Annihilator condition on modules (II)

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Abstract.Let R be a commutative ring with $1 \neq 0$ and M be a unital R-module. M satisfies Property (A) if for each finitely generated ideal J of R contained in $Z_R(M)$, there exists $0 \neq m \in M$ such that $Jm = (0)$. M satisfies strong Property (A) if for any $r_1, r_2, ..., r_n \in Z_R(M)$, there exists $0 \neq m \in M$ such that $r_1m = r_2m = ... = r_nm = 0$. M is satisfies Property (T) if for each finitely generated submodule N of M contained in $T(M)$, there exists $0 \neq a \in R$ such that $aN = 0$. An R-module M satisfies strong Property (T) if each subset $N \subseteq T(M)$ with $|N| \leq m$ for some $m \in \mathbb{N}$, then $0 \neq a \in R$ such that $aN = (0)$. M is strongly coprimal if for arbitrary $a, b \in Z_R(M)$, the annihilator of $\{a, b\}$ in M is nonzero. M is super coprimal if for every finite subset F of $Z_R(M)$, the annihilator of the set F in M is nonzero. M is coprimal if $Z_R(M)$ forms an ideal of R. In this paper, we study the transfer of certain annihilator conditions on modules such as Property (A), strong Property (A), Property (T), strong Property (T) and strongly coprimal Property to amalgamated duplication module along an ideal. As an application of our results, we establish the transfer of super coprimal module property to duplication modules.

1 Introduction

Throughout this whole paper, we focus only on commutative rings with nonzero identity and nonzero unital modules. This paper concerns with certain annihilator conditions on modules such as that Property (A), strong Property (A), Property (T), strong Property (T) and strongly coprimal property. In [2], Anderson and Chun studied rings and modules satisfying Property (A), strong Property (A), Property (T) and strong Property (T). Among other things, they established some characterizations and provided some examples using the method of idealizations. In [1], the authors introduced and investigated a new class of rings lying properly between the class of (A)-rings and the class of SA-rings. The new class of rings, termed the class of PSA-rings, turned out to share common characteristics with both (A)-rings and SA-rings. Numerous properties and characterizations of this class are given as well as the module-theoretic version of PSA-rings is introduced and studied. In [10], the authors showed that if $\{R_i\}_{i \in I}$ is a family of commutative rings and $M_{i \, i \in I}$ is a family of modules such that each M_i is an R_i-module, then the direct product $\prod_{i \in I} M_i$ is an A-module over $\prod_{i \in I} R_i$ if and only if each M_i is an A-module over R_i, $i \in I$. This result extends the work of Hong, Kim, Lee and Ryu in rings with Property (A) and their extensions, J. Algebra 315 (2007), 612–628. Moreover, they characterized Property (A) for modules in terms of Property (A) of their total quotient modules, extending the work of Dobbs and Shapiro in On the strong (A)-ring of Mahdou and Hassani, Mediterr. J. Math. 10 (2013), 1995–1997. In [11], the author gave some results on these classes of modules. Also, he provided a relationship between the families of coprimal, strongly coprimal and super coprimal modules. He proved that if M is a coprimal module of finite Goldie dimension over a commutative ring, then M is super coprimal. He also showed that every proper submodule

of a module over a Prüfer domain of finite character can be expressed as a finite intersection of strongly primal submodules.

In 2006, M. D'Anna and M. Fontana [15] introduced a new construction, called amalgamated duplication of a ring A along an $A-$submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated duplication $A \bowtie E$ of A along an $A-$submodule E of $Q(A)$ are discussed in more details, especially in the particular case where E is an ideal of A, in recent papers, for instance, see [12, 13, 14, 15]. Consider a ring R, I an ideal of R, and M an R-module. In [7], Bouba, Mahdou and Tamekkante defined the duplication of an R-module M along an ideal I as follows:

$M \bowtie I := \{(m, m') \in M \times M/m - m' \in IM\}$ which is an $R \bowtie I$-module with the multiplication is given by:

$(r, r + i)(m, m') = (rm, (r + i)m')$, where $r \in R$, $i \in I$, and $(m, m') \in M \bowtie I$. They studied some basic properties of the duplication of an R-module M along an ideal I. More precisely, they investigated about when $M \bowtie I$ is a Noetherian, an Artinian $R \bowtie I$-module. They also examined some basic homological properties of $M \bowtie I$ such as the injectivity, the projectivity of the duplication of module along an ideal.

