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ON CERTAIN SUBCLASSES OF MULTIVALENT ANALYTIC FUNCTIONS OF COMPLEX ORDER INVOLVING HYPER-GEOMETRIC FUNCTION

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Abstract *An operator $\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)$ is designed by the convolution of a hypergeometric function with a suitable function. Using this operator, two sub-classes, $S_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$ and $\mathcal{R}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho)$, of multivalent functions were defined. By taking their intersection with $\mathcal{T}_p(n)$, two new subclasses of multivalent functions are obtained: $\widetilde{S}p,q,s^{\alpha,\beta,n}(e_1,\rho)$ and $\widetilde{\mathcal{R}}p,n^{b,q,s}(e_1,\alpha,\mu;\rho)$. For these two latter subclasses of multivalent functions, the following results are obtained: a necessary and sufficient condition for a function in $\mathcal{T}_p(n)$ to be in the class $\widetilde{S}p,q,s^{\alpha,\beta,n}(e_1,\rho)$ or $\widetilde{\mathcal{R}}p,n^{b,q,s}(e_1,\alpha,\mu;\rho)$; inclusion relationships of the sub-classes with respect to parameters. Following the earlier investigations of different authors, the (n,δ)-neighborhood of a function $f\in\mathcal{A}_p(n)$ is defined, and inclusion relations involving the neighborhood are studied. Some inclusion relationships involving modified Hadamard products are also established. Finally, the subordination relation involving the above-defined operator is studied.*

1 Introduction

Let $\mathcal{A}_p(n)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\qquad (p,n\in\mathbb{N}=\{1,2,\dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For convenience, we write $\mathcal{A}_p(1)=\mathcal{A}_p$ and $\mathcal{A}_1(1)=\mathcal{A}$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f(z)\prec g(z)\quad(z\in\mathbb{U})$, if there exists a function ω, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that $f(z)=g(\omega(z)),\ z\in\mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence (cf., e.g., [6]; see also [7]):

$$f(z)\prec g(z)\ (z\in\mathbb{U})\Longleftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

For a function $f\in\mathcal{A}_p(n)$, given by (1.1) and $g\in\mathcal{A}_p(n)$ defined by $g(z)=z^p+\sum_{k=n}^{\infty}b_{k+p}z^{k+p}$, we define the Hadamard product (or convolution) of f and g by

$$f(z)*g(z)=(f*g)(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}b_{p+k}z^{k+p}\quad(p\in\mathbb{N}).$$

For real or complex numbers

$$e_1, e_2, \cdots, e_q \quad and \quad d_1, d_2, \cdots, d_s \; \left(e_i,\ d_j \notin \mathbb{I}_0^- = \{0, -1, -2, \cdots\};\ j = 1, 2, \cdots, s\right),$$

we consider the generalized hypergeometric function ${}_qF_s$ *(see, for example, ([14], p.19)) defined as follows:*

$$ {}_qF_s(e_1, \cdots, e_q; d_1, \cdots, d_s; z) = \sum_{k=0}^{\infty} \frac{(e_1)_k \cdots (e_q)_k}{(d_1)_k \cdots (d_s)_k} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \Gamma_k(e_1) z^k \tag{1.2}$$

$$(q \leqq s + 1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \mathbb{U}),$$

where $\Gamma_k(e_1) = \frac{(e_1)_k \cdots (e_q)_k}{(d_1)_k \cdots (d_s)_k} \frac{1}{k!}$ *and* $(x)_m$ *denotes the* Pochhammer *symbol (or the* shifted factorial*) defined as, in terms of the Gamma function* Γ *by*

$$(x)_m = \frac{\Gamma(x+m)}{\Gamma(x)} = \begin{cases} x(x+1)(x+2)\cdots(x+m-1) & (m \in \mathbb{N}) \\ 1 & (m = 0). \end{cases}$$

Corresponding to the function $t_p(e_1, \cdots, e_q; d_1. \cdots, d_s; z)$ *given by*

$$t_p(e_1, \cdots, e_q; d_1, \cdots, d_s; z) = z^p\ {}_qF_s(e_1, \cdots, e_q; d_1, \cdots, d_s; z), \tag{1.3}$$

we introduce a function $t_{p,\alpha}(e_1, \cdots, e_q; d_1, \cdots, d_s; z)$ *defined by*

$$t_p(e_1, \cdots, e_q; d_1, \cdots, d_s; z) * t_{p,\alpha}(e_1, \cdots, e_q; d_1, \cdots, d_s; z) = \frac{z^p}{(1-z)^{\alpha+p}} \quad (\alpha > -p;\ z \in \mathbb{U}). \tag{1.4}$$

We now define a linear operator $\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1, \cdots, e_q; d_1, \cdots, d_s) : \mathcal{A}_p(n) \longrightarrow \mathcal{A}_p(n)$ *by*

$$\begin{aligned} &\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1, \cdots, e_q; d_1, \cdots, d_s) f(z) \\ &= t_{p,\alpha}(e_1, \cdots, e_q; d_1, \cdots, d_s; z) * f(z) \\ &= z^p + \sum_{k=n}^{\infty} \frac{(\alpha+p)_k}{k!} \times \Gamma_k^{-1}(e_1) a_{p+k} z^{p+k} \\ &= z^p + \sum_{k=n}^{\infty} \frac{(\alpha+p)_k (d_1)_k \cdots (d_s)_k}{(e_1)_k \cdots (e_q)_k} a_{p+k} z^{p+k} \\ &= z^p + \sum_{k=n}^{\infty} \eta_k(\alpha, e_1) a_{p+k} z^{p+k} \\ &= \mathcal{Q}^{n,\alpha}_{p,q,s}(e_1) f(z)\ (say) \\ &\left(e_i, d_j \in \mathbb{C} \setminus \mathbb{I}_0^-;\ i = 1, 2 \cdots, q;\ j = 1, 2, \cdots, s;\ \alpha > -p;\ f \in \mathcal{A}_p(n);\ z \in \mathbb{U}\right). \end{aligned} \tag{1.5}$$

We take a notational convention as

$$\frac{(\alpha+p)_k}{k!} \times \Gamma_k^{-1}(e_1) = \eta_k(\alpha, e_1) \text{ and } \mathcal{Q}^{1,\alpha}_{p,q,s}(e_1) = \mathcal{Q}^{\alpha}_{p,q,s}(e_1).$$

We can verify following two equality on $z \in \mathbb{U}$ *using* (1.5)

$$z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1) f\right)'(z) = (\alpha+p)\mathcal{Q}^{n,\alpha+1}_{p,q,s}(e_1) f(z) - \alpha \mathcal{Q}^{n,\alpha}_{p,q,s}(e_1) f(z), \tag{1.6}$$

and

$$z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1+1) f\right)'(z) = e_1 \mathcal{Q}^{n,\alpha}_{p,q,s}(e_1) f(z) - (e_1 - p)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1+1) f(z). \tag{1.7}$$

We shall see the operator $\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)$ *generalizes many operators*

$\mathfrak{A}$. *For* $s = 1, q = 2;\ e_2 = 1$ *we get* $\mathcal{Q}^{n,\alpha}_{p,2,1}(e_1)f(z) = \mathcal{I}^{\alpha}_{p}(d_1,e_1)f(z)\ (e_1,d_1 \in \mathbb{R}\setminus\mathbb{I}^{-}_{0}\ ;\alpha > -p)$. *The operator (Cho-Kwon-Srivatava)was introduced by cho et al.[8].*

i $\mathcal{I}^{1}_{p}(n+p,1) = \mathcal{I}_{n,p}\ (n > -p)$ *is the integral operator of order* $n+p-1$ *which was studied by Liu and Noor [11](see also [9, 10]). Choi et al.,[12] introduced* $\mathcal{I}^{\alpha}_{1}(\mu+2,1)\ (\alpha > -1;\mu > -2)$.

