \varpi_1\mathfrak{s}_1(\varsigma) + e^{-\varrho\chi(\varsigma,a)} \sum_{j=0}^{n-1} \frac{\widehat{d}_j - \varpi_1 k^j \widehat{c}_j}{\Gamma_k(k(j+1))}\chi(\varsigma, a)^j. \tag{4.3}$$

We apply the operator ${}^{T}_{k}\mathcal{I}^{\mu_1,\varrho;\phi}_{a^+}(\cdot)$ to both sides of Eq (4.3), and use the results of Theorem 2.19 and Lemma 2.15, we get

$$\begin{aligned} \mathfrak{s}_1(\varsigma) &= {}^{T}_{k}\mathcal{I}^{\vartheta_1+\mu_1,\varrho;\phi}_{a^+} f(\varsigma) + e^{-\varrho\chi(\varsigma,a)} \sum_{j=0}^{n-1} \frac{\widehat{d}_j - \varpi_1 k^j \widehat{c}_j}{\Gamma_k(kj+k+\mu 1)}\chi(\varsigma, a)^{\frac{\mu_1}{k}+j} \\ &\quad + \varpi_1 {}^{T}_{k}\mathcal{I}^{\mu_1,\varrho;\phi}_{a^+}\mathfrak{s}_1(\varsigma) + e^{-\varrho\chi(\varsigma,a)} \sum_{l=0}^{m-1} \frac{\left[D^{[l],\varrho;\phi}\mathfrak{s}_1(\varsigma)\right]_{\varsigma=a}}{k^{-l}\Gamma_k(k(l+1))}\chi(\varsigma, a)^l, \end{aligned}$$

Hence

$$\begin{aligned} \mathfrak{s}_1(\varsigma) &= {}^{T}_{k}\mathcal{I}^{\vartheta_1+\mu_1,\varrho;\phi}_{a^+} f(\varsigma) + e^{-\varrho\chi(\varsigma,a)} \sum_{j=0}^{n-1} \frac{\widehat{d}_j - \varpi_1 k^j \widehat{c}_j}{\Gamma_k(kj+k+\mu 1)}\chi(\varsigma, a)^{\frac{\mu_1}{k}+j} \\ &\quad + \varpi_1 {}^{T}_{k}\mathcal{I}^{\mu_1,\varrho;\phi}_{a^+}\mathfrak{s}_1(\varsigma) + e^{-\varrho\chi(\varsigma,a)} \sum_{l=0}^{m-1} \frac{k^l \widehat{c}_l}{\Gamma_k(k(l+1))}\chi(\varsigma, a)^l, \end{aligned}$$

This finishes the proof. □

Theorem 4.2. *Suppose that*

(A1) The functions $\mathfrak{f}_i$ are random Carathéodory on $\mathfrak{I} \times \mathbb{O} \times \mathbb{O} \times \Omega$.

(A2) There exists random variables $\mathfrak{X}_{i,j}:\Omega\to(0,\infty)$*;* $i,j=1,2$ *such that:*

$$\|\mathfrak{f}_i(\varsigma,x_1,x_2,\omega)-\mathfrak{f}_i(\varsigma,\widehat{x}_1,\widehat{x}_2,\omega)\|\le\mathfrak{X}_{i,1}(\omega)\|x_1-\widehat{x}_1\|+\mathfrak{X}_{i,2}(\omega)\|x_2-\widehat{x}_2\|,\ i=1,2,$$

for $x_1,x_2,\widehat{x}_1,\widehat{x}_2\in\mathbb{O}$*,* $(\varsigma,\omega)\in\mathfrak{I}'\times\Omega$*.*

Then, system (1.1) admits a unique random solution.

Proof. Firstly, endowing the product Banach space $\mathbb{J}=C(\mathfrak{I},\mathbb{O})\times C(\mathfrak{I},\mathbb{O})$ by the vector-norm

$$\|(\mathfrak{s}_1,\mathfrak{s}_2)\|_{\mathbb{J}}=\begin{pmatrix}\|\mathfrak{s}_1\|_\infty\\ \|\mathfrak{s}_2\|_\infty\end{pmatrix}. \tag{4.4}$$

Next, according to Lemma 4.1, system (1.1) is equivalent to the operator equation $\mathcal{Q}(\mathfrak{s}_1,\mathfrak{s}_2,\omega)=(\mathfrak{s}_1,\mathfrak{s}_2)$ where $\mathcal{Q}:\mathbb{J}\times\Omega\to\mathbb{J}$ be the operator given by:

$$\mathcal{Q}(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)=\left(\mathcal{Q}_1(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega),\mathcal{Q}_2(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\right). \tag{4.5}$$

where,

$$\begin{aligned}&\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\\ &\quad=\textstyle\sum_{l=0}^{m-1}\frac{k^l z_{i,l}(\omega)}{\Gamma_k(kl+k)}\mathfrak{T}^l_{\varrho,\chi}(\varsigma,a)+\sum_{j=0}^{n-1}\frac{w_{i,j}(\omega)-\varpi_1 k^j z_{i,j}(\omega)}{\Gamma_k(kj+k+\mu_i)}\mathfrak{T}^{\frac{\mu_i}{k}+j}_{\varrho,\chi}(\varsigma,a)\\ &\quad+\frac{1}{k\Gamma_k(\vartheta_i+\mu_i)}\int_a^\varsigma\phi'(s)\mathfrak{T}^{\frac{\vartheta_i+\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\mathfrak{f}_i(s,\mathfrak{s}_1(s,\omega),\mathfrak{s}_2(s,\omega),\omega)ds\\ &\quad+\frac{\varpi_i}{k\Gamma_k(\mu_i)}\int_a^\varsigma\phi'(s)\mathfrak{T}^{\frac{\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\mathfrak{s}_i(s,\omega)ds,\quad i=1,2.\end{aligned} \tag{4.6}$$

Since the function $\mathfrak{f}_i$, $i=1,2$ are absolutely continuous for all $\omega\in\Omega$ and $\varsigma\in\mathfrak{I}$, then $(\mathfrak{s}_1,\mathfrak{s}_2)$ is a random solution for the problem (1.1) if and only if $(\mathfrak{s}_1,\mathfrak{s}_2)=(\mathcal{Q}(\mathfrak{s}_1,\mathfrak{s}_2))(\varsigma,\omega)$.

We need to demonstrate that the operator $\mathcal{Q}$ is a contraction mapping on $\mathbb{J}$ using Bielecki's vector-norm.

Step 1. $\mathcal{Q}$ is a random operator on $\mathbb{J}$.

Using (A1), the functions $\omega\to\mathfrak{f}_i(\xi,\mathfrak{s}_1,\mathfrak{s}_2,\omega)$ are measurable for $i=1,2$. In view of Lemma 2.7, the products

$$\mathfrak{T}^{\frac{\vartheta_i+\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\mathfrak{f}_i(s,\mathfrak{s}_1(s,\omega),\mathfrak{s}_2(s,\omega),\omega)\quad\text{and}\quad\mathfrak{T}^{\frac{\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\mathfrak{s}_i(s,\omega).$$

are again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the maps

$$\omega\to\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega),\ \ i=1,2,$$

are measurable. Accordingly, $\mathcal{Q}$ is a random operator on $\mathbb{J}\times\Omega$ into $\mathbb{J}$.

Step 2. $\mathcal{Q}$ is a contraction mapping on $\mathbb{J}$.

For any $\omega\in\Omega$ and each $(\mathfrak{s}_1,\mathfrak{s}_2),(\mathfrak{r}_1,\mathfrak{r}_2)\in\mathbb{J}$, we have

$$\begin{aligned}&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)-\mathcal{Q}_i(\mathfrak{r}_1(\varsigma,\omega),\mathfrak{r}_2(\varsigma,\omega),\omega)\|\\ &\ =\int_a^\varsigma\frac{\phi'(s)\mathfrak{T}^{\frac{\vartheta_i+\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)}{k\Gamma_k(\vartheta_i+\mu_i)}\|\mathfrak{f}_i(s,\mathfrak{s}_1(s,\omega),\mathfrak{s}_2(s,\omega),\omega)-\mathfrak{f}_i(s,\mathfrak{r}_1(s,\omega),\mathfrak{r}_2(s,\omega),\omega)\|ds\\ &\quad+\frac{|\varpi_i|}{k\Gamma_k(\mu_i)}\int_a^\varsigma\phi'(s)\mathfrak{T}^{\frac{\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\|\mathfrak{s}_i(s,\omega)-\mathfrak{r}_i(s,\omega)\|ds,\quad i=1,2.\end{aligned}$$

By (A2) and remark 2.24, can be written as

$$\begin{aligned}&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)-\mathcal{Q}_i(\mathfrak{r}_1(\varsigma,\omega),\mathfrak{r}_2(\varsigma,\omega),\omega)\|\\ &\quad\le \textstyle\sum_{r=1}^{2}\mathfrak{X}_{i,r}(\omega)\|\mathfrak{s}_r(\cdot,\omega)-\mathfrak{r}_r(\cdot,\omega)\|_{\mathfrak{B}}\left({}_k^T\mathcal{I}_{a^+}^{\vartheta_i+\mu_i,\varrho;\phi}e^{\xi\chi(\varsigma,a)}\right)\\ &\qquad+|\varpi_i|\|\mathfrak{s}_i(\cdot,\omega)-\mathfrak{r}_i(\cdot,\omega)\|_{\mathfrak{B}}\left({}_k^T\mathcal{I}_{a^+}^{\mu_i,\varrho;\phi}e^{\xi\chi(\varsigma,a)}\right),\quad i=1,2.\end{aligned}$$

By Lemma 2.23, one obtains

$$\begin{aligned}&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)-\mathcal{Q}_i(\mathfrak{r}_1(\varsigma,\omega),\mathfrak{r}_2(\varsigma,\omega),\omega)\|\\ &\quad\le (k\varrho+k\xi)^{-\frac{\vartheta_i+\mu_i}{k}}e^{\xi\chi(\varsigma,a)}\textstyle\sum_{r=1}^{2}\mathfrak{X}_{i,r}(\omega)\|\mathfrak{s}_r(\cdot,\omega)-\mathfrak{r}_r(\cdot,\omega)\|_{\mathfrak{B}}\\ &\qquad+|\varpi_i|(k\varrho+k\xi)^{-\frac{\mu_i}{k}}e^{\xi\chi(\varsigma,a)}\|\mathfrak{s}_i(\cdot,\omega)-\mathfrak{r}_i(\cdot,\omega)\|_{\mathfrak{B}},\quad i=1,2.\end{aligned}$$