Let R be a ring and M be an R-module. Recall that a ring R is said to satisfy Property (A) if for each finitely generated ideal I which is consisting of entirely of zero divisors of R, then there exists $0 \neq a \in R$ such that $aI = (0)$ [18]. The class of commutative rings with Property (A) is quite large; for example, Noetherian rings, rings whose prime ideals are maximal, the polynomial ring $R[x]$ and rings whose total ring of quotients are von Neumann regular.

For a ring R, an R-module M and a submodule N of M, we will denote by $(N :_R M)$ the residual of N by M which is the set of all r in R such that $rM \subseteq N$. The annihilator of M, denoted by $Ann_R(M)$, is $(0 :_R M)$. For every subset S of R, we denote by $Ann_M(S)$, the set of elements $m \in M$ such that $ma = 0$ for each $a \in S$. An element $r \in R$ is called a zero-divisor on M if there exists $0 \neq m \in M$ such that $rm = 0$, that is $Ann_M(r) \neq 0$. We denote by $Z_R(M)$, the set of all zero-divisors of R on M. If we consider R as an R-module, then we write $Z(R)$ instead of $Z_R(R)$.

2 Main Results

We start this section by recalling the following definitions:

Definition 2.1. (i) An R-module M is said to satisfy Property (A) if for each finitely generated ideal J of R contained in $Z_R(M)$, there exists $0 \neq m \in M$ such that $Jm = 0$ [2].

(ii) An R-module M is said to satisfy strong Property (A) if for any $r_1, r_2, ..., r_n \in Z_R(M)$, there exists $0 \neq m \in M$ such that $r_1 m = r_2 m = ... = r_n m = 0$ [2].

(iii) An R-module M is said to satisfy Property (T) if for each finitely generated submodule N of M contained in $T(M)$, there exists $0 \neq a \in R$ such that $aN = 0$ [2].

(iv) an R-module M is said to satisfy strong Property (T) if each subset $N \subseteq T(M)$ with $|N| \leq m$ for some $m \in \mathbb{N}$, then $0 \neq a \in R$ such that $aN = (0)$ [2].

(v) An R-module M is said to be coprimal if $Z_R(M)$ forms an ideal of R [11].

(vi) An R-module M is said to be strongly coprimal if for arbitrary $a, b \in Z_R(M)$, the annihilator of $\{a, b\}$ in M is nonzero [11].

Now, we establish the main result of this paper which examines the transfer of Property (A), strong Property (A), Property (T), coprimal and strongly coprimal module properties to the amalgamated duplication of a module along an ideal.

Theorem 2.2. *Let (R, I) be a local ring with maximal ideal I such that $I^2 = 0$ and let M be an R-module. Then the following assertions hold:*

(i) $M \bowtie I$ satisfies Property (A) over $R \bowtie I$ if and only if M satisfies Property (A) over R.

(ii) $M \bowtie I$ satisfies strong Property (A) if and only if M satisfies strong Property (A) over R.

(iii) $M \bowtie I$ *satisfies Property* (T) *if and only if* M *satisfies Property* (T) *over* R.

(iv) $M \bowtie I$ *is coprimal* $R \bowtie I$*-module if and only if* M *is coprimal* R*-module.*

(v) $M \bowtie I$ *is strongly coprimal* $R \bowtie I$*-module if and only if* M *is strongly coprimal* R*-module.*

To prove the previous theorem, we need the following lemma.

Lemma 2.3. *Let* R, I *and* M *as in the above theorem. Then following assertions hold:*

(i) $Z_{R \bowtie I}(M \bowtie I) = Z_R(M) \bowtie I$.

(ii) $T_{R \bowtie I} = T_R(M) \bowtie I$.