ii . $\mathcal{I}^{1}_{p}(p+1,1)f(z) = \mathcal{I}^{\lambda}_{p}(\lambda+p,1)f(z) = f(z)$.

iii . $\mathcal{I}^{1}_{p}(p,1)f(z) = \dfrac{zf'(z)}{p}$.

iv . $\mathcal{I}^{n}_{p}(a,a)f(z) = \mathcal{D}^{n+p-1}f(z)\ (n > -p)$ *(see Goel and Sohi [26]).*

v . $\mathcal{I}^{1-\mu}_{p}(p-\mu,p+1)f(z) = \Omega^{(\mu,p)}f(z)\ (-\infty < \mu < p+1)$ *(see Patel and Mishra [17]).*

vi . $\mathcal{I}^{\delta}_{p}(\delta+p+1,1)f(z) = \mathcal{F}_{\delta,p}(f)(z)\ (\delta > -p; z\in\mathbb{U})$, *the familiar Bernardi-Libera-Livingston integral operator (see, for example [12]).*

We note that for the function $f\in\mathcal{A}$, *the action of* $\mathcal{F}_{\delta,p}$ *as follows*

$$\mathcal{F}_{\delta,p}(f)(z) = \frac{\delta+p}{z^{\delta}}\int_0^z t^{\delta+p-1}f(t)dt \tag{1.8}$$

$$= z^p + \sum_{k=1}^{\infty}\frac{\delta+p}{\delta+k+p}a_{p+k}z^{p+k}\ (\delta > -p; z\in\mathbb{U}).$$

$\mathfrak{B}$ *In* $\mathcal{I}^{\alpha}_{p}(d_1,e_1)f(z)\ (e_1,d_1 \in \mathbb{R}\setminus\mathbb{I}^{-}_{0})$, *if* $\alpha = 1-p, e_2 = 1$ *will be taken then it will be the linear operator* $\mathcal{L}_p(d_1,e_1)$ *introduced by Saitoh [23] on* $\mathcal{A}_p(n)$. *The linear operator* $\mathcal{L}_1(d_1,e_1) = \mathcal{L}(d_1,e_1)$ *introduced by Carlson and Shaffer [25] in their systematic investigations of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions. We also note that for* $f\in\mathcal{A}_p$,

(i) $\mathcal{L}_p(e_1,e_1)f(z) = f(z)$;

(ii) $\mathcal{L}_p(p+1,p)f(z) = \dfrac{z^2f''(z)+2zf'(z)}{p(p+1)}$;

(iii) $\mathcal{L}_p(p+2,p)f(z) = \dfrac{zf'(z)}{p}$;

(iv) $\mathcal{L}_p(m+p,1)f(z) = D^{m+p-1}f(z)\ (m\in\mathbb{Z}, m > -p)$, *the operator studied by Goel and Sohi [26]. In the case* $p = 1$, $D^m f$ *is the familiar Ruscheweyh derivative [27] of* $f\in\mathcal{A}$.

(v) $\mathcal{L}_p(\nu+p,1)f(z) = D^{\nu,p}f(z)\ (\nu > -p)$ *an extended linear derivative operator of Rusheweyh type introduced by Raina and Srivastava [28]. In particular, when* $\nu = m$ *we get operator* $D^{m+p-1}f(z)\ (m\in\mathbb{Z}, m > -p)$, *studied by Goel and Sohi [26].*

(vi) $\mathcal{L}_p(p+1,m+p)f(z) = \mathcal{I}_{m,p}f(z)\ (m\in\mathbb{Z}, m > -p)$, *the extended Noor integral operator considered by Liu and Noor [18].*

(vii) $\mathcal{L}_p(p+1,p+1-\lambda)f(z) = \Omega^{(\lambda,p)}f(z)\ (-\infty < \lambda < p+1)$, *the extended fractional differintegral operator considered by Patel and Mishra [17]. Note that*

$$\Omega^{0,p}f(z) = f(z),\ \Omega^{1,p}f(z) = \frac{zf'(z)}{p}\ \text{and}\ \Omega^{2,p}f(z) = \frac{z^2f''(z)}{p(p-1)}\ (p\ge 2;\ \ z\in\mathbb{U}).$$

Cho et al.[8] established some inclusion relationships and argument properties for certain subclasses of $\mathcal{A}_p$, *which were defined in terms of their operator* $\mathcal{I}^{\alpha}_{p}(e_1,d_1)$ *(see also [16]). For the choices* $\alpha = d_1 = 1$ *and* $e_1 = n+p$, *the Cho-Kwon-Srivastava operator* $\mathcal{I}^{\alpha}_{p}(e_1,d_1)$ *reduces to the operator* $\mathcal{I}^{1}_{p}(n+p,1) = \mathcal{I}_{n,p}(n > -p)$, *where* $\mathcal{I}_{n,p}$ *is the integral operator studied by Liu and Noor [18] (for details, see [9] and [10]). Using Horadam polynomials*

Moslemi and Motamednezhad [38] construct a sub-class of bi-univalent functions and solve Fekete-Szegö problem of functions belonging to this family. The Choi-Saigo-Srivastava operator $\mathcal{I}_1^{\lambda}(\mu+2,1)(\alpha>-1;\mu>-2)$ was studied in [12]. The operator $\Omega^{(\mu,p)}$ for $0\le\mu<1$ was investigated by Srivastava and Aouf[20] and studied by Srivastava and Mishra [15]. Müge Sakar et al. [42] used the Noor type differential operator to define a new class $\Omega(\lambda,\alpha;\psi)$ of multivalent functions in open unit disk and find out some interesting applications of this operator for multivalent functions by using the method of convolution. Patel and Mishra [17] also studied certain classes of multivalent analytic functions involving the extended differintegral operator $\Omega^{(\mu,p)}$ when $-\infty<\mu<p+1$. We further observe that $\Omega^{(\mu,1)}=\Omega^{\mu}$ is the operator introduced and studied by Owa and Srivastava [21]. In [19] using Al-Oboudi differential operator, authors designed a subclass of analytic functions related to Poisson distribution series and studied various properties. Similarly in [47] and [51] different sub-classes of analytic functions are formed and Fekete-Szegö problem is studied in both the works. Aouf et al. [22] studied a certain subclass of $\mathcal{A}_p$ using the operator $\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)$ and investigated subordination results and coefficient bounds related to the class. Using the operator $\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)$ and the principle of subordination between analytic functions, we now define the following subclasses of $\mathcal{A}_p(n)$, p-valent functions. Srivastava et al. [2] used a hypergeometric function and designed a subclass of meromorphically multivalent functions, and studied the inclusion property of the designed subclass. Aldawish et al. [4], using a combination of confluent hypergeometric functions and the binomial series, created a subclass of multivalent functions in which a specific inclusion property was established. Verma et al. [3] designed a subclass $\mathcal{K}K_q$ of the class $\mathcal{K}_q$(the class of q-convex functions) and proved that class is closed under convolution with convex function.