Hence

$$\begin{aligned}&\|\mathcal{Q}_i(\mathfrak{s}_1(\cdot,\omega),\mathfrak{s}_2(\cdot,\omega),\omega)-\mathcal{Q}_i(\mathfrak{r}_1(\cdot,\omega),\mathfrak{r}_2(\cdot,\omega),\omega)\|_{\mathfrak{B}}\\ &\quad\le (k\varrho+k\xi)^{-\frac{\vartheta_i+\mu_i}{k}}\textstyle\sum_{r=1}^{2}\mathfrak{X}_{i,r}(\omega)\|\mathfrak{s}_r(\cdot,\omega)-\mathfrak{r}_r(\cdot,\omega)\|_{\mathfrak{B}}\\ &\qquad+|\varpi_i|(k\varrho+k\xi)^{-\frac{\mu_i}{k}}\|\mathfrak{s}_i(\cdot,\omega)-\mathfrak{r}_i(\cdot,\omega)\|_{\mathfrak{B}},\quad i=1,2.\end{aligned}$$

Therefore, we have

$$d((\mathcal{Q}(\mathfrak{s}_1,\mathfrak{s}_2))(\cdot,\omega),(\mathcal{Q}(\mathfrak{r}_1,\mathfrak{r}_2))(\cdot,\omega))\le\mathbb{K}_\xi(\omega)d\left((\mathfrak{s}_1(\cdot,\omega),\mathfrak{s}_2(\cdot,\omega)),(\mathfrak{r}_1(\cdot,\omega),\mathfrak{r}_2(\cdot,\omega))\right),$$

where:

$$\mathbb{K}_\xi(\omega)=\begin{pmatrix}\frac{|\varpi_1|}{(k\varrho+k\xi)^{\frac{\mu_1}{k}}}+\frac{\mathfrak{X}_{1,1}(\omega)}{(k\varrho+k\xi)^{\frac{\vartheta_1+\mu_1}{k}}} & \frac{\mathfrak{X}_{1,2}(\omega)}{(k\varrho+k\xi)^{\frac{\vartheta_1+\mu_1}{k}}}\\ \frac{\mathfrak{X}_{2,1}(\omega)}{(k\varrho+k\xi)^{\frac{\vartheta_2+\mu_2}{k}}} & \frac{|\varpi_2|}{(k\varrho+k\lambda)^{\frac{\mu_2}{k}}}+\frac{\mathfrak{X}_{2,2}(\omega)}{(k\varrho+k\xi)^{\frac{\vartheta_2+\mu_2}{k}}}\end{pmatrix},$$

and

$$d\left((\mathfrak{s}_1(\cdot,\omega),\mathfrak{s}_2(\cdot,\omega)),(\mathfrak{r}_1(\cdot,\omega),\mathfrak{r}_2(\cdot,\omega))\right)=\begin{pmatrix}\|\mathfrak{s}_1(\cdot,\omega)-\mathfrak{r}_1(\cdot,\omega)\|_{\mathfrak{B}}\\ \|\mathfrak{s}_2(\cdot,\omega)-\mathfrak{r}_2(\cdot,\omega)\|_{\mathfrak{B}}\end{pmatrix}.$$

Choosing $\xi>0$ large enough, the matrix $\mathbb{K}_\xi(\omega)$ converges to zero. Then, according to Theorem 2.25, $\mathcal{Q}$ possesses a unique random fixed-point, serving as the unique random solution to system (1.1). □

Theorem 4.3. *Suppose that*

(A1) The functions $\mathfrak{f}_i$ are random Carathéodory on $\mathfrak{I}\times\mathbb{O}\times\mathbb{O}\times\Omega$.

(A3) There exist $\varphi_i:\mathfrak{I}\times\Omega\to L^\infty(\mathfrak{I},\mathbb{R}_+)$, $i=1,2$ such that

$$\|\mathfrak{f}_i(\varsigma,x_1,x_2,\omega)\|\le\varphi_i(\varsigma,\omega)(\|x_1\|+\|x_2\|+1),\quad i=1,2,$$

for all $(\varsigma,x_1,x_2,\omega)\in\mathfrak{I}\times\mathbb{O}\times\mathbb{O}\times\Omega$.

(A4) There exists a constant random variable $\mathbb{G}_{i,r}:\Omega\to[0,\infty)$, $i,r=1,2$ such that for each $S^r\subset\mathcal{P}(C(\mathfrak{I},\mathbb{O}))$,

$$\Lambda(\mathfrak{f}_i(\varsigma,S^1(\varsigma),S^2(\varsigma),\omega))\le\mathbb{G}_{i,1}(\omega)\Lambda(S^1(\varsigma))+\mathbb{G}_{i,2}(\omega)\Lambda(S^2(\varsigma)),\ \textit{for all } (\varsigma,\omega)\in\mathfrak{I}\times\Omega.$$

Then, the system (1.1) possesses at least one random solution.

For brevity, let $\varphi_i^*=\sup_{\omega\in\Omega}\|\varphi_i(\cdot,\omega)\|_{L^\infty},\quad i=1,2.$

Proof. Consider a closed ball

$$\mathbb{S}_N = \{(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{J} \ : \ \|\mathfrak{s}_1(\cdot,\omega)\|_\infty \leq N, \ \|\mathfrak{s}_2(\cdot,\omega)\|_\infty \leq N\}. \tag{4.7}$$

The proof of Theorem 4.3 will proceed through several steps.

Step 1. $\mathcal{Q}$ transforms bounded sets into bounded sets in $\mathbb{J}$.

Let $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{S}_N$. Then, by the fact $e^{-\varrho\chi(\varsigma,a)} \leq 1$ for each $\varsigma \in \mathfrak{I}$, we have

$$\begin{aligned}
&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \\
&\quad \leq \sum_{l=0}^{m-1} \frac{k^l |z_{i,l}(\omega)|}{\Gamma_k(kl+k)} \chi(\varsigma,a)^l + \sum_{j=0}^{n-1} \frac{|w_{i,j}(\omega) - \varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)} \chi(\varsigma,a)^{\frac{\mu_i}{k}+j} \\
&\quad + \frac{1}{k\Gamma_k(\vartheta_i+\mu_i)} \int_a^\varsigma \phi'(s) \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma,s) \|\mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| ds \\
&\quad + \frac{|\varpi_i|}{k\Gamma_k(\mu_i)} \int_a^\varsigma \phi'(s) \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma,s) \|\mathfrak{s}_i(s,\omega)\| ds, \quad i = 1,2.
\end{aligned}$$

Thanks to hypothesis (A3), we get

$$\begin{aligned}
&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \\
&\quad \leq \sum_{l=0}^{m-1} \frac{k^l |z_{i,l}(\omega)|}{\Gamma_k(kl+k)} \chi(\varsigma,a)^l + \sum_{j=0}^{n-1} \frac{|w_{i,j}(\omega) - \varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)} \chi(\varsigma,a)^{\frac{\mu_i}{k}+j} \\
&\quad + \varphi_i^*(1+2N) \left({}_k^T\mathcal{I}_{a^+}^{\vartheta_i+\mu_i,\varrho;\phi} 1 \right)(\varsigma) + N|\varpi_i| \left({}_k^T\mathcal{I}_{a^+}^{\mu_i,\varrho;\phi} 1 \right)(\varsigma), \quad i = 1,2.
\end{aligned}$$

Using Lemma 2.16 and the fact $e^{-\varrho\chi(\varsigma,a)} \leq 1$ for each $\varsigma \in \mathfrak{I}$, we have

$$\begin{aligned}
&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \\
&\quad \leq \sum_{l=0}^{m-1} \frac{k^l |z_{i,l}(\omega)|}{\Gamma_k(kl+k)} \chi(\varsigma,a)^l + \varphi_i^*(2N+1) \chi(\varsigma,a)^{\frac{\vartheta_i+\mu_i}{k}} \mathbb{E}_{k,k+\vartheta_i+\mu_i}^k(k\varrho\chi(\varsigma,a)) \\
&\quad + \sum_{j=0}^{n-1} \frac{|w_{i,j}(\omega) - \varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)} \chi(\varsigma,a)^{\frac{\mu_i}{k}+j} + N|\varpi_i| \chi(\varsigma,a)^{\frac{\mu_i}{k}} \mathbb{E}_{k,k+\mu_i}^k(k\varrho\chi(\varsigma,a)), \ i = 1,2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \\
&\quad \leq \sum_{l=0}^{m-1} \frac{k^l |z_{i,l}(\omega)|}{\Gamma_k(kl+k)} \chi(b,a)^l + \varphi_i^*(2N+1) \chi(b,a)^{\frac{\vartheta_i+\mu_i}{k}} \mathbb{E}_{k,k+\vartheta_i+\mu_i}^k(k\varrho\chi(b,a)) \\
&\quad + \sum_{j=0}^{n-1} \frac{|w_{i,j}(\omega) - \varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)} \chi(b,a)^{\frac{\mu_i}{k}+j} + N|\varpi_i| \chi(b,a)^{\frac{\mu_i}{k}} \mathbb{E}_{k,k+\mu_i}^k(k\varrho\chi(b,a)), \ i = 1,2.
\end{aligned}$$

This shows that $\mathcal{Q}$ transforms bounded sets into bounded sets in $\mathbb{J}$.

Step 2. $\mathcal{Q}$ is continuous.

Let $\{\mathfrak{s}_{1,n}, \mathfrak{s}_{2,n}\}$ be a sequence satisfying $\{\mathfrak{s}_{1,n}, \mathfrak{s}_{2,n}\} \to (\mathfrak{s}_1, \mathfrak{s}_2)$ in $\mathbb{S}_N$ as $n \to \infty$. For each $(\varsigma, \omega) \in \mathfrak{I} \times \Omega$, making use of (A1), we easily have

$$\|\mathfrak{f}_i(s, \mathfrak{s}_{1,n}(s,\omega), \mathfrak{s}_{2,n}(s,\omega), \omega) - \mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| \to 0, \text{ as } n \longrightarrow \infty, \ i = 1,2.$$

Next, in view of (A3), one gets

$$\begin{aligned}
&\|\mathfrak{f}_i(s, \mathfrak{s}_{1,n}(s,\omega), \mathfrak{s}_{2,n}(s,\omega), \omega) - \mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| \\
&\quad \leq \|\mathfrak{f}_i(s, \mathfrak{s}_{1,n}(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| + \|\mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| \\
&\quad \leq 2\varphi_i(s,\omega) \left(\|\mathfrak{s}_1(s,\omega)\| + \|\mathfrak{s}_2(s,\omega)\| + 1 \right) \\
&\quad \leq 2(2N+1)\varphi_i(s,\omega), \quad i = 1,2.
\end{aligned}$$

Since, the functions $s \mapsto \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)\varphi_i(s,\omega)$ and $s \mapsto \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma, s)$, $i = 1, 2$ are Lebesgue integrable over $[a, \varsigma]$. Then it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\|\mathcal{Q}_i(\mathfrak{s}_{1,n}(\varsigma,\omega), \mathfrak{s}_{2,n}(\varsigma,\omega), \omega) - \mathcal{Q}_i(\mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \\
&\leq \int_a^\varsigma \frac{\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\kappa_i}{k}-1}(\varsigma,s)}{k\Gamma_k(\vartheta_i+\mu_i)} \|\mathfrak{f}_i(s, \mathfrak{s}_{1,n}(s,\omega), \mathfrak{s}_{2,n}(s,\omega), \omega) - \mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| ds \\
&\quad + \frac{|\varpi_i|}{k\Gamma_k(\mu_i)} \int_a^\varsigma \phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma, s)\|\mathfrak{s}_{i,n}(s,\omega) - \mathfrak{s}_i(s,\omega)\| dsds \\
&\xrightarrow[n\to\infty]{} 0, \text{ for all } \varsigma \in \mathfrak{I},\ i = 1, 2.
\end{aligned}$$

Therefore,

$$\|\mathcal{Q}_i(\cdot, \mathfrak{s}_{1,n}(\cdot,\omega), \mathfrak{s}_{2,n}(\cdot,\omega), \omega) - \mathcal{Q}_i(\cdot, \mathfrak{s}_1(\cdot,\omega), \mathfrak{s}_2(\cdot,\omega), \omega)\|_\infty \xrightarrow[n\to\infty]{} 0, \quad i = 1, 2.$$

Accordingly, the operator $\mathcal{Q}(\cdot, \cdot)$ is continuous.