Proof. (1) We first want to show that $Z_{R \bowtie I}(M \bowtie I) \subseteq Z_R(M) \bowtie I$.
Let $(r, r + i) \in Z_{R \bowtie I}(M \bowtie I)$. Then there exists $(0, 0) \neq (m, m') \in M \bowtie I$ such that $(r, r + i)(m, m') = (0, 0)$ and so $rm = 0$ and $(r + i)m' = 0$. Two cases are then possible:
    Case 1: $m \neq 0$:
In this case, $r \in Z_R(M)$.
    Case 2: $m = 0$:
In this case, $m' \in IM$ since $m - m' \in IM$. We have $im' \in iIM \subseteq I^2 M = 0$ since $I^2 = 0$ and so $im' = 0$. Using the fact that $(r + i)m' = 0$, it follows that $rm' = 0$. Consequently, $m' \neq 0$ since $(m, m') \neq (0, 0)$ and $m = 0$. Hence, $r \in Z_R(M)$.
In both cases, we obtain $r \in Z_R(M)$ and so $(r, r + i) \in Z_R(M) \bowtie I$.
    Conversely, we show that $Z_R(M) \bowtie I \subseteq Z_{R \bowtie I}(M \bowtie I)$ and let $(r, r + i) \in Z_R(M) \bowtie I$. Then $r \in Z_R(M)$ and so there exists $0 \neq m \in M$ such that $rm = 0$. Two cases are then possible:
    Case 1: $im = 0$.
In this case, we have $(r, r + i)(m, m) = (rm, rm) = (0, 0)$ (since $im = 0$) and so $(r, r + i) \in Z_{R \bowtie I}(M \bowtie I)$ (as $(m, m) \neq (0, 0)$).
    Case 2: $im \neq 0$.
In this case, we have $(r, r + i)(im, im) = (0, 0)$ and so $(r, r + i) \in Z_{R \bowtie I}(M \bowtie I)$ (since $(im, im) \neq (0, 0)$).
In both cases, we obtain $(r, r + i) \in Z_{R \bowtie I}(M \bowtie I)$, as desired.
(2) We first show that $T_{R \bowtie I}(M \bowtie I) \subseteq T_R(M) \bowtie I$ and let $(m, m') \in T_{R \bowtie I}(M \bowtie I)$. Hence, there exists $(0, 0) \neq (r, r + i) \in R \bowtie I$ such that $(r, r + i)(m, m') = (0, 0)$ and so $rm = 0$ and $(r + i)m' = 0$. Two cases are then possible:
    Case 1: $r \neq 0$.
In this case, $m \in T_R(M)$ and so $(m, m') \in T_R(M) \bowtie I$.
    Case 2: $r = 0$.
Using the fact the $m - m' \in IM$, it follows that $im - im' = i(m - m') \in iIM \subseteq I^2 M = 0$. Therefore, $im = im' = 0$ (since $(r + i)m' = 0$ and $r = 0$ in this case) and so $m \in T_R(M)$ since $i \neq 0$ (as $(r, r + i) \neq (0, 0)$ and $r = 0$ in this case). Hence, $(m, m') \in T_R(M) \bowtie I$.
In both cases, we obtain $(m, m') \in T_R(M) \bowtie I$, as desired.
    Conversely, we need to prove that $T_R(M) \bowtie I \subseteq T_{R \bowtie I}(M \bowtie I)$ and let $(m, m') \in T_R(M) \bowtie I$. Hence, $m \in T_R(M)$ and so there exists $0 \neq r \in R$ such that $rm = 0$. Two cases are then possible:
    Case 1: $r \notin I$.
In this case, r is invertible in R (as (R, I) is a local ring) and so $m = 0$. Hence, $m' \in IM$ since $m - m' \in IM$. Therefore, for each $0 \neq i \in I$, we have $(0, i)(m, m') = (0, 0)$ and so $(m, m') \in T_{R \bowtie I}(M \bowtie I)$.
    Case 2: $r \in I$.
In this case, we have $rm - rm' \in rIM \subseteq I^2 M = 0$ (as $m - m' \in IM$) and so $rm = rm' = 0$. Hence, $(m, m') \in T_{R \bowtie I}(M \bowtie I)$.
In both cases, we obtain $(m, m') \in T_{R \bowtie I}(M \bowtie I)$, as desired. $\qquad\square$

Proof. of Theorem 2.2:
(1) Assume that $M \bowtie I$ satisfies Property (A) and let J_0 be a finitely generated ideal of R such that $J_0 \subseteq Z_R(M)$. Our aim is to show that there exists $0 \neq m \in M$ such that $J_0 m = 0$.
Assume that $J_0 = \sum_{i=1}^n R a_i$, where $a_i \in J_0$, and set $J := \sum_{i=1}^n (R \bowtie I)(a_i, a_i)$ which is a

finitely generated ideal of $R \bowtie I$ and $J \subseteq Z_R(M) \bowtie I$. Hence, there exists $(0,0) \neq (m, m') \in M \bowtie I$ such that $J(m, m') = (0,0)$ since $M \bowtie I$ satisfies Property (A). Two cases are then possible:

Case 1: $m \neq 0$.

We have $J_0 m = 0$ since $J(m, m') = (0,0)$, as desired.

Case 2: $m = 0$.

In this case, $m' \neq 0$ since $(m, m') \neq (0,0)$. Also, we have $J_0 m' = 0$ since $J(m, m') = (0,0)$, as desired.

In all cases, there exists $m \in M$ such that $J_0 m = 0$.

Hence, M satisfies Property (A).