Definition 1.1. A function $f\in\mathcal{A}_p(n)$ is said to be in class $S^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$, $0\le\beta\le1$ and $\alpha+p>0$, if it satisfies the following subordination condition:

$$\left|\frac{1}{b}\left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'+\beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)+\beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}-p\right)\right|<p-\rho \tag{1.9}$$

$$(e_j,d_j\in\mathbb{R}\setminus\mathbb{I}_0^-,\alpha>-p;z\in\mathbb{U}:0\le\rho<p,\ q,s,p\in\mathbb{N},\ q\le s+1).$$

Using the definition of subordination, it is easily seen that the condition (1.9) is equivalent to the following subordination relation,

$$\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'+\beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)+\beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}\prec p+b(p-\rho)z\quad(b\in\mathbb{C}^*,0\le\rho<p;z\in\mathbb{U}).$$

Definition 1.2. A function $f\in\mathcal{A}_p(n)$ is said to be in the class $\mathcal{R}^{b,q,s}_{p,n}(e_1,\alpha,\mu;\rho)$, if it satisfies the following inequality:

$$\left|\frac{1}{b}\left\{p(1-\mu)\frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)}{z^p}+\mu\frac{\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'(z)}{z^{p-1}}-p\right\}\right|<p-\rho \tag{1.10}$$

$$(b\in\mathbb{C}^*,0\le\mu\le1,0\le\rho<p;z\in\mathbb{U}).$$

It may be noted that by suitably choosing the parameters involved in Definition 1.1 and Definition 1.2, the classes $S^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$ and $\mathcal{R}^{b,q,s}_{p,n}(e_1,\alpha,\mu;\rho)$ extend several subclasses of p-valent analytic functions in $\mathbb{U}$. The following are the different subclasses of $S^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$:

P. $S^{b,q,s}_{p,n}(e_1,\alpha,0;\rho)=S^{b,q,s}_{p,n}(e_1,\alpha;\rho)$, *is the subclass of $\mathcal{A}_p(n)$ consists of functions, which are satisfies*

$$\left|\frac{1}{b}\left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)}-p\right)\right|<p-\rho \tag{1.11}$$

$$(e_j,d_j\in\mathbb{R}\setminus\mathcal{I}_0^-,\alpha>-p;z\in\mathbb{U}:0\le\rho<p).$$

Q. $S_{p,n}^{b,q,s}(e_1,\alpha,1;\rho) \ = \mathcal{K}_{p,n}^{b,q,s}(e_1,\alpha;\rho)$, *is the subclass of $\mathcal{A}_p(n)$ consists of functions, which are satisfies*

$$\left|\frac{1}{b}\left(1+\frac{z\left(\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)f(z)\right)''}{\left(\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)f(z)\right)'}-p\right)\right|<p-\rho \tag{1.12}$$

$$(e_j,d_j\in\mathbb{R}\setminus\mathbb{I}_0^-,\alpha>-p;z\in\mathbb{U}:0\le\rho<p).$$

R. $S_{p,n}^{b,2,1}(e_1,\alpha,\beta;\rho)=S_{p,n}^{b}(e_1,\alpha,\beta;\rho)=\{f\in\mathcal{A}_p(n):f$ *satisfies* (1.13)$\}$

$$\left|\frac{1}{b}\left(\frac{z\left(\mathcal{I}_p^{\alpha}(e_1,d_1)f(z)\right)'+\beta z^2\left(\mathcal{I}_p^{\alpha}(e_1,d_1)f(z)\right)''}{(1-\beta)\mathcal{I}_p^{\alpha}(e_1,d_1)f(z)+\beta z\left(\mathcal{I}_p^{\alpha}(e_1,d_1)f(z)\right)'}-p\right)\right|<p-\rho \tag{1.13}$$

$$(e_1,d_1\in\mathbb{R}\setminus\mathbb{I}_0^-,\alpha>-p;z\in\mathbb{U}:0\le\rho<p).$$

Using specification of the parameter as taken in $\mathfrak{B}$

T. $S_{p,n}^{b}(e_1,1-p,0;\rho)=S_{p,n}^{b}(e_1,d_1;\rho)=\{f\in\mathcal{A}_p(n):f$ *satisfies* (1.14)$\}$*[30]*

$$\left|\frac{1}{b}\left(\frac{z\left(\mathcal{L}_p(d_1,e_1)f(z)\right)'}{\mathcal{L}_p(d_1,e_1)f(z)}-p\right)\right|<p-\rho \tag{1.14}$$

$$(e_1,d_1\in\mathbb{R}\setminus\mathbb{I}_0^-,\alpha>-p;z\in\mathbb{U}:0\le\rho<p).$$

$S_{p,n}^{b}(e_1,d_1;\rho) \ =\{f\in\mathcal{A}_p(n):f$ *satisfies* (1.15)$\}$

$$\left|\frac{1}{b}\left(\frac{z\left(\mathcal{L}_p^{\alpha}(d_1,e_1)f(z)\right)'}{\mathcal{L}_p^{\alpha}(d_1,e_1)f(z)}-p\right)\right|<p-\rho \tag{1.15}$$

$$(e_1,d_1\in\mathbb{R}\setminus\mathbb{I}_0^-,\alpha>-p;z\in\mathbb{U}:0\le\rho<p).$$

The subclass defined in (1.14) *generalizes many subclasses of analytic functions as seen below*

Example 1.3. $\mathcal{S}_{p,n}^{b}(p+1,p+1-\lambda,\rho)=\mathcal{S}_{p,n}^{b}(\lambda,\rho)\ (b\in\mathbb{C}^*,-\infty<\lambda<p+1,0\le\rho<p)$

$$=\left\{f\in\mathcal{A}_p(n):\left|\frac{1}{b}\left(\frac{z(\Omega^{(\lambda,p)}f(z))'}{\Omega^{(\lambda,p)}f(z)}-p\right)\right|<p-\rho,z\in\mathbb{U}\right\},$$

which reduces to the class $\mathcal{K}_n(p,\lambda,b,\beta)\ \ (b\in\mathbb{C}^*,0\le\lambda\le1,0<\beta\le1)$ studied by Aouf [31] for $\rho=p-\beta$, the class

$$\mathcal{S}_{p,n}^{b}(\rho)=\left\{f\in\mathcal{A}_p(n):\left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right|<p-\rho,b\in\mathbb{C}^*,0\le\rho<p;z\in\mathbb{U}\right\}$$

for $\lambda=0$ and the class

$$\mathcal{C}_{p,n}^{b}(\rho)=\left\{f\in\mathcal{A}_p:\left|\frac{1}{b}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right|<p-\rho,b\in\mathbb{C}^*,0\le\rho<p;z\in\mathbb{U}\right\}.$$

for $\lambda=1$. The classes $\mathcal{S}_{p,n}^{b}(\rho)$ and $\mathcal{C}_{p,n}^{b}(\rho)$ are the subclasses of p-valently starlike and p-valently convex functions of complex order b and type $\rho\,(b\in\mathbb{C}^*,0\le\rho<p)$ in $\mathbb{U}$.

The following subclasses are derived from, Definition -1.2

U. Taking $\mu=0$ we get
$\mathcal{R}_{p,n}^{b,q,s}(e_1,\alpha,0;\rho)=\mathcal{R}_{p,n}^{b,q,s}(e_1,\alpha;\rho)$ *is the class of $f\in\mathcal{A}_p(n)$ which satisfy*

$$\left|\frac{1}{b}\left\{p\frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)f(z)}{z^p}-p\right\}\right|<p-\rho \tag{1.16}$$

$$(b\in\mathbb{C}^*,0\le\mu\le1,0\le\rho<p;z\in\mathbb{U}).$$

V. Upon putting $\mu = 1$ in the above class we get $\mathcal{R}_{p,n}^{b,q,s}(e_1, \alpha, 1, \rho) = \mathcal{C}_{p,n}^{b,q,s}(e_1, \alpha, \rho)$, consists of functions whose members satisfies the following inequality:

$$\left| \frac{1}{b} \left\{ \frac{(\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1) f(z))'}{z^{p-1}} - p \right\} \right| < p - \rho \tag{1.17}$$

$$(b \in \mathbb{C}^*, 0 \le \mu \le 1, 0 \le \rho < p; z \in \mathbb{U}).$$

W. Upon taking specification as mentioned in $\mathfrak{B}$ we get the class $\mathcal{R}_{p,n}^{b,q,s}(d_1, e_1, \mu, \rho)$ member of this class satisfies the following inequality:

$$\left| \frac{1}{b} \left\{ p(1-\mu) \frac{\mathcal{L}_p^{\alpha}(d_1, e_1) f(z)}{z^p} + \mu \frac{\left(\mathcal{L}_p^{\alpha}(d_1, e_1) f(z)\right)'(z)}{z^{p-1}} - p \right\} \right| < p - \rho$$

$$(b \in \mathbb{C}^*, 0 \le \mu \le 1, 0 \le \rho < p; z \in \mathbb{U}).$$

The subclass mentioned in W generalizes many other subclasses of $p-$valent functions as given below.