Step 3. $\mathcal{Q}(\mathbb{S}_N)$ is equicontinuous.

On one hand, for all $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{S}_N$ and $a < \varsigma_1 < \varsigma_2 \leq b$, we have

$$\begin{aligned}
&\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma_2,\omega), \mathfrak{s}_2(\varsigma_2,\omega), \omega) - \mathcal{Q}_i(\mathfrak{s}_1(\varsigma_1,\omega), \mathfrak{s}_2(\varsigma_1,\omega), \omega)\| \\
&\leq \sum_{l=0}^{m-1} \frac{k^l|z_{i,l}(\omega)|}{\Gamma_k(kl+k)} \left|\mathfrak{T}_{\varrho,\chi}^l(\varsigma_2, a) - \mathfrak{T}_{\varrho,\chi}^l(\varsigma_1, a)\right| \\
&\quad + \sum_{j=0}^{n-1} \frac{|w_{i,j}(\omega) - \varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)} \left|\mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}+j}(\varsigma_2, a) - \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}+j}(\varsigma_1, a)\right| \\
&\quad + \frac{|\varpi_i|}{k\Gamma_k(\mu_i)} I_{1,i} + \frac{|\varpi_i|}{k\Gamma_k(\mu_i)} I_{2,i} + \frac{I_{3,i}}{k\Gamma_k(\vartheta_i+\mu_i)} + \frac{I_{4,i}}{k\Gamma_k(\vartheta_i+\mu_i)}, \quad i = 1, 2,
\end{aligned} \tag{4.8}$$

where,

$$I_{1,i} = \int_{\varsigma_1}^{\varsigma_2} \phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma_2, s)\|\mathfrak{s}_i(s,\omega)\| ds,\ i = 1, 2,$$

$$I_{2,i} = \int_a^{\varsigma_1} \phi'(s) \left|\mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma_2, s) - \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma_1, s)\right| \|\mathfrak{s}_i(s,\omega)\| ds,\ i = 1, 2,$$

$$I_{3,i} = \int_{\varsigma_1}^{\varsigma_2} \phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma_2, s)\|\mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| ds,\ i = 1, 2,$$

and

$$I_{4,i} = \int_a^{\varsigma_1} \phi'(s) \left|\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma_2, s) - \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma_1, s)\right| \|\mathfrak{f}_i(s, \mathfrak{s}_1(s,\omega), \mathfrak{s}_2(s,\omega), \omega)\| ds,\ i = 1, 2.$$

On the other hand, using Lemma 2.16, we get

$$\begin{aligned}
\frac{|\varpi_i|}{k\Gamma_k(\mu_i)} I_{1,i} &\leq N|\varpi_i| \left({}_k^T\mathcal{I}_{\varsigma_1}^{\mu_i,\varrho,\phi}1\right)(\varsigma_2) \\
&= \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}}(\varsigma_2, \varsigma_1)\mathbb{E}_{k,k+\mu_i}^k(k\varrho\chi(\varsigma_2, \varsigma_1)),\ i = 1, 2,
\end{aligned}$$

This produces that,

$$\frac{|\varpi_i| I_{1,i}}{k\Gamma_k(\mu_i)} \to 0 \quad \text{as} \quad \varsigma_2 \to \varsigma_1,\ i = 1, 2. \tag{4.9}$$

From (A3), and by Lemma 2.16, we get

$$\begin{aligned}
\frac{I_{3,i}}{k\Gamma_k(\vartheta_i+\mu_i)} &\leq (2N+1)\varphi_i^* \left({}_k^T\mathcal{I}_{\varsigma_1}^{\vartheta_i+\mu_i,\varrho,\phi}1\right)(\varsigma_2) \\
&= (2N+1)\varphi_i^*\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}}(\varsigma_2, \varsigma_1)\mathbb{E}_{k,k+\vartheta_i+\mu_i}^k(k\varrho\chi(\varsigma_2, \varsigma_1)),\ i = 1, 2,
\end{aligned}$$

So

$$\frac{I_{3,i}}{k\Gamma_k(\vartheta_i+\mu_i)} \to 0 \quad \text{as} \quad \varsigma_2 \to \varsigma_1,\ i=1,2. \tag{4.10}$$

From the fact $0 < e^{-\varrho\chi(\varsigma_1,s)} \le 1$, for $a < s < \varsigma_1$, we have

$$\begin{aligned}\left|\mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma_2,s) - \mathfrak{T}_{\varrho,\chi}^{\frac{\mu_i}{k}-1}(\varsigma_1,s)\right| &= e^{-\varrho\chi(\varsigma_1,s)}\left|e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\mu_i}{k}-1} - \chi(\varsigma_1,s)^{\frac{\mu_i}{k}-1}\right| \\ &\le \left|e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\mu_i}{k}-1} - \chi(\varsigma_1,s)^{\frac{\mu_i}{k}-1}\right|,\ i=1,2.\end{aligned}$$

If $e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\mu_i}{k}-1} \ge \chi(\varsigma_1,s)^{\frac{\mu_i}{k}-1}$, then,

$$\frac{|\varpi_i|}{k\Gamma_k(\mu_i)}I_{2,i} \le \frac{N|\varpi_i|}{\Gamma_k(\mu_i+k)}\left[e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\left(\chi(\varsigma_2,a)^{\frac{\mu_i}{k}} - \chi(\varsigma_2,\varsigma_1)^{\frac{\mu_i}{k}}\right) - \chi(\varsigma_1,a)^{\frac{\mu_i}{k}}\right],\ i=1,2,$$

If $e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\mu_i}{k}-1} \le \chi(\varsigma_1,s)^{\frac{\mu_i}{k}-1}$, then

$$\frac{|\varpi_i|}{k\Gamma_k(\mu_i)}I_{2,i} \le \frac{N|\varpi_i|}{\Gamma_k(\mu_i+k)}\left[\chi(\varsigma_1,a)^{\frac{\mu_i}{k}} + e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\left(\chi(\varsigma_2,\varsigma_1)^{\frac{\mu_i}{k}} - \chi(\varsigma_2,a)^{\frac{\mu_i}{k}}\right)\right],\ i=1,2,$$

Thus,

$$\frac{|\varpi_i|I_{2,i}}{k\Gamma_k(\mu_i)} \to 0 \quad \text{as} \quad \varsigma_2 \to \varsigma_1,\ i=1,2. \tag{4.11}$$

And

$$\left|\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma_2,s) - \mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma_1,s)\right| \le \left|e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\vartheta_i+\mu_i}{k}-1} - \chi(\varsigma_1,s)^{\frac{\vartheta_i+\mu_i}{k}-1}\right|,\ i=1,2.$$

If $e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\vartheta_i+\mu_i}{k}-1} \ge \chi(\varsigma_1,s)^{\frac{\vartheta_i+\mu_i}{k}-1}$, then

$$\begin{aligned}&\frac{I_{4,i}}{k\Gamma_k(\vartheta_i+\mu_i)} \\ &\le \frac{(2N+1)\varphi_i^*}{\Gamma_k(\vartheta_i+\mu_i+k)}\left[e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\left(\chi(\varsigma_2,a)^{\frac{\vartheta_i+\mu_i}{k}} - \chi(\varsigma_2,\varsigma_1)^{\frac{\vartheta_i+\mu_i}{k}}\right) - \chi(\varsigma_1,a)^{\frac{\vartheta_i+\mu_i}{k}}\right],\ i=1,2.\end{aligned}$$

If $e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\chi(\varsigma_2,s)^{\frac{\vartheta_i+\mu_i}{k}-1} \le \chi(\varsigma_1,s)^{\frac{\vartheta_i+\mu_i}{k}-1}$, then

$$\begin{aligned}&\frac{I_{4,i}}{k\Gamma_k(\vartheta_i+\mu_i)} \\ &\le \frac{(2N+1)\varphi_i^*}{\Gamma_k(\vartheta_i+\mu_i+k)}\left[\chi(\varsigma_1,a)^{\frac{\vartheta_i+\mu_i}{k}} + e^{-\varrho\chi(\varsigma_2,\varsigma_1)}\left(\chi(\varsigma_2,\varsigma_1)^{\frac{\vartheta_i+\mu_i}{k}} - \chi(\varsigma_2,a)^{\frac{\vartheta_i+\mu_i}{k}}\right)\right],\ i=1,2.\end{aligned}$$

Thus,

$$\frac{I_{4,i}}{k\Gamma_k(\vartheta_i+\mu_i)} \to 0 \quad \text{as} \quad \varsigma_2 \to \varsigma_1,\ i=1,2. \tag{4.12}$$

Equations (4.9)-(4.12) along with (4.8) imply

$$\|\mathcal{Q}_i(\mathfrak{s}_1(\varsigma_2,\omega),\mathfrak{s}_2(\varsigma_2,\omega),\omega) - \mathcal{Q}_i(\mathfrak{s}_1(\varsigma_1,\omega),\mathfrak{s}_2(\varsigma_1,\omega),\omega)\| \xrightarrow[\varsigma_2\to\varsigma_1]{} 0, \quad i=1,2.$$

This proves that, $\mathcal{Q}(\mathbb{S}_N)$ is equicontinuous.

Step 4. $\mathcal{Q}$ is $\Xi_{\mathbb{J}}$-condensing.

First, for every $S^1 \times S^2 \subset \mathcal{P}(\mathbb{J})$, we define the MNC as

$$\Xi_{\mathbb{J}}(S^1 \times S^2) = \begin{pmatrix}\Xi(S^1) \\ \Xi(S^2)\end{pmatrix}, \tag{4.13}$$

where

$$\Xi(S^i) = \sup_{\varsigma\in\mathfrak{I}} e^{-\gamma\varsigma}\Lambda(S^i(\varsigma));\quad \gamma > 0,\ i=1,2, \tag{4.14}$$

The MNC $\Xi_{\mathbb{J}}$ is well defined and gives a semiadditive, monotone, nonsingular and regular MNC in $\mathbb{J}$.