Conversely, assume that M satisfies Property (A) and let J be a finitely generated ideal of $R \bowtie I$ such that $J \subseteq Z_{R\bowtie I}(M \bowtie I)$. Our aim is to show that there exists $(0,0) \neq (m, m') \in M \bowtie I$ such that $J(m, m') = 0$.

Set $J_0 = \{j \in R / (j, j + i) \in J \text{ for some } i \in I\}$. Two cases are then possible:

Case 1: $J_0 \neq 0$.

Since J_0 is a finitely generated ideal of R (as J is a finitely generated ideal of $R \bowtie I$) and $J_0 \subseteq Z_R(M)$ (as $J \subseteq Z_{R\bowtie I}(M \bowtie I)$). So, there exists $0 \neq m \in M$ such that $J_0 m = 0$ since M satisfies Property (A). Two subcases are then possible:

Subcase 1: Assume that there exists $0 \neq i \in I$ such that $im \neq 0$.

In this case, we have $J(im, 0) = 0$, as desired since $(im, 0) \neq (0,0)$.

Subcase 2: Assume that $im = 0$ for each $0 \neq i \in I$ such that $im \neq 0$.

In this case, we have $J(m, m) = (0,0)$ since $Im = J_0 m = 0$ and $J \subseteq J_0 \bowtie I$, as desired.

Case 2: $J_0 = 0$.

In this case, we have $J = \sum_{i=1}^{n}(R \bowtie I)(0, j_i) = 0 \bowtie J_1$, where $J_1 = \sum_{i=1}^{n} RJ_i$ is a finitely generated ideal of R. But $(0, j_i) \in Z_R(M) \bowtie I$ since $J \subseteq Z_R(M) \bowtie I$. Hence, there exists $(0,0) \neq (m, m') \in M \bowtie I$ such that $(0, j_i)(m, m') = (0,0)$ and so $j_i m' = 0$.

Therefore, $j_i m = j_i m' = 0$ for each $i = 1, ..., n$ since $m - m' \in IM$ and $j_i \in I$ and so $J_1 m = J_1 m' = 0$, which means that $J_1 \subseteq Z_R(M)$. Hence, $J(m, m') = (0 \bowtie J_1)(m, m') = 0$, as desired since $(m, m') \neq (0,0)$.

In all cases, it follows that there exists $(0,0) \neq (m, m') \in M \bowtie I$ such that $J(m, m') = 0$. Hence, $M \bowtie I$ satisfies Property (A), as desired.

(2) Assume that $M \bowtie I$ satisfies strong Property (A). We claim that R satisfies strong Property (A). Let $r_1, r_2, ..., r_n \in Z_R(M)$. Then by Lemma 2.3(1), $(r_1, r_1), (r_2, r_2), ..., (r_n, r_n) \in Z_{R\bowtie I}(M \bowtie I)$. So, there exists $0 \neq (m, m') \in M \bowtie I$ such that $(r_1, r_1)(m, m') = (r_2, r_2)(m, m') = ... = (r_n, r_n)(m, m') = (0,0)$. Two cases are then possible:

Case 1: $m \neq 0$. Then $r_1 m = r_2 m = ... = r_n m = 0$. So, there is $0 \neq m \in M$ such that $r_1 m = r_2 m = ... = r_n m = 0$, as desired.

Case 2: $m = 0$. Then $0 \neq m'$ and $r_1 m' = r_2 m' = r_3 m' = ... = r_n m' = 0$, as desired.

Hence, in all cases, it follows that M satisfies strong Property (A).

Conversely, assume that M satisfies strong Property (A). Let $(r_1, r_1+i_1), (r_2, r_2+i_2), ..., (r_n, r_n+i_n) \in Z_{R\bowtie I}(M \bowtie I) = Z_R(M) \bowtie I$. Then by Lemma 2.3(1), $r_1, r_2, ..., r_n \in Z_R(M)$. Since M satisfies strong Property (A), there exists $0 \neq m \in M$ such that $r_1 m = r_2 m = r_3 m = ... = r_n m = 0$. Two cases are then possible:

Case 1: $i_j m = 0$ for all $j \in \{1, 2, ..., n\}$.

In this case, there exists $(0,0) \neq (m, m) \in M \bowtie I$ (as $m - m = 0 \in IM$) such that $(r_1, r_1 + i_1)(m, m) = (r_2, r_2 + i_2)(m, m) = ... = (r_n, r_n + i_n)(m, m) = (0,0)$, as desired.

Case 2: There is $j \in \{1, 2, ..., n\}$ such that $i_j m \neq 0$.