Example 1.4. $\mathcal{R}_{p,n}^{b}(p+1, p+1-\lambda, \mu, \rho) = \mathcal{R}_{p,n}^{b}(\lambda, \mu, \rho)$ $(b \in \mathbb{C}^*, -\infty < \lambda < p, 0 \le \mu)$, which yields the class considered by Aouf [31] for $\rho = p - \beta \quad (0 < \beta \le 1,\ 0 \le \rho < p)$.

Special cases of the parameters p, λ and ρ in the class $\mathcal{R}_{p,n}^{b}(\lambda, \mu, \rho)$ yields the following subclasses of $\mathcal{A}_p$.
(i) $\mathcal{R}_{p,n}^{b}(0, \mu, \rho) = \mathcal{R}_{p,n}^{b}(\mu, \rho)$

$$= \left\{ f \in \mathcal{A}_p : \left| \frac{1}{b} \left(p(1-\mu) \frac{f(z)}{z^p} + \mu \frac{f'(z)}{z^{p-1}} - p \right) \right| < p - \rho, \mu \ge 0, 0 \le \rho < p; z \in \mathbb{U} \right\}.$$

(ii) $\mathcal{R}_{p,n}^{b}(1, \mu, \rho) = \mathcal{P}_{p,n}^{b}(\mu, \rho)$

$$= \left\{ f \in \mathcal{A}_p : \left| \frac{1}{b} \left((\mu + \mu(1-p)) \frac{f'(z)}{pz^{p-1}} + \mu \frac{f''(z)}{pz^{p-2}} - p \right) \right| < p - \rho,\ \mu \ge 0, 0 \le \rho < p; z \in \mathbb{U} \right\}.$$

(iii) $\mathcal{R}_{1,n}^{b}(1, \mu, 1-\beta) = \mathcal{R}_{n}^{b}(\mu, \beta)$

$$= \left\{ f \in \mathcal{A}_p : \left| \frac{1}{b} \left(f'(z) + \mu z f''(z) - 1 \right) \right| < \beta, \mu \ge 0, 0 < \beta \le 1; z \in \mathbb{U} \right\}.$$

The class $\mathcal{R}_n^b(\mu, \beta)$ was studied by Altintas et al.[32]. Kumar and Srivastav[34] designed a subclass of multivalent functions using a differintegral operator, which can be generalized using the operator $\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1) f(z)$. They found coefficient bounds, inclusion relations, and particular neighborhoods of subclasses of analytic and multivalent functions with negative coefficients, Hadamard products, and integral means. Sarah A. Al-Ameedee et al.[35] used the generalized hypergeometric function (1.2) and studied various strong differential subordinations of higher order. Ali et al.[1] used the generalized hypergeometric function (1.2) to define a subclass of p-valent functions using convex combinations and studied various properties. El-Ashwah et al.[36] studied coefficient inequalities, growth and distortion theorems, and extreme points of different subclasses of multivalent functions with respect to symmetric points. Using the generalized Libera integral operator and the principle of Aouf et al.[43], a subclass of multivalent functions was defined, and different sufficient conditions for membership were established. The properties of the operator in connection with that subclass were also studied.
Let $\mathcal{T}_p(n)$ be the subclass of $\mathcal{A}_p(n)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (a_{p+k} \ge 0; p, n \in \mathbb{N}). \tag{1.18}$$

We write $\mathcal{T}_1(1) = \mathcal{T}$. We denote by $\widetilde{S}p, n^{b,q,s}(e_1, \alpha, \beta; \rho)$ and $\widetilde{\mathcal{R}}p, n^{b,q,s}(e_1, \alpha, \mu; \rho)$, respectively, the classes obtained by taking the intersections of $S_{p,n}^{b,q,s}(e_1, \alpha, \beta; \rho)$ and $\mathcal{R}_{p,n}^{b,q,s}(e_1, \alpha, \mu; \rho)$ with

$\mathcal{T}_p(n)$. In the present investigation, we introduce and study the subclasses $S_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$, $\mathcal{R}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho)$, $\widetilde{S}p,n^{b,q,s}(e_1,\alpha,\beta;\rho)$, and $\widetilde{\mathcal{R}}p,n^{b,q,s}(e_1,\alpha,\mu;\rho)$ of $\mathcal{A}_p(n)$ involving the operator $\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)$. We first find a necessary and sufficient condition for a function in $\mathcal{T}_p(n)$ to be in the subclasses $\widetilde{S}p,n^{b,q,s}(e_1,\alpha,\beta;\rho)$ and $\widetilde{\mathcal{R}}p,n^{b,q,s}(e_1,\alpha,\mu;\rho)$, respectively. These conditions are then used to establish some inclusion relationships involving these classes. We also obtain several neighborhood properties for functions belonging to the classes $S_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$, $\widetilde{S}p,n^{b,q,s}(e_1,\alpha,\beta;\rho)$, and $\widetilde{\mathcal{R}}p,n^{b,q,s}(e_1,\alpha,\mu;\rho)$. Certain results on neighborhoods involving quasi-Hadamard products have also been established for the class $\widetilde{S}p,n^{b,q,s}(e_1,\alpha,\beta;\rho)$. Finally, subordination results involving the operator $\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)$ are obtained.

2 Preliminary Lemmas

In this section, we give some lemmas which are required in the proof of our main results. First, we give the following definition

Definition 2.1. A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence, if for any $g(z)=z+\sum_{k=2}^{\infty} d_k z^k \in \mathcal{C}$

$$\sum_{k=1}^{\infty} c_k d_k z^k \prec g(z) \qquad (d_1=1; z\in\mathbb{U}).$$

Lemma 2.2. *([37]) A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence, if and only if*

$$Re\left\{1+2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \qquad (z\in\mathbb{U}). \tag{2.1}$$

We need the following lemmas to establish our main results.

Lemma 2.3. *([50]). Let $\widetilde{\beta}\in\mathbb{C}$ and $\widetilde{\gamma}\in\mathbb{C}^*$. Let q be a convex univalent function in $\mathbb{U}$ such that*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\widetilde{\beta}}{\widetilde{\gamma}}\right)\right\}.$$

If ϕ is analytic in $\mathbb{U}$ with $\phi(0)=q(0)$ and

$$\widetilde{\beta}\phi(z)+\widetilde{\gamma}z\phi'(z) \prec \widetilde{\beta}q(z)+\widetilde{\gamma}zq'(z) \quad (z\in\mathbb{U}),$$

then

$$\phi(z)\prec q(z) \quad (z\in\mathbb{U}) \tag{2.2}$$

and the function q is the best dominant of (2.2).