Secondly, let $S^1 \times S^2 \subset \mathcal{P}(\mathbb{J})$ be such that

$$\Xi_{\mathbb{J}}\left(\mathcal{Q}_i(S^1 \times S^2)\right) \geq \Xi_{\mathbb{J}}(S^1 \times S^2), \quad i = 1, 2. \tag{4.15}$$

We will show that (4.15) implies the relative compactness of $S^1 \times S^2$.

There exists a countable set $\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^{\infty}$ such that

$$\mathfrak{z}_{i,n}(\varsigma, \omega) \quad = \mathcal{Q}_i\left(\{\mathfrak{s}_{1,n}(\varsigma, \omega), \mathfrak{s}_{2,n}(\varsigma, \omega), \omega\}\right), \, i = 1, 2,$$

where $\{(\mathfrak{s}_{1,n}, \mathfrak{s}_{2,n})\}_{n=1}^{\infty} \subset \mathbb{J}$. From the properties of the MNC, one gets

$$\begin{aligned} \Xi(\{\mathfrak{z}_{i,n}\}_{n=1}^{\infty}) \quad &\leq \Xi\left(\left\{{}_{k}^{T}\mathcal{I}_{a^+}^{\vartheta_i+\mu_i,\varrho;\phi}\mathfrak{f}_i(\varsigma, \mathfrak{s}_{1,n}(\varsigma, \omega), \mathfrak{s}_{2,n}(\varsigma, \omega), \omega)\right\}_{n=1}^{\infty}\right) \\ &+ |\varpi_i| \Xi\left(\left\{{}_{k}^{T}\mathcal{I}_{a^+}^{\mu_i,\varrho;\phi}\mathfrak{s}_{i,n}(\varsigma, \omega)\right\}_{n=1}^{\infty}\right), \, i = 1, 2. \end{aligned} \tag{4.16}$$

Now, we will find an estimate for $\Xi(\{\mathfrak{z}_{i,n}\}_{n=1}^{\infty})$, $i = 1, 2$. By using (A4), for all $\varsigma \in \mathcal{I}$ and $s \leqslant \varsigma$, one has

$$\begin{aligned} &\Lambda\left(\left\{\frac{\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)}{k\Gamma_k(\vartheta_i+\mu_i)}\mathfrak{f}_i(s, \mathfrak{s}_{1,n}(s, \omega), \mathfrak{s}_{2,n}(s, \omega), \omega)\right\}_{n=1}^{\infty}\right) \\ &\leq \sum_{r=1}^{2} \frac{\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)}{k\Gamma_k(\vartheta_i+\mu_i)}\mathbb{G}_{i,r}(\omega)\Lambda\left(\{\mathfrak{s}_{r,n}(s, \omega)\}_{n=1}^{\infty}\right) \\ &\leq \sum_{r=1}^{2} \frac{\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)}{k\Gamma_k(\vartheta_i+\mu_i)}\mathbb{G}_{i,r}(\omega)e^{\gamma s}\sup_{a\leq s\leq \varsigma} e^{-\gamma s}\Lambda\left(\{\mathfrak{s}_{r,n}(s, \omega)\}_{n=1}^{\infty}\right) \\ &\leq \sum_{r=1}^{2} \frac{\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)}{k\Gamma_k(\vartheta_i+\mu_i)}\mathbb{G}_{i,r}(\omega)e^{\gamma s}\Xi\left(\{\mathfrak{s}_{r,n}(\cdot, \omega)\}_{n=1}^{\infty}\right), \, i = 1, 2. \end{aligned}$$

Then, applying Lemma 2.11, we obtain

$$\begin{aligned} &\Lambda\left(\left\{{}_{k}^{T}\mathcal{I}_{a}^{\vartheta_i+\mu_i,\varrho;\phi}\mathfrak{f}_i(s, \mathfrak{s}_{1,n}(s, \omega), \mathfrak{s}_{2,n}(s, \omega), \omega)\right\}_{n=1}^{\infty}\right) \\ &\leq \sum_{r=1}^{2} \Xi\left(\{\mathfrak{s}_{r,n}(\cdot, \omega)\}_{n=1}^{\infty}\right) \frac{2\mathbb{G}_{i,r}(\omega)}{k\Gamma_k(\vartheta_i+\mu_i)} \int_a^{\varsigma} \phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma, s)e^{\gamma s}ds, \, i = 1, 2. \end{aligned}$$

Multiplying both sides by $e^{-\gamma\varsigma}$ and taking $\sup_{\varsigma\in\mathfrak{I}}$, one obtains

$$\begin{aligned} &\sup_{\varsigma\in\mathfrak{I}} e^{-\gamma\varsigma}\Lambda\left(\left\{{}_{k}^{T}\mathcal{I}_{a}^{\vartheta_i+\mu_i,\varrho;\phi}\mathfrak{f}_i(\varsigma, \mathfrak{s}_{1,n}(\varsigma, \omega), \mathfrak{s}_{2,n}(\varsigma, \omega), \omega)\right\}_{n=1}^{\infty}\right) \\ &\leq \sum_{r=1}^{2} \Xi\left(\{\mathfrak{s}_{r,n}(\cdot, \omega)\}_{n=1}^{\infty}\right)\mathfrak{h}_{i,r}(\gamma, \omega), \quad i = 1, 2. \end{aligned}$$

where $\mathfrak{h}_{i,r}(\gamma, \omega)$, $i, r = 1, 2$ are defined in (2.6).

Hence,

$$\begin{aligned} &\Xi\left(\left\{{}_{k}^{T}\mathcal{I}_{a}^{\vartheta_i+\mu_i,\varrho;\phi}\mathfrak{f}_i(\zeta, \mathfrak{s}_{1,n}(\varsigma, \omega), \mathfrak{s}_{2,n}(\varsigma, \omega), \omega)\right\}_{n=1}^{\infty}\right) \\ &\leq \sum_{r=1}^{2} \Xi\left(\{\mathfrak{s}_{j,n}(\cdot, \omega)\}_{n=1}^{\infty}\right)\mathfrak{h}_{i,r}(\gamma, \omega), \quad i = 1, 2. \end{aligned} \tag{4.17}$$

By employing a similar reasoning, we get

$$\Xi\left(\left\{{}_{k}^{T}\mathcal{I}_{a}^{\mu_i,\varrho;\phi}\mathfrak{s}_{i,n}(\varsigma, \omega)\right\}_{n=1}^{\infty}\right) \quad \leq \Xi\left(\{\mathfrak{s}_{i,n}(\cdot, \omega)\}_{n=1}^{\infty}\right)\mathfrak{g}_i(\gamma, \omega), \quad i = 1, 2. \tag{4.18}$$

where $\mathfrak{g}_i(\gamma, \omega)$, $i = 1, 2$ are defined in (2.7).

Next, by (4.16)-(4.18), we derive

$$\Xi(\{\mathfrak{z}_{i,n}\}_{n=1}^{\infty}) \quad \leq |\varpi_i|\Xi\left(\{\mathfrak{s}_{i,n}(\cdot, \omega)\}_{n=1}^{\infty}\right)\mathfrak{g}_i(\gamma, \omega) + \sum_{r=1}^{2} \Xi\left(\{\mathfrak{s}_{r,n}(\cdot, \omega)\}_{n=1}^{\infty}\right)\mathfrak{h}_{i,r}(\gamma, \omega), \, i = 1, 2.$$

which implies

$$\Xi_{\mathbb{J}}(\mathcal{Q}(\{\mathfrak{s}_{1,n}(\cdot,\omega),\mathfrak{s}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty}) = \begin{pmatrix}\Xi(\mathcal{Q}_1(\{\mathfrak{s}_{1,n}(\cdot,\omega),\mathfrak{s}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty})\\ \Xi(\mathcal{Q}_2(\{\mathfrak{s}_{1,n}(\cdot,\omega),\mathfrak{s}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty})\end{pmatrix} \le \widehat{\mathcal{W}}_{\gamma}(\omega)\begin{pmatrix}\Xi\left(\{\mathfrak{s}_{1,n}(\cdot,\omega)\}_{n=1}^{\infty}\right)\\ \Xi\left(\{\mathfrak{s}_{2,n}(\cdot,\omega)\}_{n=1}^{\infty}\right)\end{pmatrix},$$

where

$$\widehat{\mathcal{W}}_{\gamma}(\omega) = \begin{pmatrix}|\varpi_1|\mathfrak{g}_1(\gamma,\omega)+\mathfrak{h}_{1,1}(\gamma,\omega) & \mathfrak{h}_{1,2}(\gamma,\omega)\\ \\ \mathfrak{h}_{2,1}(\gamma,\omega) & |\varpi_2|\mathfrak{g}_2(\gamma,\omega)+\mathfrak{h}_{2,2}(\gamma,\omega)\end{pmatrix}.$$

By Lemma 2.27, one can choose γ such that the spectral radius $\rho(\widehat{\mathcal{W}}_{\gamma}(\omega))<1$, therefore

$$\Xi(\mathcal{Q}_i(\{\mathfrak{s}_{1,n}(\cdot,\omega),\mathfrak{s}_{2,n}(\cdot,\omega),\omega\}_{n=1}^{+\infty}) = 0, \quad i=1,2.$$

This implies that

$$\Xi(\mathcal{Q}_i(\{\mathfrak{s}_{1,n}(\varsigma,\omega),\mathfrak{s}_{2,n}(\varsigma,\omega),\omega\}_{n=1}^{+\infty}) = 0,\ \text{for } \varsigma\in\mathfrak{I},\ i=1,2.$$

Finally,

$$\Xi_{\mathbb{J}}(S^1\times S^2) = (0,0),$$

which proves the compactness of the set $\overline{S^1\times S^2}$.

Step 5. The set $\mathcal{W}$ (see Theorem 2.26 (2)) is bounded.