In this case, $(0,0) \neq (i_j m, i_j m) \in M \bowtie I$ such that $(r_1, r_1 + i_1)(i_j m, i_j m) = (r_2, r_2 + i_2)(i_j m, i_j m) = ... = (r_n, r_n + i_n)(i_j m, i_j m) = (0,0)$, as desired.

Hence, in all cases, it follows that $M \bowtie I$ satisfies strong Property (A).

(3) Assume that $M \bowtie I$ satisfies Property (T) and let N be a finitely generated submodule of M such that $N \subseteq T(M)$. Our aim is to show that there exists $0 \neq a \in R$ such that $aN = 0$.

Assume that $N = \sum_{i=1}^{n} Rn_i$ for some positive integer n and some $n_i \in N$. Hence, $N_1 := \sum_{i=1}^{n}(R \bowtie I)(n_i, n_i)$ is a finitely generated submodule of $M \bowtie I$. Also, we have $N_1 \subseteq N \bowtie I \subseteq T(M) \bowtie I = T(M \bowtie I)$. Therefore, there exists $(0,0) \neq (a, a + i) \in R \bowtie I$ such that

$(a, a + i)N_1 = 0$ since $M \bowtie I$ satisfies Property (T). Two cases are then possible:

Case 1: $a \neq 0$. We have the desired result since $aN = 0$.

Case 2: $a = 0$. We have also the desired result since $iN = 0$ and $i \neq 0$ (as $(a, a+i) \neq (0, 0)$). Therefore, we can conclude that M satisfies Property (T), as desired.

Conversely, assume that M satisfies Property (T) and let N be a finitely generated submodule of $M \bowtie I$ such that $N \subseteq T(M \bowtie I)$. Our aim is to show that there exists $(0, 0) \neq (a, a + i) \in R \bowtie I$ such that $(a, a + i)N = 0$.

We have $N \subseteq T(M \bowtie I) = T(M) \bowtie I$ by Lemma 2.3(2). Set $N_0 := \{m \in M / (m, m') \in N$ for some $m' \in M\}$ which is a finitely generated submodule of M since N is a finitely generated submodule of $M \bowtie I$. Two cases are then possible:

Case 1: $N_0 \neq 0$.

In this case, there exists $0 \neq a \in R$ such that $aN_0 = 0$ since $N_0 \subseteq T(M)$. Hence, $a \in I$ since (R, I) is a local ring and $N_0 \neq 0$. Therefore, $(a, a)N = 0$ since for each $(m, m') \in N$, we have $(a, a)(m, m') = 0$ since $am = 0$ (as $m \in N_0$ and $a(m - m') \in aIM \subseteq I^2M = 0$ and so $am' = am = 0$), as desired.

Case 2: $N_0 = 0$.

In this case, we have $N \subseteq \{(0, m') / m' \in IM\}$. Hence, $(i, 0)N = 0$ for each $0 \neq i \in I$, as desired.

Hence, we can conclude that $M \bowtie I$ satisfies Property (T), as desired.

(4) Assume that $M \bowtie I$ is a coprimal $R \bowtie I$-module. Then $Z_{R \bowtie I}(M \bowtie I)$ is an ideal of $R \bowtie I$. From Lemma 2.3, $Z_{R \bowtie I}(M \bowtie I) = Z_R(M) \bowtie I$ is an ideal of $R \bowtie I$. It follows that $Z_R(M)$ is an ideal of R, making M, a coprimal R-module.

Conversely, assume that M is a coprimal R-module. Then $Z_R(M)$ is an ideal of R. One can easily check that $Z_R(M) \bowtie I$ is an ideal of $R \bowtie I$. By Lemma 2.3(1), $Z_{R \bowtie I}(M \bowtie I) = Z_R(M) \bowtie I$ forms an ideal of $R \bowtie I$, making $M \bowtie I$, a coprimal $R \bowtie I$-module, as desired.

(5) Assume that $M \bowtie I$ is strongly coprimal over $R \bowtie I$. We claim that M is strongly coprimal over R. Indeed, let $a, b \in Z_R(M)$. Then by Lemma 2.3(1), $(a, a), (b, b) \in Z_R(M) \bowtie I = Z_{R \bowtie I}(M \bowtie I)$. Since $M \bowtie I$ is strongly coprimal over $R \bowtie I$, then there is $(0, 0) \neq (m, m') \in M \bowtie I$ such that $(m, m') < (a, a), (b, b) >= (0, 0)$. So, $(m, m')(a, a)R \bowtie I + (m, m')(b, b)R \bowtie I = (0, 0)$. Therefore, there exist $(c, c + i), (d, d + j) \in R \bowtie I$ such that $(m, m')(a, a)(c, c + i) + (m, m')(b, b)(d, d + j) = (0, 0)$. Two cases are then possible:

Case 1: $m \neq 0$. Then $m(ac + bd) = 0$ and so $Ann_M(< a, b >) \neq 0$.