Lemma 2.4. *([52]). Let q be univalent in $\mathbb{U}$, and let θ and ϕ be analytic in a domain Ω containing $q(\mathbb{U})$ with $\phi(w)\neq 0$ for $w\in q(\mathbb{U})$. Set $Q(z)=zq'(z)\phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*
(i) Q is univalent starlike in $\mathbb{U}$,
(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z\in\mathbb{U})$.
If g is analytic in $\mathbb{U}$ with $g(0)=q(0), g(\mathbb{U})\subseteq\Omega$ and

$$\theta(g(z))+zg'(z)\phi(g(z)) \prec \theta(q(z))+zq'(z)\phi(q(z)) \quad (z\in\mathbb{U}),$$

then

$$g(z)\prec q(z) \quad (z\in\mathbb{U}) \tag{2.3}$$

and the function q is the best dominant of (2.3).

3 Derivation of Sufficient Condition

Now, we derive a necessary and sufficient condition for a function in $\mathcal{T}_p(n)$ to be in the class $\widetilde{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho)$ and $\widetilde{\mathcal{R}}^{b,q,s}_{p,n}(e_1, \alpha, \mu; \rho)$.

Lemma 3.1. *Let the function f, be given by (1.18). Then $f \in \widetilde{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho)$, if and only if*

$$\sum_{k=n}^{\infty} \frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\,|\eta_k(\alpha, e_1)|}{(p-\rho)|b|\left\{(1+\beta(p-1)) - \sum\limits_{k=n}^{\infty}(1+\beta(p+k-1))\,|\eta_k(\alpha, e_1)|\,a_{p+k}\right\}} a_{p+k} \leq 1. \quad (3.1)$$

The result is sharp for

$$f_k(z) = z^p - \frac{(p-\rho)|b|}{(1+\beta(p+k-1))(k+(p-\rho)|b|)\,|\eta_k(\alpha, e_1)|} z^{p+k} \ (k \geq n, e_i, d_j \in \mathbb{R}\backslash\mathbb{Z}_0^-; z \in \mathbb{U}).$$

.

Proof. Since f given by (1.18) and is in $\widetilde{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho)$. Then from (1.9), it follows that

$$\mathrm{Re}\left\{\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} - p\right\} > -|b|(p-\rho) \ \ (0 \leq \rho; z \in \mathbb{U}),$$

using series expansion of (1.5)in the above expression, we get

$$\mathrm{Re}\left\{\frac{-\sum\limits_{k=n}^{\infty} k(1+\beta(p+k-1))\eta_k(\alpha, e_1)a_{p+k}z^k}{(1+\beta(p-1)) - \sum\limits_{k=n}^{\infty}(1+\beta(p+k-1))\eta_k(\alpha, e_1)a_{p+k}z^k}\right\} > -(p-\rho)|b| \qquad (z \in \mathbb{U}). \quad (3.2)$$

Setting $|z| = r(0 \leq r < 1)$ in (3.2) and noting the fact that for $r = 0$, the resulting expression in the denominator is positive, and remains so for all $r \in (0, 1)$, the desired inequality (3.1) follows upon letting $r \to 1^-$ through real values.
Conversely, letting $|z| = 1$, we find from (1.5) that

$$\left|\left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} - p\right)\right| \quad (3.3)$$

$$= \left\{\left|\frac{-\sum\limits_{k=n}^{\infty} k(1+\beta(p+k-1))\eta_k(\alpha, e_1)a_{p+k}z^k}{(1+\beta(p-1)) - \sum\limits_{k=n}^{\infty}(1+\beta(p+k-1))\eta_k(\alpha, e_1)a_{p+k}z^k}\right|\right\}$$

$$\leq$$

$$\left\{\frac{\sum\limits_{k=n}^{\infty} k(1+\beta(p+k-1))\,|\eta_k(\alpha, e_1)|\,a_{p+k}z^k}{(1+\beta(p-1)) - \sum\limits_{k=n}^{\infty}(1+\beta(p+k-1))\,|\eta_k(\alpha, e_1)|\,a_{p+k}z^k}\right\}.$$

The expression in (3.3) is bounded by $(p-\rho)|b|$ provided

$$\sum_{k=n}^{\infty} k(1+\beta(p+k-1))\,|\eta_k(\alpha, e_1)|\,a_{p+k}$$
$$\leq (p-\rho)|b|\left\{(1+\beta(p-1)) - \sum_{k=n}^{\infty}(1+\beta(p+k-1))\,|\eta_k(\alpha, e_1)|\,a_{p+k}\right\}$$

which can be justified by using maximum modulus theorem and assertion (3.1). Using (1.9), we deduce that $f \in \widetilde{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho)$ which complete the proof of lemma 3.1. □

Remark 3.2. R1. Using result of lemma 3.1 in the class $S^b_{p,n}(d_1, e_1; \rho)$ as define in (1.15) we get $\sum_{k=n}^{\infty} \frac{k + (p-\rho)|b|}{(p-\rho)|b|} \left| \frac{(a)_k}{(c)_k} \right| a_{p+k}$ here a substitution $e_1 = c,\ d_1 = a$ are taken to make a notational match with the result obtained in lemma-2 of [30].

R1a. For $a = \lambda + 1,\ (\lambda > 1)$ and $p = n = b = c = 1$ we get the result obtained by [5].

R1b. For $a = 2, n = p = b = 1$, we set $c = 2$ and $c = 1$, we4 shall obtain the familiar results of Silverman [24].

R1c. Putting $a = p+1, c = p+1-\lambda\,(-\infty < \lambda < p+1)$ and $\rho = (p-\beta)(0 < \beta \le 1)$, we get the result obtained by Aouf[31, Lemma 1].

R1d. With $a = p+1, n = b = 1$, if we set $c = p+1$ and $c = p$ respectively, we obtain the results of Owa [40].

R1e. Setting $a = \lambda + 1\ (\lambda > -1), c = p = 1$ and $\rho = 1 - \beta\,(0 < \beta \le 1)$, we get the result obtained by Murugusundarmoorthy and Srivastava [39].

Our proof of Lemma 3.3 given below is much akin to that of Lemma 3.1, so we omit the details.

Lemma 3.3. *Let the function f be given by (1.18). Then $f \in \widetilde{\mathcal{R}}^{b,q,s}_{p,n}(e_1, \alpha, \mu; \rho)$, if and only if*

$$\sum_{k=n}^{\infty} \frac{(p+\mu k)}{(p-\rho)|b|} |\eta_k(\alpha, e_1)| a_{p+k} \le 1. \tag{3.4}$$

The result is sharp for the functions

$$f_k(z) = z^p - \frac{(p-\rho)|b|}{(p+\mu k)|\eta_k(\alpha, e_1)|} z^{p+k}\ (k \ge n; z \in \mathbb{U}).$$

Remark 3.4. R2. Using parameter specification as in W we get the above result as

$$\sum_{k=n}^{\infty} \frac{(p+\mu k)}{(p-\rho)|b|} \frac{|(a)_k|}{|(c)_k|} a_{p+k} \le 1. \tag{3.5}$$

where a substitution $d_1 = a,\ e_1 = c$ are taken to make a notational match with the result obtained in lemma-3 of [30].

R2a. With $a = p+1, b = n = 1$, we set $c = p+1$ and $c = p$, we shall obtain the results by Lee et al. [41, Lemma 2] and Aouf [44, Theorem 1], respectivlely.

R2b. Putting $a = \lambda + 1, c = p = 1$ and $\rho = 1 - \beta(0 < \beta \le 1)$, we get the result due to Murugusundarmoorthy and Srivastava [39, Lemma 2].

R2c. When $a = p+1, c = p+1-\lambda\,(-\infty < \lambda < p+1)$ and $\rho = p(1-\beta)\,(0 < \beta \le 1)$, we get the corresponding result obtained by Aouf [31, Lemma 2].