Let $(\mathfrak{s}_1,\mathfrak{s}_2)\in\mathbb{J}$ and $(\mathfrak{s}_1,\mathfrak{s}_2)=\kappa(\omega)\mathcal{Q}(\mathfrak{s}_1,\mathfrak{s}_2)$ for some $\kappa(\omega)\in(0,1)$. Then, for each $\varsigma\in\mathfrak{I}$, we obtain

$$\begin{aligned}\mathfrak{s}_i(\varsigma,\omega) &= \kappa(\omega)\Bigg[\sum_{l=0}^{m-1}\frac{k^l z_{i,l}(\omega)}{\Gamma_k(kl+k)}\mathfrak{T}^{l}_{\varrho,\chi}(\varsigma,a)+\sum_{j=0}^{n-1}\frac{w_{i,j}(\omega)-\varpi_i k^j z_{i,j}(\omega)}{\Gamma_k(kj+k+\mu_i)}\mathfrak{T}^{\frac{\mu_i}{k}+j}_{\varrho,\chi}(\varsigma,a)\\ &\quad+\frac{1}{k\Gamma_k(\vartheta_i+\mu_i)}\int_a^{\varsigma}\phi'(s)\mathfrak{T}^{\frac{\vartheta_i+\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\mathfrak{f}_i(s,\mathfrak{s}_1(s,\omega),\mathfrak{s}_2(s,\omega),\omega)ds\\ &\quad+\frac{\varpi_i}{k\Gamma_k(\mu_i)}\int_a^{\varsigma}\phi'(s)\mathfrak{T}^{\frac{\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\mathfrak{s}_i(s,\omega)ds\Bigg], \quad i=1,2.\end{aligned}$$

Using (A3) and the fact $e^{-\varrho\chi(\varsigma,a)}\le 1$ for all $\varsigma\in\mathfrak{I}$, and Lemma 2.16, we obtain

$$\begin{aligned}\|\mathfrak{s}_i(\varsigma,\omega)\| &\le \sum_{l=0}^{m-1}\frac{k^l|z_{i,l}(\omega)|}{\Gamma_k(kl+k)}\chi(\varsigma,a)^l+\sum_{j=0}^{n-1}\frac{|w_{i,j}(\omega)-\varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)}\chi(\varsigma,a)^{\frac{\mu_i}{k}+j}\\ &\quad+\varphi_i^*\chi(\varsigma,a)^{\frac{\vartheta_i+\mu_i}{k}}\mathbb{E}^k_{k,k+\vartheta_i+\mu_i}(k\varrho\chi(\varsigma,a))\\ &\quad+\frac{\varphi_i^*}{k\Gamma_k(\vartheta_i+\mu_i)}\int_a^{\varsigma}\phi'(s)\mathfrak{T}^{\frac{\vartheta_i+\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\left(\|\mathfrak{s}_1(s,\omega)\|+\|\mathfrak{s}_2(s,\omega)\|\right)ds\\ &\quad+\frac{|\varpi_i|}{k\Gamma_k(\mu_i)}\int_a^{\varsigma}\phi'(s)\mathfrak{T}^{\frac{\mu_i}{k}-1}_{\varrho,\chi}(\varsigma,s)\|\mathfrak{s}_i(s,\omega)\|ds, \quad i=1,2.\end{aligned}$$

We observe that, for any $a\le s<\varsigma$,

$$\chi(\varsigma,s)^{\frac{\mu_i}{k}-1}\le\chi(\varsigma,a)^{\frac{\mu_i}{k}-\frac{\mu_{\min}}{k}}\chi(\varsigma,s)^{\frac{\mu_{\min}}{k}-1}, \quad i=1,2,$$

where $\mu_{\min}=\min\{\mu_1,\mu_2\}$. Therefore

$$\begin{aligned}&\|\mathfrak{s}_1(\varsigma,\omega)\|+\|\mathfrak{s}_2(\varsigma,\omega)\|\\ &\quad\le \widehat{\mathbf{A}}(\varsigma)+\sum_{i=1}^{2}\mathbf{B}_i\int_a^{\varsigma}\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\vartheta_i+\mu_i}{k}-1}(\varsigma,s)\left(\|\mathfrak{s}_1(s,\omega)\|+\|\mathfrak{s}_2(s,\omega)\|\right)ds\\ &\quad\quad+\mathbf{C}_{\max}\int_a^{\varsigma}\phi'(s)\mathfrak{T}_{\varrho,\chi}^{\frac{\mu_{\min}}{k}-1}(\varsigma,s)\left(\|\mathfrak{s}_1(s,\omega)\|+\|\mathfrak{s}_2(s,\omega)\|\right)ds,\end{aligned}$$

where

$$\widehat{\mathbf{A}}(\varsigma)=\mathbf{A}_1(\varsigma)+\mathbf{A}_2(\varsigma)$$

$$\begin{aligned}\mathbf{A}_i(\varsigma)\quad &=\sum_{l=0}^{m-1}\frac{k^l|z_{i,l}(\omega)|}{\Gamma_k(kl+k)}\chi(\varsigma,a)^l+\sum_{j=0}^{n-1}\frac{|w_{i,j}(\omega)-\varpi_i k^j z_{i,j}(\omega)|}{\Gamma_k(kj+k+\mu_i)}\chi(\varsigma,a)^{\frac{\mu_i}{k}+j}\\ &\quad+\varphi_i^*\chi(\varsigma,a)^{\frac{\vartheta_i+\mu_i}{k}}\mathbb{E}^k_{k,k+\vartheta_i+\mu_i}(k\varrho\chi(\varsigma,a)),\quad i=1,2,\end{aligned}$$

$$\mathbf{B}_i=\frac{\varphi_i^*}{k\Gamma_k(\vartheta_i+\mu_i)},\ \ \mathbf{C}_{\max}=\max\{\mathbf{C}_1,\mathbf{C}_2\},\ \ \mathbf{C}_i=\frac{|\varpi_i|\chi(\varsigma,a)^{\frac{\mu_i}{k}-\frac{\mu_{\min}}{k}}}{k\Gamma_k(\mu_i)}$$

Applying Corollary 3.3, we obtain

$$\begin{aligned}\|\mathfrak{s}_1(\varsigma,\omega)\|+\|\mathfrak{s}_2(\varsigma,\omega)\|\quad&\le\sum_{i=1}^{2}\widehat{\mathbf{A}}(b)\mathbb{E}^k_{\vartheta_i+\mu_i}\left(\mathbf{B}_i\Gamma_k(\vartheta_i+\mu_i)\chi(b,a)^{\frac{\vartheta_i+\mu_i}{k}}\right)\\ &\quad+\widehat{\mathbf{A}}(b)\mathbb{E}^k_{\mu_{\min}}\left(\mathbf{C}_{\max}\Gamma_k(\mu_{\min})\chi(b,a)^{\frac{\mu_{\min}}{k}}\right)\\ &:=\mathbf{D}\end{aligned}$$

Hence

$$\|(\mathfrak{s}_1(\cdot,\omega),\mathfrak{s}_2(\cdot,\omega))\|_{\mathbb{J}}\le\widehat{\mathbf{D}}:=\begin{pmatrix}\mathbf{D}\\ \mathbf{D}\end{pmatrix}$$

Which achieves the desired estimate. Therefore, Theorem 2.26 ensures the existence of a random solution for the system (1.1).

□

5 Ulam–Hyers stability analysis

In this section, we study Ulam–Hyers (U-H), generalized Ulam–Hyers (G-U-H), Ulam–Hyers–Rassias (U-H-R), and generalized Ulam–Hyers–Rassias (G-U-H-R) stability of system (1.1).

Let $\epsilon=(\epsilon_1,\epsilon_2)>0$ and $\mathcal{K}_1,\mathcal{K}_1:\mathfrak{I}\to[0,\infty)$ are a continuous and increasing functions. We consider for $i=1,2$ the following inequalities:

$$\left\|{}^{Tc}_{k}\mathcal{D}^{\vartheta_i,\varrho;\phi}_{a^+}\left({}^{Tc}_{k}\mathcal{D}^{\mu_i,\varrho;\phi}_{a^+}-\varpi_i\right)\mathfrak{u}_i(\varsigma,\omega)-\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega)\right\|\le\epsilon_i,\ \ (\varsigma,\omega)\in\mathfrak{I}\times\Omega\tag{5.1}$$

$$\left\|{}^{Tc}_{k}\mathcal{D}^{\vartheta_i,\varrho;\phi}_{a^+}\left({}^{Tc}_{k}\mathcal{D}^{\mu_i,\varrho;\phi}_{a^+}-\varpi_i\right)\mathfrak{u}_i(\varsigma,\omega)-\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega)\right\|\le\mathcal{K}_i(\varsigma),\ \ (\varsigma,\omega)\in\mathfrak{I}\times\Omega\tag{5.2}$$

$$\left\|{}^{Tc}_{k}\mathcal{D}^{\vartheta_i,\varrho;\phi}_{a^+}\left({}^{Tc}_{k}\mathcal{D}^{\mu_i,\varrho;\phi}_{a^+}-\varpi_i\right)\mathfrak{u}_i(\varsigma,\omega)-\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega)\right\|\le\epsilon_i\mathcal{K}_i(\varsigma)\ \ (\varsigma,\omega)\in\mathfrak{I}\times\Omega\tag{5.3}$$

Definition 5.1. [39] The system (1.1) is U-H stable if there exists $\mathfrak{n}_r>0$, $r=\overline{1,4}$, with the following property: For each $\epsilon_i>0$, $i=1,2$ and for each random solution $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{J}$ of the inequality (5.1), there exists a solution $(\mathfrak{s}_1,\mathfrak{s}_2)\in\mathbb{J}$ of (1.1) where

$$\begin{cases}\left(D^{[\ell],\varrho;\phi}\mathfrak{s}_1(a,\omega),D^{[\ell],\varrho;\phi}\mathfrak{s}_2(a,\omega)\right)=\left(D^{[\ell],\varrho;\phi}\mathfrak{u}_1(a,\omega),D^{[\ell],\varrho;\phi}\mathfrak{u}_2(a,\omega)\right)\\ \left({}^{Tc}_{k}\mathcal{D}^{\mu_1+k\ell,\varrho;\phi}_{a^+}\mathfrak{s}_1(a,\omega),{}^{Tc}_{k}\mathcal{D}^{\mu_2+k\ell,\varrho;\phi}_{a^+}\mathfrak{s}_2(a,\omega)\right)=\left({}^{Tc}_{k}\mathcal{D}^{\mu_1+k\ell,\varrho;\phi}_{a^+}\mathfrak{u}_1(a,\omega),{}^{Tc}_{k}\mathcal{D}^{\mu_2+k\ell,\varrho;\phi}_{a^+}\mathfrak{u}_2(a,\omega)\right)\end{cases}\tag{5.4}$$

complying with

$$\begin{cases} \|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty \le \mathfrak{n}_1\epsilon_1+\mathfrak{n}_2\epsilon_2 \\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty \le \mathfrak{n}_3\epsilon_1+\mathfrak{n}_4\epsilon_2 \end{cases}, \quad \omega\in\Omega.$$

Definition 5.2. The system (1.1) is G-U-H stable if there exists $\aleph_r \in C(\mathbb{R}_+,\mathbb{R}_+)$, $r=\overline{1,4}$ with $\aleph_i(0)=0$ such that for each $\epsilon_i>0$, $i=1,2$ and for each solution $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{J}$ of the inequality (5.1), there exists a random solution $(\mathfrak{s}_1,\mathfrak{s}_2)\in\mathbb{J}$ of (1.1) where (5.4) with

$$\begin{cases} \|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty \le \aleph_1(\epsilon_1)+\aleph_2(\epsilon_2) \\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty \le \aleph_3(\epsilon_1)+\aleph_4(\epsilon_2) \end{cases}, \quad \omega\in\Omega.$$

Definition 5.3. The system (1.1) is U-H-R stable with respect to $\mathcal{K}=(\mathcal{K}_1,\mathcal{K}_2)$ if there exists a real number $\tilde{\mathfrak{n}}_r>0$, $r=\overline{1,4}$, such that, for each $\epsilon_i>0$, $i=1,2$ and for each solution $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{J}$ of inequalities (5.3), there exists a solution $(\mathfrak{s}_1,\mathfrak{s}_2)\in\mathbb{J}$ of (1.1) where (5.4) with

$$\begin{cases} \|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty \le \epsilon_1\tilde{\mathfrak{n}}_1\mathcal{K}_1(\varsigma)+\epsilon_2\tilde{\mathfrak{n}}_2\mathcal{K}_2(\varsigma) \\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty \le \epsilon_1\tilde{\mathfrak{n}}_3\mathcal{K}_1(\varsigma)+\epsilon_2\tilde{\mathfrak{n}}_4\mathcal{K}_2(\varsigma) \end{cases}, \quad \omega\in\Omega.$$