Case 2: $m = 0$. Then $0 \neq m'$. Since $m - m' \in IM$, then $-m' \in IM$ and so $im', jm' \in iIM \subseteq I^2M = 0$. Consequently, $m'a(c + i) + m'b(d + j) = m'ac + m'bd = m'(ac + bd) = 0$. Hence, $Ann_M(< a, b >) \neq 0$.

Hence, in all cases, it follows that $Ann_M(< a, b >) \neq 0$, making M, a strongly coprimal R-module, as desired.

Conversely, assume that M is a strongly coprimal R-module. Let $(a, a + i), (b, b + j) \in Z_{R \bowtie I}(M \bowtie I)$. By Lemma 2.3(1), $a, b \in Z_R(M)$ and from assumption, it follows that there is $0 \neq m \in M$ such that $m < a, b >= 0$. So, $mac + mbd = 0$ for some $c, d \in R$. One can easily check that $m(a, a)(c, c) + m(b, b)(d, d) = (0, 0)$ for some $(c, c), (d, d) \in R \bowtie I$. Two cases are then possible:

Case 1: $m \in Ann_R(J)$. Then $(m, m)(a, a + i)(c, c) + (m, m)(b, b + j)(d, d) = (0, 0)$ and so $Ann_M(< (a, a + i), (b, b + j) >) \neq 0$.

Case 2: $m \notin Ann_R(J)$. Then there is $0 \neq k \in J$ such that $km \neq 0$. So, $(km, km)(a, a + i)(c, c) = (0, 0)$ and $(km, km)(b, b + j)(d, d) = (0, 0)$. Hence, there is $(0, 0) \neq (km, km) \in M \bowtie I$ such that $(km, km)(a, a + i)(c, c) + (km, km)(b, b + j)(d, d) = (0, 0)$ and so $Ann_M(< (a, a + i), (b, b + j) >) \neq 0$.

Hence, in all cases, it follows that $Ann_{M \bowtie I}((a, a + i), (b, b + j)) \neq 0$. Finally, $M \bowtie I$ is strongly coprimal over $R \bowtie I$, as desired.

$\square$

It is worthwhile recalling that the class of (A)-rings includes the class of rings whose total ring of quotient is Von Neumann. However, there are examples showing that an (A)-ring need not to have a Von Neumann total quotient ring. Theorem 2.2 allows us to construct an example of

a non-Von Neumann regular $R \bowtie I$-module satisfying Property (A). Recall from [19, Example 1] that a ring R is Von Neumann regular R-module if and only if R is a Von Neuman Regular ring.

Example 2.4. Let $(R, I) := (\mathbb{Z}_4, 2\mathbb{Z}_4)$ be a local ring such that $I^2 = 0$. Then:

 (i) $R \bowtie I$ satisfies Property (A) over $R \bowtie I$.

 (ii) $R \bowtie I$ is not a Von Neumann regular $R \bowtie I$-module.

Proof. (1) By Theorem 2.2(1), $R \bowtie I$ satisfies (A) Property over $R \bowtie I$, as $\mathbb{Z}_4$ satisfies (A) Property as $\mathbb{Z}_4$-module and and $I^2 = 0$.
(2) $R \bowtie I$ is not a Von Neumann regular $R \bowtie I$-module since $R \bowtie I$ is not a Von Neumann regular ring (as R is not reduced, $I^2 = 0$). □

    The following corollary is a consequence of Theorem 2.2 on the transfer of super coprimal module property to the special setting of the duplication of module along an ideal. Recall from [11, Proposition 2.2(3)] that an R-module is super coprimal if and only if it is coprimal and $F - McCoy$ (i.e satisfies Property (A)).

Corollary 2.5. *Let (R, I) be a local ring with maximal ideal I such that $I^2 = 0$ and let M be an R-module. Then $M \bowtie I$ is a super coprimal $R \bowtie I$-module if and only if M is a super coprimal R-module.*

Proof. This follows from a combination of assertions (1) and (4) of Theorem 2.2. □

    The next proposition establishes another characterization of $M \bowtie I$ to inherit the (A) Property and strong (A) Property. Recall that, a maximal prime of an R-module M is a prime ideal of R maximal within $Z_R(M)$.