4 Containment Relation

In this section, we derive inclusion relationships involving the classes $\widetilde{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta, \rho)$ and $\widetilde{\mathcal{R}}^{b,q,s}_{p,n}(e_1, \alpha, \mu, \rho)$.
Unless otherwise mentioned, we assume throughout the sequel that $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},\ e_i, d_j > 0,\ i = 1, \cdots q,\ j = 1, \cdots s,\ \alpha + p > 0,\ 0 \le \mu \le 1$ and $0 \le \rho \le 1$. We first prove

Theorem 4.1. *If*

$$\xi = p - \frac{e_1(p-\rho)}{(e_1+n) + (p-\rho)|b|},$$

then

$$\widetilde{S}^{b,q,s}_{p,n}(e_1+1, \alpha, \beta, \rho) \subset \widetilde{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta, \xi).$$

The result is the best possible.

Proof. Let the function f given by (1.18) be in the class $\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta,\rho)$.
Then by (3.1),

$$\sum_{k=n}^{\infty}\frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\,|\eta_k(\alpha,e_1+1)|}{(p-\rho)(1+\beta(p-1))|b|}a_{p+k}\leq 1. \tag{4.1}$$

To show that $f\in\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta,\xi)$, in view of (4.1), we need to find the best possible value of ξ such that

$$\frac{(1+\beta(p+k-1))(k+(p-\xi)|b|)\,|\eta_k(\alpha,e_1)|}{(p-\xi)|b|}a_{p+k}$$
$$\leq\frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\,|\eta_k(\alpha,e_1+1)|}{(p-\rho)|b|}a_{p+k}\quad(k\geq n)$$

which is equivalent to

$$\xi\leq p-\frac{e_1(p-\rho)}{e_1+k+(p-\rho)|b|}\qquad(k\geq n) \tag{4.2}$$

Since the right hand side of (4.2) is an increasing function of k, letting $k=n$ in (4.2), we get the required result.

It is easily seen that the result is the best possible for the function

$$f(z)=z^p-\frac{(p-\rho)|b|}{(n+(p-\rho)|b|)\eta_k(\alpha,e_1+1)}z^{p+n}\quad(z\in\mathbb{U}). \tag{4.3}$$

□

For $\alpha=1-p, q=2, s=1, e_2=1$ and $\beta=0$ we find the above result is consistent with Sahoo and Patel [30] which in turn slide to Silverman [24, Theorem 7].
Similarly, by using Lemma 3.3 we can prove the following result.

Theorem 4.2. *If*

$$\varsigma=p-\frac{a(p-\rho)}{a+n},$$

then

$$\widetilde{\mathcal{R}}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho)\subset\widetilde{\mathcal{R}}_{p,n}^{b,q,s}(e_1+1,\alpha,\mu;\rho).$$

The result is best possible for the function

$$f(z)=z^p-\frac{(p-\rho)|b|}{(p+\mu n)\eta_n(\alpha,e_1+1)}z^{p+n}\quad(z\in\mathbb{U}).$$

5 Neighborhood Properties

Following the earlier investigations by Goodman [48], Ruscheweyh [46] and others including Altintas and Owa [4], Altintas et al. ([49] and [32]), we now define and establish inclusion relations involving the (n,δ)-neighborhood of a function $f\in\mathcal{A}_p(n)$ given by (1.1) as follows:

$$\mathcal{T}_{n,\delta}(f)=\Bigg\{g\in\mathcal{A}_p(n):g(z)=z^p+\sum_{k=n}^{\infty}b_{p+k}z^{p+k}\text{ and} \tag{5.1}$$
$$\sum_{k=n}^{\infty}\frac{[1+\beta(p+k-1)](k+(p-\rho)|b|)}{(p-\rho)|b|(1+\beta(p-1))}\eta_k(\alpha,e_1)\,|b_{p+k}-a_{p+k}|\leq\delta;\delta>0\Bigg\},$$

$$\mathcal{N}_{n,\delta}(f)=\Bigg\{g\in\mathcal{A}_p(n):g(z)=z^p+\sum_{k=n}^{\infty}b_{p+k}z^{p+k}\text{ and} \tag{5.2}$$
$$\sum_{k=n}^{\infty}(p+k)[1+\beta(p+k-1)]\,|b_{p+k}-a_{p+k}|\leq\delta;\delta>0\Bigg\}.$$

In particular, for the identity function $e(z) = z^p$ $(p \in \mathbb{N}; z \in \mathbb{U})$ *, we immediately have*

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k} \text{ and } \sum_{k=n}^{\infty} [1 + \beta(p+k-1)](p+k)|b_{p+k}| \leq \delta; \delta > 0 \right\}. \tag{5.3}$$

We, further denote

$$\mathcal{T}^{+}_{n,\delta}(f) = \mathcal{T}_{n,\delta}(f) \cap \mathcal{T}_p(n), \quad \mathcal{N}^{+}_{n,\delta}(f) = \mathcal{N}_{n,\delta}(f) \cap \mathcal{T}_p(n) \quad \text{and} \quad \mathcal{N}^{+}_{n,\delta}(e) = \mathcal{N}_{n,\delta}(e) \cap \mathcal{T}_p(n).$$

To establish our results, we need the following lemma.

Lemma 5.1. *Let the function* $f \in \mathcal{A}_p(n)$ *be given by* (1.1) *Then* $f \in \mathcal{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho)$*, if and only if*

$$\begin{aligned} &\frac{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}{z^p} \\ &\star \left\{ \frac{(p-1)}{1-z} + \frac{1}{(1-z)^2} - \frac{(p+(p-\rho)b\,x}{1-z} \right\} \neq 0 \quad (0 < |z| < 1) \end{aligned} \tag{5.4}$$

Proof. From (1.9), it follows that

$$f \in \mathcal{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho) \Longleftrightarrow \left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2 \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} \right) \neq p + (p-\rho)b\,x \tag{5.5}$$

for all $z \in \mathbb{U}$ with $|x| = 1$ and $x \neq 1$. We have

$$\begin{aligned} &z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2 \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'' = \\ &(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' \star \left\{ \frac{(p-1)}{1-z} + \frac{1}{(1-z)^2} \right\} \end{aligned}$$

Since $\left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2 \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} \right)$ takes the value p at $z = 0$, (5.5) is equivalent to

$$\begin{aligned} &(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' \\ &\star \\ &\left\{ \frac{(p-1)z^p}{1-z} + \frac{z^p}{(1-z)^2} \right\} - (p+(p-\rho)b\,x) \left((1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z \left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' \right) \neq 0 \\ &\qquad (0 < |z| < 1) \end{aligned}$$

which reduces to our assertion (5.4). The converse part of the lemma5.1 follows easily by retracing back the steps that proved (5.4). The proof of Lemma 5.1 is completed. □

Theorem 5.2. *If* $f \in \mathcal{A}_p(n)$ *satisfies*

$$\frac{f(z) + \varepsilon z^p}{1+\varepsilon} \in \mathcal{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho) \quad (\varepsilon \in \mathbb{C}, |\varepsilon| < \delta; \delta > 0), \tag{5.6}$$

then

$$\mathcal{T}_{n,\delta}(f) \subset \mathcal{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho).$$

Proof. In view of Lemma 5.1, we note that a function $g \in \mathcal{S}^{b,q,s}_{p,n}(e_1, \alpha, \beta; \rho)$, if and only if

$$\frac{(g \star h)(z)}{z^p} \neq 0 \quad (z \in \mathbb{U}), \tag{5.7}$$

where for convenience

$$h(z)=z^p+\sum_{k=n}^{\infty}c_{p+k}z^{p+k} \text{ with } c_{p+k}=-\frac{[(\beta(p+k-1)+1](k-(p-\rho)b\,x)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)b\,x}$$
$$(|x|=1, x\neq 1, k\geq n).$$