Definition 5.4. The system (1.1) is G-U-H-R stable with respect to $\mathcal{K}=(\mathcal{K}_1,\mathcal{K}_2)$ if there exists a real number $\tilde{\mathfrak{n}}_r>0$, $r=\overline{1,4}$, such that, for each solution $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{J}$ of inequalities (5.2), there exists a solution $(\mathfrak{s}_1,\mathfrak{s}_2)\in\mathbb{J}$ of (1.1) where (5.4) with

$$\begin{cases} \|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty \le \tilde{\mathfrak{n}}_1\mathcal{K}_1(\varsigma)+\tilde{\mathfrak{n}}_2\mathcal{K}_2(\varsigma) \\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty \le \tilde{\mathfrak{n}}_3\mathcal{K}_1(\varsigma)+\tilde{\mathfrak{n}}_4\mathcal{K}_2(\varsigma) \end{cases}, \quad \omega\in\Omega.$$

Remark 5.5. It is clear that

(i) Definition 5.1 $\Rightarrow$ Definition 5.2,

(ii) Definition 5.3 $\Rightarrow$ Definition 5.4,

(iii) Definition 5.3 for $(\mathcal{K}_1(\cdot),\mathcal{K}_2(\cdot))=(1,1)$ $\Rightarrow$ Definition 5.1.

Remark 5.6. We say that $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{J}$ is a random solution of the inequality (5.1), if there exist functons $\mathcal{L}_i\in C(\mathfrak{I},\mathbb{R})$, $i=1,2$, which depend on $\mathfrak{u}_i$, such that

(i) $\mathcal{L}_i(\varsigma)\le\epsilon_i$, $\varsigma\in\mathfrak{I}$, $i=1,2$ and

(ii) For all $(\varsigma,\omega)\in\mathfrak{I}\times\Omega$,

$$ {}^{Tc}_{k}\mathcal{D}^{\vartheta_i,\varrho;\phi}_{a^+}\left({}^{Tc}_{k}\mathcal{D}^{\mu_i,\varrho;\phi}_{a^+}-\varpi_i\right)\mathfrak{s}_i(\varsigma,\omega)=\mathfrak{f}_i(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)+\mathcal{L}_i(\varsigma),\ i=1,2. \quad (5.5)$$

Remark 5.7. We say that $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{J}$ is a random solution of the inequality (5.3), if there exist functons $\mathcal{L}_i\in C(\mathfrak{I},\mathbb{R})$, $i=1,2$, which depend on $\mathfrak{u}_i$, such that

(i) $\mathcal{L}_i(\varsigma)\le\epsilon_i\mathcal{K}_i(\varsigma)$, $\varsigma\in\mathfrak{I}$, $i=1,2$ and

(ii) For all $\varsigma\in\mathfrak{I}$,

$$ {}^{Tc}_{k}\mathcal{D}^{\vartheta_i,\varrho;\phi}_{a^+}\left({}^{Tc}_{k}\mathcal{D}^{\mu_i,\varrho;\phi}_{a^+}-\varpi_i\right)\mathfrak{s}_i(\varsigma,\omega)=\mathfrak{f}_i(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)+\mathcal{L}_i(\varsigma),\ i=1,2. \quad (5.6)$$

Theorem 5.8. *Assume that*

(i) The assumptions of Theorem 4.2 are fulfilled.

(ii) For any $\epsilon=(\epsilon_1,\epsilon_2)>0$, the inequality (5.1) have at least one solution,

(iii) The $\Psi=\mathcal{T}_{1,1}(b,\omega)\mathcal{T}_{2,2}(b,\omega)-\mathfrak{d}_{1,2}(b,\omega)\mathfrak{d}_{2,1}(b,\omega)\neq0$ and $\mathcal{T}_{i,i}(b,\omega)>0$, $i=1,2$ hold,

Then, the solution of the system (1.1) is U-H stable and G-U-H stable, where,

$$\mathcal{T}_{i,i}(b,\omega) = 1 - \mathfrak{d}_{i,i}(b,\omega) - |\varpi_i|\mathfrak{N}(b,\mu_i), \quad \mathfrak{N}(b,\mu_i) = \chi(b,a)^{\frac{\mu_i}{k}}\mathbb{E}^k_{k,k+\mu_i}(k\varrho\chi(b,a)),$$

$$\mathfrak{N}(b,\vartheta_i+\mu_i) := \chi(b,a)^{\frac{\vartheta_i+\mu_i}{k}}\mathbb{E}^k_{k,k+\vartheta_i+\mu_i}(k\varrho\chi(b,a)), \quad i = 1,2.$$

and

$$\mathfrak{d}_{i,r}(b,\omega) := \mathfrak{X}_{i,r}(\omega)\mathfrak{N}(b,\vartheta_i+\mu_i), \quad i,r = 1,2$$

Proof. Let $(\mathfrak{u}_1,\mathfrak{u}_2) \in \mathbb{J}$ satisfies (5.1) and $(\mathfrak{s}_1,\mathfrak{s}_2) \in \mathbb{J}$ be the unique random solution of the system (1.1) with (5.4), then in view of Lemma 4.1, we have

$$\begin{aligned}\mathfrak{s}_i(\varsigma,\omega) &= \sum_{l=0}^{m-1}\frac{k^l z_{i,l}(\omega)}{\Gamma_k(kl+k)}\mathfrak{T}^l_{\varrho,\chi}(\varsigma,a) + \sum_{j=0}^{n-1}\frac{w_{i,j}(\omega)-\varpi_1 k^j z_{i,j}(\omega)}{\Gamma_k(kj+k+\mu_i)}\mathfrak{T}^{\frac{\mu_i}{k}+j}_{\varrho,\chi}(\varsigma,a)\\ &\quad + {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\mathfrak{f}_i(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega) + \varpi_i\,{}^T_k\mathcal{I}^{\mu_i,\varrho;\varphi}_{a^+}\mathfrak{s}_i(\varsigma,\omega),\ i=1,2,\end{aligned} \tag{5.7}$$

Since we have assumed that $(\mathfrak{u}_1,\mathfrak{u}_2) \in \mathbb{J}$ satisfies is a solution of (5.1) with (5.4), hence we have by Remark 5.6 and Lemma 4.1,

$$\begin{aligned}\mathfrak{u}_i(\varsigma,\omega) &= \sum_{l=0}^{m-1}\frac{k^l z_{i,l}(\omega)}{\Gamma_k(kl+k)}\mathfrak{T}^l_{\varrho,\chi}(\varsigma,a) + \sum_{j=0}^{n-1}\frac{w_{i,j}(\omega)-\varpi_1 k^j z_{i,j}(\omega)}{\Gamma_k(kj+k+\mu_i)}\mathfrak{T}^{\frac{\mu_i}{k}+j}_{\varrho,\chi}(\varsigma,a)\\ &\quad + {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\left(\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega) + \mathcal{L}_i(\varsigma)\right) + \varpi_i\,{}^T_k\mathcal{I}^{\mu_i,\varrho;\varphi}_{a^+}\mathfrak{u}_i(\varsigma,\omega),\ i=1,2,\end{aligned} \tag{5.8}$$

On the other hand, by (5.8) and (5.7), for each $\varsigma \in \mathfrak{I}$, it follows

$$\begin{aligned}&\|\mathfrak{u}_i(\varsigma,\omega) - \mathfrak{s}_i(\varsigma,\omega)\|\\ &\le {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathcal{L}_i(\varsigma)\| + {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega) - \mathfrak{f}_i(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\|\\ &\quad + \varpi_i\,{}^T_k\mathcal{I}^{\mu_i,\varrho;\varphi}_{a^+}\|\mathfrak{u}_i(\varsigma,\omega) - \mathfrak{s}_i(\varsigma,\omega)\|, \quad i=1,2.\end{aligned}$$

By Remark 5.6, Lemma 2.16 and the fact $e^{-\varrho\chi(\varsigma,a)} \le 1$, for $\varsigma \in \mathfrak{I}$, we get

$$\begin{aligned}{}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathcal{L}_i(\varsigma)\| &\le \epsilon_i\left({}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}1\right)(\varsigma)\\ &\le \epsilon_i\mathfrak{T}^{\frac{\vartheta_i+\mu_i}{k}}_{\varrho,\chi}(\varsigma,a)\mathbb{E}^k_{k,k+\vartheta_i+\mu_i}(k\varrho\chi(\varsigma,a))\\ &\le \mathfrak{N}(b,\vartheta_i+\mu_i)\epsilon_i, \quad i=1,2.\end{aligned}$$

So

$$\begin{aligned}&\|\mathfrak{u}_i(\varsigma,\omega) - \mathfrak{s}_i(\varsigma,\omega)\|\\ &\le \mathfrak{N}(\varsigma,\vartheta_i+\mu_i)\epsilon_i + {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega) - \mathfrak{f}_i(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\|\\ &\quad + \varpi_i\,{}^T_k\mathcal{I}^{\mu_i,\varrho;\varphi}_{a^+}\|\mathfrak{u}_i(\varsigma,\omega) - \mathfrak{s}_i(\varsigma,\omega)\|, \quad i=1,2.\end{aligned}$$

Therefore, by (A2) and using Lemma 2.16, we get

$$\begin{aligned}\|\mathfrak{u}_i(\varsigma,\omega) - \mathfrak{s}_i(\varsigma,\omega)\| &\le \left(\sum_{i=1}^2\mathfrak{X}_{i,r}(\omega)\|\mathfrak{u}_r(\cdot,\omega) - \mathfrak{s}_r(\cdot,\omega)\|_\infty\right){}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}(1)(\varsigma)\\ &\quad + |\varpi_i|\|\mathfrak{u}_i(\cdot,\omega) - \mathfrak{s}_i(\cdot,\omega)\|_\infty\,{}^T_k\mathcal{I}^{\mu_i,\varrho;\phi}_{a^+}(1)(\varsigma) + \mathfrak{N}(b,\vartheta_i+\mu_i)\epsilon_i\\ &\le \mathfrak{N}(b,\vartheta_i+\mu_i)\left(\sum_{r=1}^2\mathfrak{X}_{i,r}(\omega)\|\mathfrak{u}_r(\cdot,\omega) - \mathfrak{s}_r(\cdot,\omega)\|_\infty\right)\\ &\quad + |\varpi_i|\mathfrak{N}(b,\mu_i)\|\mathfrak{u}_i(\cdot,\omega) - \mathfrak{s}_i(\cdot,\omega)\|_\infty + \mathfrak{N}(b,\vartheta_i+\mu_i)\epsilon_i, \quad i=1,2,\end{aligned}$$