Proposition 2.6. *Let (R,I) be a local ring with maximal ideal I such that $I^2 = 0$ and let M be an R-module. Then the following assertions are equivalent:*

 (i) $M \bowtie I$ *satisfies the strong Property (A) over $R \bowtie I$.*

 (ii) $M \bowtie I$ *satisfies Property (A) over $R \bowtie I$ and $Z_{R\bowtie I}(M \bowtie I)$ is an ideal of $R \bowtie I$.*

 (iii) M *satisfies Property (A) over R and $Z_R(M)$ is an ideal of R.*

 (iv) M *satisfies Property (A) over R and R has only one maximal prime of M.*

 (v) M *satisfies Property (A) over R and $Q_R(M)$ is local.*

Proof. (1) $\iff$ (2) This follows from [10, Theorem 2.4].
(2) $\iff$ (3) Assume that $M \bowtie I$ satisfies Property (A) over $R \bowtie I$ and $Z_{R\bowtie I}(M \bowtie I)$ is an ideal of $R \bowtie I$. Then by Theorem 2.2(2), M satisfies Property (A) over R and by Lemma 2.3(1), $Z_{R\bowtie I}(M \bowtie I) = Z_R(M) \bowtie I$ is an ideal of $R \bowtie I$ and so it follows that $Z_R(M)$ is an ideal of R.
Conversely, assume that M satisfies Property (A) over R and $Z_R(M)$ is an ideal of R. Then by Theorem 2.2(1), $M \bowtie I$ satisfies Property (A) over $R \bowtie I$. Using the fact that $Z_R(M)$ is an ideal of R, it follows that $Z_R(M) \bowtie I = Z_{R\bowtie I}(M \bowtie I)$ is an ideal of $R \bowtie I$, as desired.
(3) $\iff$ (4) $\iff$ (5) follow from [10, Lemma 2.5].
                                                                                          □

    The following proposition gives another characterization of about when the duplication module along an ideal satisfies the Property (A) in term of $R \bowtie I$.

Proposition 2.7. *Let (R, I) be a local ring with maximal ideal I such that $I^2 = 0$. Let M be an R-module. Then the following assertions are equivalent:*

 (i) $M \bowtie I$ *satisfies Property (A) over $R \bowtie I$.*

 (ii) $R \bowtie I$ *satisfies Property (A).*

 (iii) M *satisfies Property (A) over R.*

The proof of the previous proposition requires the following Lemma.

Lemma 2.8. *Let R, I and M as in the above proposition. Then $Q_{R\bowtie I}(M \bowtie I) = R \bowtie I$.*

Proof. From [1, Lemma 2.1], it suffices to show that any element of $R \bowtie I$ is either a zero divisor of $R \bowtie I$ on $M \bowtie I$ or invertible in $R \bowtie I$. Let $(r, r+i) \in R \bowtie I$. If $r \notin I$, then $(r, r+i) \notin I \bowtie I$. So, $(r, r+i)$ is invertible in $R \bowtie I$. Now, we may assume that $r \in I$. We claim that $r \in Z_R(M)$. Assume by the way of contradiction that $r \notin Z_R(M)$. Then for every $0 \neq m \in M$, $rm \neq 0$. And so $(r, r+i)(rm, rm) = (r^2m, r^2m + irm) = (0,0)$. Using the fact that $rm \in M$, it follows that $(r, r+i) \in Z_{R\bowtie I}(M \bowtie I)$. From Lemma 2.3(1), it follows that $r \in Z_R(M)$, which is absurd. Hence, $r \in Z_R(M)$, forcing $(r, r+i)$ to be an element of $Z_{R\bowtie I}(M \bowtie I)$ by Lemma 2.3(1). $\qquad\square$

Proof of Proposition 2.7

(1) $\iff$ (2) This follows from a combination of Lemma 2.8 and [2, Theorem 2.1(3)].

(1) $\iff$ (3) By Theorem 2.2(1), we have the desired result.

Proposition 2.9. *Let R be a ring, M be an A-module and I be an ideal of R such that $I \subseteq Ann_R(M)$. Then $M \bowtie I$ satisfies Property (A) over $R \bowtie I$ if and only if $M \bowtie I$ satisfies Property (A) over $\frac{R}{Ann_R(M)}$.*