It is easily seen that

$$|c_{p+k}|\leq \frac{[(\beta(p+k-1)+1](k+(p-\rho)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|} \quad (k\geq n).$$

Using (5.6) and (5.7), we deduce that

$$\frac{\dfrac{f(z)+\varepsilon z^p}{1+\varepsilon}*h(z)}{z^p}\neq 0 \quad (z\in\mathbb{U})$$

or, $(f\star h)(z)/z^p\neq -\varepsilon$, which is equivalent to

$$\left|\frac{(f*h)(z)}{z^p}\right|\geq\delta \quad (\delta>0; z\in\mathbb{U}). \tag{5.8}$$

Letting $g(z)=z^p+\sum_{k=n}^{\infty}b_{p+k}z^{p+k}\in\mathcal{T}_{n,\delta}(f)$, we find that

$$\left|\frac{((g-f)\star h)(z)}{z^p}\right|=\left|\sum_{k=n}^{\infty}(b_{p+k}-a_{p+k})c_{p+k}z^k\right|$$
$$\leq |z|^k\sum_{k=n}^{\infty}\frac{[(\beta(p+k-1)+1])(k+(p-\rho)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|}|b_{p+k}-a_{p+k}|<\delta \quad (z\in\mathbb{U}). \tag{5.9}$$

so that by using (5.8) and (5.9), we obtain

$$\left|\frac{(g*h)(z)}{z^p}\right|\geq\left|\frac{(f\star h)(z)}{z^p}\right|-\left|\frac{((g-f)\star h)(z)}{z^p}\right|>0 \quad (z\in\mathbb{U}).$$

Thus, in view of (5.8), we get $g\in\mathcal{S}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$ and this complete the proof of Theorem 5.2. □

Theorem 5.3. *If $f\in\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1+1,\alpha,\beta;\rho)$, then*

$$\mathcal{T}_{n,\delta_1}^{+}(f)\subset\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$$

where $\delta_1=n/(e_1+n)$. The result is the best possible in the sense that δ_1 cannot be increased.

Proof. Let f, given by (1.18) be in the class $\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1+1,\alpha,\beta;\rho)$. Then by (3.1), we have

$$\sum_{k=n}^{\infty}\frac{(k+(p-\rho)|b|)(1+\beta(p+k-1))}{(1+\beta(p-1))(p-\rho)|b|}\eta_k(\alpha,e_1+1)a_{p+k}\leq 1$$

so that

$$\sum_{k=n}^{\infty}\frac{(k+(p-\rho)|b|)(1+\beta(p+k-1))}{(1+\beta(p-1))(p-\rho)|b|}\eta_k(\alpha,e_1)a_{p+k}\leq\frac{e_1}{e_1+n}. \tag{5.10}$$

Assuming that the function $g\in\mathcal{T}_p(n)$ defined in $\mathbb{U}$ by

$$g(z)=z^p-\sum_{k=n}^{\infty}b_{p+k}z^{p+k} \quad (z\in\mathbb{U}), \tag{5.11}$$

is in the set $\mathcal{T}^{+}_{n,\delta_1}(f)$, we deduce from (5.1) that

$$\sum_{k=n}^{\infty}\frac{(k+(p-\rho)|b|)(1+\beta(p+k-1))}{(1+\beta(p-1))(p-\rho)|b|}\eta_k(\alpha,e_1)|b_{p+k}-a_{p+k}|\leq\delta_1. \tag{5.12}$$

Now, with the aid of (5.10) and (5.12), we obtain

$$\begin{aligned}&\sum_{k=n}^{\infty}\frac{(k+(p-\rho)|b|)(1+\beta(p+k-1))}{(1+\beta(p-1))(p-\rho)|b|}\eta_k(\alpha,e_1)b_{p+k}\\&\leq\sum_{k=n}^{\infty}\frac{(k+(p-\rho)|b|)(1+\beta(p+k-1))}{(1+\beta(p-1))(p-\rho)|b|}\eta_k(\alpha,e_1)|b_{p+k}-a_{p+k}|\\&+\sum_{k=n}^{\infty}\frac{(k+(p-\rho)|b|)(1+\beta(p+k-1))}{(1+\beta(p-1))(p-\rho)|b|}\eta_k(\alpha,e_1)a_{p+k}\\&\leq\frac{e_1}{e_1+n}+\delta_1=1,\end{aligned}$$

which shows that $g\in\widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$. To see that the result is the best possible, we consider the function f given by (4.3) and the function g defined by

$$\begin{aligned}g(z)=&z^p-\left[\frac{(p-\rho)|b|(1+\beta(p-1))}{(n+(p-\rho)|b|)(1+\beta(p+n-1)\eta_n(\alpha,e_1+1)}\right.\\&\left.+\frac{(p-\rho)|b|(1+\beta(p-1))\delta'}{(n+(p-\rho)|b|)(1+\beta(p+n-1)\eta_n(\alpha,e_1)}\right]z^{p+n}\quad(\delta'>\delta_1;z\in\mathbb{U}).\end{aligned}$$

It is easily verified that $f\in\widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1+1,\alpha,\beta;\rho)$, $g\in\mathcal{T}^{+}_{n,\delta'}(f)$, but $g\notin\widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$. This completes the proof of Theorem 5.3. □

In forthcoming derivation of various results, unless otherwise mentioned, we shall assume $d_i\geq e_j>0$ $i=1,\cdots q,\ j=1,\cdots s,\ q\leq s.$

Theorem 5.4. *If* $|b|<p/(p-\rho)$ *then*

$$\widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)\subset\mathcal{N}^{+}_{n,\delta_3}(e),$$

where

$$\delta_3=\frac{(p+n)(p-\rho)|b|(1+\beta(p-1))}{(n+(p-\rho)|b|)\eta_n(\alpha,e_1)}.$$

Proof. For a function $f\in\widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$ of the form (1.18), the assertion (3.1) immediately yields

$$(n+(p-\rho)|b|)\eta_n(\alpha,e_1)\sum_{k=n}^{\infty}\frac{(1+\beta(p+k-1))}{1+\beta(p-1)}a_{p+k}\leq(p-\rho)|b|,$$

so that

$$\sum_{k=n}^{\infty}(1+\beta(p+k-1))a_{p+k}\leq\frac{(p-\rho)|b|(1+\beta(p-1))}{(n+(p-\rho)|b|)\eta_n(\alpha,e_1)}. \tag{5.13}$$

Making use of (3.1) again, in conjunction with (5.13), we get

$$\begin{aligned}&\eta_n(\alpha,e_1)\sum_{k=n}^{\infty}(p+k)(1+\beta(p+k-1))a_{p+k}\\&\leq(p-\rho)|b|(1+\beta(p-1))+(p-(p-\rho)|b|)\eta_n(\alpha,e_1)\sum_{k=n}^{\infty}(1+\beta(p+k-1))a_{p+k}\\&\leq(p-\rho)|b|(1+\beta(p-1))+(p-(p-\rho)|b|)\frac{(p-\rho)|b|(1+\beta(p-1))}{(n+(p-\rho)|b|)}\\&=\frac{(n+p)(1+\beta(p-1))(p-\rho)|b|}{n+(p-\rho)|b|}\end{aligned}$$

that is,

$$\sum_{k=n}^{\infty}(1+\beta(p+k-1))(p+k)a_{p+k} \leq \frac{(n+p)(p-\rho)|b|(1+\beta(p-1))}{(n+(p-\rho)|b|)\eta_n(\alpha,e_1)} = \delta_3,$$

which, in view of (5.3) establishes the inclusion relation asserted by Theorem 5.4. □

Remark 5.5. Using specifications of various parameters as mentioned earlier, it was verified that the result is consistent with the subclass $S^{b}_{p,n}(d_1,e_1;\rho)$ defined in (1.14) [30], which in turn generalizes the following results.