Thus

$$\|\mathfrak{u}_i(\cdot,\omega)-\mathfrak{s}_i(\cdot,\omega)\|_\infty \quad \leq \sum_{r=1}^{2}\mathfrak{d}_{i,r}(b,\omega)\|\mathfrak{u}_r(\cdot,\omega)-\mathfrak{s}_r(\cdot,\omega)\|_\infty$$

$$+|\varpi_i|\mathfrak{N}(b,\mu_i)\|\mathfrak{u}_i(\cdot,\omega)-\mathfrak{s}_i(\cdot,\omega)\|_\infty+\mathfrak{N}(b,\vartheta_i+\mu_i)\epsilon_i, \quad i=1,2.$$

which implies

$$\begin{aligned}&\mathcal{T}_{i,i}(b,\omega)\|\mathfrak{u}_i(\cdot,\omega)-\mathfrak{s}_i(\cdot,\omega)\|_\infty-\mathfrak{d}_{i,3-i}(b,\omega)\|\mathfrak{u}_{3-i}(\cdot,\omega)-\mathfrak{s}_{3-i}(\cdot,\omega)\|_\infty\\&\leq \mathfrak{N}(b,\vartheta_i+\mu_i)\epsilon_i, \quad i=1,2,\end{aligned} \tag{5.9}$$

Now, we shall express the formula in (5.9) in matrix form as follows:

$$\begin{pmatrix}\mathcal{T}_{1,1}(b,\omega) & -\mathfrak{d}_{1,2}(b,\omega)\\ -\mathfrak{d}_{2,1}(b,\omega) & \mathcal{T}_{2,2}(b,\omega)\end{pmatrix}\begin{pmatrix}\|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty\\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty\end{pmatrix}\leq\begin{pmatrix}\mathfrak{N}(b,\vartheta_1+\mu_1)\epsilon_1\\ \mathfrak{N}(b,\vartheta_2+\mu_2)\epsilon_2\end{pmatrix}.$$

On solving above inequality, we obtain

$$\begin{pmatrix}\|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty\\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty\end{pmatrix}\leq\frac{1}{\Psi}\begin{pmatrix}\mathcal{T}_{2,2}(b,\omega) & \mathfrak{d}_{1,2}(b,\omega)\\ \mathfrak{d}_{2,1}(b,\omega) & \mathcal{T}_{1,1}(b,\omega)\end{pmatrix}\begin{pmatrix}\mathfrak{N}(b,\vartheta_1+\mu_1)\epsilon_1\\ \mathfrak{N}(b,\vartheta_2+\mu_2)\epsilon_2\end{pmatrix}.$$

Therefore,

$$\begin{cases}\|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty\leq\mathfrak{n}_1\epsilon_1+\mathfrak{n}_2\epsilon_2\\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty\leq\mathfrak{n}_3\epsilon_1+\mathfrak{n}_4\epsilon_2.\end{cases} \tag{5.10}$$

where

$$\mathfrak{n}_1=\frac{\mathcal{T}_{2,2}(b,\omega)\mathfrak{N}(b,\vartheta_1+\mu_1)}{\Psi}, \quad \mathfrak{n}_2=\frac{\mathfrak{d}_{2,1}(b,\omega)\mathfrak{N}(b,\vartheta_1+\mu_1)}{\Psi},$$

and

$$\mathfrak{n}_3=\frac{\mathfrak{d}_{1,2}(b,\omega)\mathfrak{N}(b,\vartheta_2+\mu_2)}{\Psi}, \quad \mathfrak{n}_4=\frac{\mathcal{T}_{1,1}(b,\omega)\mathfrak{N}(b,\vartheta_2+\mu_2)}{\Psi}.$$

This proves that system (1.1) is U-H stable. Furthermore, we could put into writing inequality (5.10) as

$$\begin{cases}\|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty\leq\aleph_1(\epsilon_1)+\aleph_2(\epsilon_2)\\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty\leq\aleph_3(\epsilon_1)+\aleph_4(\epsilon_2)\end{cases}, \quad \varsigma\in\mathfrak{I}$$

where $\aleph_1(\epsilon_1)=\mathfrak{n}_1\epsilon_1$, $\aleph_2(\epsilon_2)=\mathfrak{n}_2\epsilon_2$, $\aleph_3(\epsilon_1)=\mathfrak{n}_3\epsilon_1$, $\aleph_4(\epsilon_2)=\mathfrak{n}_4\epsilon_2$, with $\aleph_r(0)=0$, $r=\overline{1,4}$. This proves that system (1.1) is G-U-H stable. □

Theorem 5.9. *Assume that*

(i) The assumptions of Theorem 4.2 are fulfilled.

(ii) For any $\epsilon=(\epsilon_1,\epsilon_2)>0$, the inequality (5.3) have at least one solution,

(iii) There exists an increasing functions $\mathcal{K}_i\in C(\mathfrak{I},\mathbb{R}_+)$, $i=1,2$ and there exists $\mathfrak{q}=(\mathfrak{q}_1,\mathfrak{q}_2)>0$ such that for any $\varsigma\in\mathfrak{I}$

$${}_{k}^{T}\mathcal{I}_{a^+}^{\vartheta_i+\mu_i,\varrho;\phi}\mathcal{K}_i(\varsigma)\leq\mathfrak{q}_i\mathcal{K}_i(\varsigma), \quad i=1,2,$$

(iv) The $\Psi=\mathcal{T}_{1,1}(b,\omega)\mathcal{T}_{2,2}(b,\omega)-\mathfrak{d}_{1,2}(b,\omega)\mathfrak{d}_{2,1}(b,\omega)\neq0$ and $\mathcal{T}_{i,i}(b,\omega)>0$, $i=1,2$ hold,

Then, the solution of the system (1.1) is U-H-R stable and G-U-H-R stable.

Proof. Let $(\mathfrak{u}_1,\mathfrak{u}_2) \in \mathbb{J}$ satisfies (5.3) and $(\mathfrak{s}_1,\mathfrak{s}_2) \in \mathbb{J}$ be the unique random solution of the system (1.1) with (5.4).

Again by (5.8) and (5.7), for each $\varsigma \in \mathfrak{I}$, it follows

$$\begin{aligned}
&\|\mathfrak{u}_i(\varsigma,\omega)-\mathfrak{s}_i(\varsigma,\omega)\| \\
&\leq {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathcal{L}_i(\varsigma)\| + {}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathfrak{f}_i(\varsigma,\mathfrak{u}_1(\varsigma,\omega),\mathfrak{u}_2(\varsigma,\omega),\omega)-\mathfrak{f}_i(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\| \\
&\quad +\varpi_i{}^T_k\mathcal{I}^{\mu_i,\varrho;\varphi}_{a^+}\|\mathfrak{u}_i(\varsigma,\omega)-\mathfrak{s}_i(\varsigma,\omega)\|, \quad i=1,2.
\end{aligned}$$

Hence using part (i) of Remark 5.7, we can get

$${}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\|\mathcal{L}_i(\varsigma)\| \quad \leq \epsilon_i\mathfrak{q}_i\mathcal{K}_i(\varsigma), \quad i=1,2.$$

Then, By (A2), Lemma 2.16 and the fact $e^{-\varrho\chi(\varsigma,a)} \leq 1$, for $\varsigma \in \mathfrak{I}$, we get

$$\begin{aligned}
\|\mathfrak{u}_i(\varsigma,\omega)-\mathfrak{s}_i(\varsigma,\omega)\| \quad &\leq \epsilon_i\mathfrak{q}_i\mathcal{K}_i(\varsigma) + \textstyle\sum_{r=1}^2 \mathfrak{d}_{i,r}(b,\omega)\|\mathfrak{u}_r(\cdot,\omega)-\mathfrak{s}_r(\cdot,\omega)\|_\infty \\
&\quad +|\varpi_i|\mathfrak{N}(b,\mu_i)\|\mathfrak{u}_i(\cdot,\omega)-\mathfrak{s}_i(\cdot,\omega)\|_\infty, \quad i=1,2.
\end{aligned}$$

In consequence, it follows that

$$\begin{aligned}
&\mathcal{T}_{i,i}(b,\omega)\|\mathfrak{u}_i(\cdot,\omega)-\mathfrak{s}_i(\cdot,\omega)\|_\infty - \mathfrak{d}_{i,3-i}(b,\omega)\|\mathfrak{u}_{3-i}(\cdot,\omega)-\mathfrak{s}_{3-i}(\cdot,\omega)\|_\infty \\
&\quad \leq \epsilon_i\mathfrak{q}_i\mathcal{K}_i(\varsigma), \quad i=1,2,
\end{aligned} \tag{5.11}$$

Hence

$$\begin{pmatrix} \|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty \\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty \end{pmatrix} \leq \frac{1}{\Psi}\begin{pmatrix} \mathcal{T}_{2,2}(b,\omega) & \mathfrak{d}_{1,2}(b,\omega) \\ \mathfrak{d}_{2,1}(b,\omega) & \mathcal{T}_{1,1}(b,\omega) \end{pmatrix}\begin{pmatrix} \mathfrak{q}_1\mathcal{K}_1(\varsigma)\epsilon_1 \\ \mathfrak{q}_2\mathcal{K}_2(\varsigma)\epsilon_2 \end{pmatrix}.$$

Therefore,

$$\begin{cases} \|\mathfrak{u}_1(\cdot,\omega)-\mathfrak{s}_1(\cdot,\omega)\|_\infty & \leq \tilde{\mathfrak{n}}_1\mathcal{K}_1(\varsigma)\epsilon_1 + \tilde{\mathfrak{n}}_2\mathcal{K}_2(\varsigma)\epsilon_2 \\ \|\mathfrak{u}_2(\cdot,\omega)-\mathfrak{s}_2(\cdot,\omega)\|_\infty & \leq \tilde{\mathfrak{n}}_3\mathcal{K}_1(\varsigma)\epsilon_1 + \tilde{\mathfrak{n}}_4\mathcal{K}_2(\varsigma)\epsilon_2 \end{cases}$$

where

$$\tilde{\mathfrak{n}}_1 = \mathfrak{q}_1\frac{\mathcal{T}_{2,2}(b,\omega)}{\Psi}, \quad \tilde{\mathfrak{n}}_3 = \mathfrak{q}_1\frac{\mathfrak{d}_{2,1}(b,\omega)}{\Psi}, \quad \tilde{\mathfrak{n}}_2 = \mathfrak{q}_2\frac{\mathfrak{d}_{1,2}(b,\omega)}{\Psi}, \quad \text{and} \quad \tilde{\mathfrak{n}}_4 = \mathfrak{q}_2\frac{\mathcal{T}_{1,1}(b,\omega)}{\Psi}.$$

Thus, the system (1.1) is U–H–R stable. Further, if we set $\epsilon_i = 1$, $i=1,2$, then the system (1.1) is G-U-H–R stable. □

6 Illustrative examples

Let $\Omega = (-\infty,0)$ be endowed with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. Consider the separable Banach space

$$\mathbb{O} = c_0 = \{\mathfrak{y} = (\mathfrak{y}^1,\mathfrak{y}^2,\cdots,\mathfrak{y}^n,\cdots) \; : \; \mathfrak{y}^n \to 0 \;\text{ as }\; n\to\infty\}$$

endowed with

$$\|\mathfrak{y}\|_{\mathbb{O}} = \sup_{n\geq 1}|\mathfrak{y}^n|.$$

(I) Illustration of Theorem 4.2.