Proof. Note from [7, Lemma 3.6] that $Ann_{R\bowtie I}(M \bowtie I) = Ann_R(M) \bowtie I \cap Ann_R(M)$. Using the fact that $I \subseteq Ann_R(M)$, it follows that $Ann_{R\bowtie I}(M \bowtie I) = Ann_R(M) \bowtie I$. Applying [2, Theorem 2.1(2)], it follows that $M \bowtie I$ satisfies Property (A) over $R \bowtie I$ if and only if $M \bowtie I$ satisfies Property (A) over $\frac{R\bowtie I}{Ann_{R\bowtie I}(M\bowtie I)}$. On the other hand, by [13, Proposition 5.1(2)], $\frac{R\bowtie I}{Ann_R(M)\bowtie I} \simeq \frac{R}{Ann_R(M)}$. Hence, it follows that $M \bowtie I$ satisfies Property (A) over $R \bowtie I$ if and only if $M \bowtie I$ satisfies Property (A) over $\frac{R}{Ann_R(M)}$, as desired. $\qquad\square$

The following proposition examines about every $R \bowtie I$-module satisfies strong Property (T). Recall from [2] that an R-module M satisfies strong Property (T) if each subset $N \subseteq T(M)$ with $|N| \leq m$ for some $m \in \mathbb{N}$, then $0 \neq a \in R$ such that $aN = (0)$.

Proposition 2.10. *Let R be a ring and I be an ideal of R. If every $R \bowtie I$-module satisfies strong Property (T), then every R-module satisfies strong Property (T). The converse holds if every ideal of $R \bowtie I$ contains $0 \times I$.*

The proof of the previous proposition requires the following lemma. Recall that an ideal I of R is an *irreducible ideal* if $J \cap K = I$ for some ideals J, K of R, then $J = I$ or $K = I$ [26].

Lemma 2.11. *Let R be a ring. Then the following assertions are equivalent:*

(i) *Every R-module satisfies strong Property (T).*

(ii) *Zero ideal is an irreducible ideal of R.*

Proof. Assume that every R-module satisfies Property (T). Now, we will show that zero ideal is irreducible. Indeed, let $I \cap J = 0$ for some ideals I and J of R. Assume that $I \nsubseteq J$ and $J \nsubseteq I$. Put $M = R/I \oplus R/J$. Then M is an R-module. As $I \nsubseteq J$ and $J \nsubseteq I$, choose $a \in I - J$ and $b \in J - I$. Consider the subset $N = \{(\overline{a}, \overline{1}), (\overline{1}, \overline{b})\}$. Then note that $b(\overline{a}, \overline{1}) = 0_M$ and $a(\overline{1}, \overline{b}) = 0_M$ and this yields $N \subseteq T(M)$. As M satisfies strong Property (T), there exists a nonzero element $r \in R$ such that $rN = (0)$. This gives $r(\overline{a}, \overline{1}) = 0_M = r(\overline{1}, \overline{b})$ and this yields $r \in I \cap J = 0$ which is a contradiction. Hence, we get either $I \subseteq J$ or $J \subseteq I$ and so that $I = 0$ or $J = 0$. Therefore, zero ideal is irreducible. The converse is trivial from [2, Theorem 3.1]. $\quad\square$

Proof. of Proposition 2.10. Assume that every $R \bowtie I$-module satisfies strong Property (T). Then by Lemma 2.11, 0×0 is an irreducible ideal of $R \bowtie I$. Let J, K be two ideals of R such that $J \cap K = 0_R$. Note that $J \times 0$ and $K \times 0$ are ideals of $R \bowtie I$ and $(J \times 0) \cap (K \times 0) = (J \cap K) \times 0 = 0 \times 0$. Using the fact that 0×0 is an irreducible ideal of $R \bowtie I$, it follows that

$J \times 0 = 0 \times 0$ or $K \times 0 = 0 \times 0$. Consequently, $J = 0$ or $K = 0$, making 0_R an irreducible ideal of R. By Lemma 2.11, Every R-module satisfies strong Property (T).

Conversely, assume that every R-module satisfies strong Property (T) and every ideal of $R \bowtie I$ contains $0 \times I$. Let J, K be two ideals of $R \bowtie I$ such that $J \cap K = 0 \times 0$. Since every ideal of $R \bowtie I$ contains $0 \times I$, then it follows that $J := J_0 \bowtie I$ and $K := K_0 \bowtie I$ for some J_0 and K_0 ideals of R. We have $J \cap K = (J_0 \bowtie I) \cap (K_0 \bowtie I) = (J_0 \cap K_0) \bowtie I = 0 \times 0$. Therefore, $J_0 \cap K_0 = 0$ and $I_0 = 0$. Consequently, $J_0 = 0$ or $K_0 = 0$ and $I = 0$. It follows that, $J := J_0 \bowtie I = 0 \times 0$ or $K := K_0 \bowtie I = 0 \times 0$, making 0×0, an irreducible ideal of $R \bowtie I$. Hence, every $R \bowtie I$-module satisfies strong Property (T), as desired.

<div align="right">□</div>

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