Let us take $\varpi_i = \frac{e^{-\mathfrak{a}}}{\mathfrak{N}(b,\mu_i)}\widehat{\varpi}_i$, $i=1,2$ where $|\widehat{\varpi}_i| < \frac{e^{\mathfrak{a}}}{2}$ and $\mathfrak{a} \geq 0$.

For $(\varsigma,\omega)\in\mathfrak{I}\times\Omega$, consider the nonlinear functions $\mathfrak{f}_i$, $i=1,2$ be defined by

$$\begin{cases}\mathfrak{f}_1(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)=\left\{\frac{|\omega|(1+|\omega|)^{-1}\mathfrak{s}^{1,n}(\varsigma,\omega)}{10\mathfrak{N}(b,\vartheta_1+\mu_1)}+\frac{|\omega|(1+|\omega|)^{-1}\sin(\mathfrak{s}^{2,n}(\varsigma,\omega))}{10\mathfrak{N}(b,\vartheta_1+\mu_1)}\right\}_{n\geq 1},\\ \mathfrak{f}_2(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)=\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_2+\mu_2)}\left\{\tan^{-1}|\mathfrak{s}^{1,n}(\varsigma,\omega)|+\frac{\sin(\mathfrak{s}^{2,n}(\varsigma,\omega))}{1+|\mathfrak{s}^{2,n}(\varsigma,\omega)|}\right\}_{n\geq 1}\end{cases}\tag{6.1}$$

Firstly, we easily see that, the functions $\mathfrak{f}_i$, $i=1,2$, satisfy (A1). Secondly, we can check that

$$\begin{aligned}&\|\mathfrak{f}_1(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)-\mathfrak{f}_1(\varsigma,\mathfrak{r}_1(\varsigma,\omega),\mathfrak{r}_2(\varsigma,\omega),\omega)\|\\&\leq\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_1+\mu_1)}\|\mathfrak{s}^{1,n}(\varsigma,\omega)-\mathfrak{r}^{1,n}(\varsigma,\omega)\|+\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_1+\mu_1)}\|\mathfrak{s}^{2,n}(\varsigma,\omega)-\mathfrak{r}^{2,n}(\varsigma,\omega)\|,\end{aligned}$$

and

$$\begin{aligned}&\|\mathfrak{f}_2(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)-\mathfrak{f}_2(\varsigma,\mathfrak{r}_1(\varsigma,\omega),\mathfrak{r}_2(\varsigma,\omega),\omega)\|\\&\leq\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_2+\mu_2)}\|\mathfrak{s}^{1,n}(\varsigma,\omega)-\mathfrak{r}^{1,n}(\varsigma,\omega)\|+\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_2+\mu_2)}\|\mathfrak{s}^{2,n}(\varsigma,\omega)-\mathfrak{r}^{2,n}(\varsigma,\omega)\|.\end{aligned}$$

So, the hypotheses (A2) holds with

$$\mathfrak{X}_{1,r}(\omega)=\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_1+\mu_1)},\quad \mathfrak{X}_{2,r}(\omega)=\frac{|\omega|(1+|\omega|)^{-1}}{10\mathfrak{N}(b,\vartheta_2+\mu_2)},\quad r=1,2,\ \text{for}\ \omega\in\Omega.$$

An application of Theorem 4.2, we deduce that system (1.1) with (6.1) has a unique random solution $(\mathfrak{s}_1,\mathfrak{s}_2)$.

However,

$$\mathcal{T}_{i,i}(b,\omega)=1-\mathfrak{X}_{i,i}(\omega)\mathfrak{N}(b,\vartheta_i+\mu_i)-|\varpi_i|\mathfrak{N}(b,\mu_i)>0,\quad i=1,2.$$

and

$$\Psi\ =\left(1-\frac{|\omega|}{10(1+|\omega|)}-\frac{\widehat{\varpi}_1}{e^{\mathfrak{a}}}\right)\left(1-\frac{|\omega|}{10(1+|\omega|)}-\frac{\widehat{\varpi}_2}{e^{\mathfrak{a}}}\right)-\left(\frac{|\omega|}{10(1+|\omega|)}\right)^2\neq 0.$$

Therefore, Theorem 5.8 ensure that system (1.1) with $\mathfrak{f}_i$, $i=1,2$, defined by (6.1) is U-H and G-U-H stable.

In addition, by letting $\mathcal{K}_i(\varsigma)=\mathbb{E}^k_{\vartheta_i+\mu_i}\left(\chi(\varsigma,a)^{\frac{\vartheta_i+\mu_i}{k}}\right)$, $i=1,2$, and by [35, Lemma 2.18] we have

$${}^T_k\mathcal{I}^{\vartheta_i+\mu_i,\varrho;\phi}_{a^+}\mathcal{K}_i(\varsigma)\leq\mathcal{K}_i(\varsigma),\quad i=1,2,$$

So condition (3) of Theorem 5.9 is satisfied with $\mathcal{K}_i(\varsigma)=\mathbb{E}^k_{\vartheta_i+\mu_i}\left(\chi(\varsigma,a)^{\frac{\vartheta_i+\mu_i}{k}}\right)$ and $\mathfrak{q}_i=1$, $i=1,2$, It follows from Theorem 5.9 that the system (1.1) is U–H–R stable and consequently it is G-U–H–R stable.

Remark 6.1. From $\mathcal{K}_i(\varsigma)=\mathbb{E}^k_{\vartheta_i+\mu_i}\left(\chi(\varsigma,a)^{\frac{\vartheta_i+\mu_i}{k}}\right)$, $i=1,2$, we say also the system (1.1) is k-Mittag-Leffler-Ulam-Hyers and k-generalized Mittag-Leffler-Ulam-Hyers stable.

(II) Illustration of Theorem 4.3.

For $(\varsigma,\omega)\in\mathfrak{I}\times\Omega$ and $\mathfrak{s}_i=\{\mathfrak{s}^{i,n}\}_n\in c_0$, consider the nonlinear forcing terms,

$$\begin{cases}\mathfrak{f}_1(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\\ \quad=\frac{2\tan^{-1}(\varsigma)}{\pi}\sin(\omega/2)\left\{\sin|\mathfrak{s}^{1,n}(\varsigma,\omega)|+\log_e(|\mathfrak{s}^{2,n}(\varsigma,\omega)|+1)+\frac{1}{13^n}\right\}_{n\geq 1}\\ \mathfrak{f}_2(\varsigma,\mathfrak{s}_1(\varsigma,\omega),\mathfrak{s}_2(\varsigma,\omega),\omega)\\ \quad=\frac{|\omega|(e^{2\varsigma}-1)}{(e^{\varsigma}+1)(1+|\omega|)}\left\{\tan^{-1}(|\mathfrak{s}^{1,n}(\varsigma,\omega)|)+|\mathfrak{s}^{2,n}(\varsigma,\omega)|\cos^2(\mathfrak{s}^{2,n}(\varsigma,\omega))+\frac{1}{13^n}\right\}_{n\geq 1}\end{cases}\tag{6.2}$$

Obviously, $\mathfrak{f}_i$, $(i = 1, 2)$ satisfy hypothesis (A1).

To illustrate (A3), let $\varsigma \in \mathfrak{I}$ and $\mathfrak{s}_i = \{\mathfrak{s}^{i,n}\}_n \in S \subset c_0$, $i = 1, 2$. Then

$$\|\mathfrak{f}_1(\varsigma, \mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \quad \leq \tfrac{2\tan^{-1}(\varsigma)}{\pi} \sin(\omega/2)\Big(\|\mathfrak{s}^{1,n}(\varsigma,\omega)\| + \|\mathfrak{s}^{2,n}(\varsigma,\omega)\| + 1\Big), \qquad (6.3)$$

and

$$\|\mathfrak{f}_2(\varsigma, \mathfrak{s}_1(\varsigma,\omega), \mathfrak{s}_2(\varsigma,\omega), \omega)\| \quad \leq \tfrac{|\omega|(e^{2\varsigma}-1)}{(e^{\varsigma}+1)(1+|\omega|)}\Big(\|\mathfrak{s}^{1,n}(\varsigma,\omega)\| + \|\mathfrak{s}^{2,n}(\varsigma,\omega)\| + 1\Big), \qquad (6.4)$$

Therefore, (H3) is verified with $\varphi_1(\varsigma,\omega) = \frac{2\tan^{-1}(\varsigma)}{\pi}\sin(\omega/2)$ and $\varphi_2(\varsigma,\omega) = \frac{|\omega|(e^{2\varsigma}-1)}{(e^{\varsigma}+1)(1+|\omega|)}$ for all $(\varsigma,\omega) \in \mathfrak{I} \times \Omega$.

Next, hypothesis (A4) is satisfied. Indeed, we recall that the Hausdorff MNC Ξ in $(c_0, \|\cdot\|_{c_0})$ can be computed by means of the formula

$$\Xi(S) = \lim_{n\to\infty} \sup_{\mathfrak{y}\in S} \|(\mathrm{I} - \mathrm{P}_n)\,\mathfrak{y}\|_\infty\,,$$

where $S \in \mathcal{P}(c_0)$, P_n represents the projection onto the linear span of the first n vectors in the standard basis (see [5]).

Using (6.3) and (6.4) (see also Example in [42]), we get

$$\Xi\left(\mathfrak{f}_i(\varsigma, S^1, S^2)\right) \leq \mathbb{G}_{i,1}(\omega)\Xi(S^1) + \mathbb{G}_{i,2}(\omega)\Xi(S^2), \quad \text{for all } (\varsigma,\omega) \in \mathfrak{I} \times \Omega.$$

where

$$\mathbb{G}_{1,1}(\omega) = \mathbb{G}_{1,2}(\omega) = \sin(\omega/2), \quad \mathbb{G}_{2,1}(\omega) = \mathbb{G}_{2,2}(\omega) = \frac{|\omega|}{1+|\omega|}, \quad \text{for all } \omega \in \Omega.$$

The conclusion of Teorem 4.3 implies that problem (1.1) with (6.2) has at least one solution $(\mathfrak{s}_1, \mathfrak{s}_2)$.

7 Conclusion

The fractional Langevin system is a crucial mathematical model for describing the random motion of particles. Consequently, we investigated a class of tempered (k,ϕ)-Caputo Langevin systems with random effects in a generalized separable Banach space. By employing the Bielecki-type vector-valued norm, we established a new uniqueness criterion. Additionally, we imposed rather mild assumptions to obtain a new existence result by utilizing a recent random version of Sadovski's fixed-point theorem. Furthermore, we have also established results on U-H, G-U-H, U-H-R and G-U-H-R stability. As a result, numerous findings in the literature can be recovered through our results.

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