(i) When $p=b=1, e_1=d_1$ and $p=b=e_1=1, d_1=2$, respectively, we get the results due to Altintaş and Owa [49, Theorem 2.1 and Theorem 2.2].

(ii) For $d_1=\lambda+1$ $(\lambda>-1)$, $e_1=p=1$, and $\rho=1-\beta$ $(0<\beta\leq 1)$, this corresponds to a result of Murugusundaramoorthy and Srivastava [39, Theorem 1].

(iii) If we set $d_1=p+1$, $e_1=p+1-\lambda$ $(0\leq\lambda\leq 1)$, and $\rho=p-\beta$ $(0<\beta\leq 1)$, we get the result of Aouf [31, Theorem 1].

(iv) If in the result of Aouf [31, Theorem 1], we set $p=1$, $e_1=d_1$, and $\rho=\beta$ $(0<\beta\leq 1)$, we get the corresponding work of Altintaş et al. [33, Theorem 1].

Using (3.4) we can prove the following theorem

Theorem 5.6. *If* $|b|<p/(p-\rho)$ *then*

$$\widetilde{\mathcal{R}}^{b,q,s}_{p,n}(e_1,\alpha,\mu,\rho) \subset \mathcal{N}_{n,\delta_4}(e),$$

where

$$\delta_4 = \frac{(p+n)(p-\rho)|b|}{(p+\mu n)\eta_n(\alpha,e_1)}.$$

Remark 5.7. Using the specification of the parameter as in W, the result is consistent with Theorem 8 of [30], which further simplifies the following results.

(i) If $d_1=\lambda+1$ $(\lambda>-1)$, $e_1=p=1$, and $\rho=1-\beta$ $(0<\beta\leq 1)$, then we get the result obtained by Murugusundaramoorthy and Srivastava [39, Theorem 2].

(ii) A result due to Aouf [31, Theorem 2] can also be deduced by setting $d_1=p+1$, $e_1=p+1-\lambda$ $(0\leq\lambda\leq 1)$, and $\rho=\beta$ $(0<\beta\leq 1)$.

6 Results Involving Modified Hadamard Product

In the following results, we find some inclusion relationships involving modified Hadamard products.

For functions f *given by* (1.18) *and* g *given by* (5.11), *we define the modified Hadamard (or quasi-Hadamard) product of* f *and* g *by*

$$(f\star_q g)(z) = z^p - \sum_{k=n}^{\infty} a_{p+k}b_{p+k}z^{p+k} = (g\star_q f)(z) \quad (p,n\in\mathbb{N}; z\in\mathbb{U}).$$

Using this notion of modified Hadamard product, for subsets $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *of* $\mathcal{T}_p(n)$, *we denote*

$$\mathcal{E}_1\otimes\mathcal{E}_2 = \{f\star_q g : f\in\mathcal{E}_1 \text{ and } g\in\mathcal{E}_2\}.$$

We now prove

Theorem 6.1. *If* $e(z)=z^p$, *then*

(i) $\mathcal{T}^{+}_{n,\delta_5}(e)\otimes\mathcal{T}^{+}_{n,\delta_5}(e) \subset \widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$, *where* $\delta_5=\sqrt{\{(n+(p-\rho)|b|)\eta_n(\alpha,e_1)\}/\{(p-\rho)|b|\}}$ *and (ii)* $\mathcal{T}^{+}_{n,\delta_6}(e)\otimes\mathcal{N}^{+}_{n,\delta_6}(e) \subset \widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$, *where* $\delta_6=\sqrt{p+n}$. *The result, in each case is the best possible.*

Proof. Let f be given by (1.18) and g be given by (5.11). Suppose that $f, g \in \mathcal{T}^{+}_{n,\delta_5}(e)$. Then by (5.1), we have

$$\sum_{k=n}^{\infty} \frac{[1+\beta(p+k-1)](k+(p-\alpha)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|} a_{p+k} \leq \delta_5$$

and

$$\sum_{k=n}^{\infty} \frac{[1+\beta(p+k-1)](k+(p-\rho)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|} b_{p+k} \leq \delta_5. \tag{6.1}$$

For $d_i \geq a_i > 0$, we note that $[1+\beta(p+k-1)](k+(p-\rho)|b|)\eta_k(\alpha,e_1)$ is an increasing function of $k(k \geq n)$ so that the first inequality in (6.1) immediately yields

$$\sum_{k=n}^{\infty} a_{p+k} \leq \frac{(1+\beta(p-1))(p-\rho)|b|\delta_5}{[1+\beta(p+n-1)](n+(p-\rho)|b|)\eta_n(\alpha,e_1)}$$

which implies that

$$a_{p+k} \leq \frac{(1+\beta(p-1))(p-\rho)|b|\delta_5}{[1+\beta(p+n-1)](n+(p-\rho)|b|)\eta_n(\alpha,e_1)} \quad (k \geq n). \tag{6.2}$$

Using (6.2) and (6.1), we get

$$\sum_{k=n}^{\infty} \frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|)} a_{p+k} b_{p+k}$$
$$\leq \frac{(1+\beta(p-1))(p-\rho)|b|\delta_5^2}{[1+\beta(p+n-1)](n+(p-\rho)|b|)\eta_n(\alpha,e_1)} = 1$$

which again in view of the assertion (3.1) implies that $(f \star_q g) \in \widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$.

To see that the result in part (i) is the best possible, we consider the functions f and g defined by

$$f(z) = g(z) = z^p - \sqrt{\frac{(1+\beta(p-1))(p-\rho)|b|}{[1+\beta(p+n-1)](n+(p-\rho)|b|)\eta_n(\alpha,e_1)}}\, z^{p+n} \quad (a \geq c > 0; z \in \mathbb{U}).$$

Clearly, $f, g \in \mathcal{T}^{+}_{n,\delta_5}(e)$ and $(f \star_q g) \in \widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$. This proves part (i) of the theorem.

To prove part (ii), we assume that $f \in \mathcal{T}^{+}_{n,\delta_6}(e)$ and $g \in \mathcal{N}^{+}_{n,\delta_6}(e)$. Then

$$\sum_{k=n}^{\infty} \frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|} a_{p+k} \leq \delta_6 \quad \text{and} \quad \sum_{k=n}^{\infty}(1+\beta(p+k-1))(p+k) b_{p+k} \leq \delta_6.$$

Thus, $b_{p+k} \leq \delta_6/(p+k)(1+\beta(p+k-1))$ for $k \geq n$ and

$$\sum_{k=n}^{\infty} \frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\eta_k(\alpha,e_1)}{(1+\beta(p-1))(p-\rho)|b|} a_{p+k} b_{p+k} \leq \frac{\delta_6^2}{(1+\beta(p+n-1))(p+n)} = 1,$$

which in view of (3.4) implies that $(f \star_q g) \in \widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$. Considering the functions f, g defined in $\mathbb{U}$ by

$$f(z) = z^p - \frac{(p-\rho)|b|\sqrt{1+\beta(p+n)}}{(n+(p-\rho)|b|)\eta_n(\alpha,e_1)(1+\beta(p+n-1))} z^{p+n} \; (a \geq c > 0)$$

and

$$g(z) = z^p - \frac{z^{p+n}}{\sqrt{1+\beta(p+n-1)(p+n)}},$$

it is easily seen that $f \in \mathcal{T}^{+}_{n,\delta_6}(e), g \in \mathcal{N}^{+}_{n,\delta_6}(e)$ and $(f \star_q g) \in \widetilde{\mathcal{S}}^{b,q,s}_{p,n}(e_1,\alpha,\beta;\rho)$ thus, the result in part (ii) is the best possible. This completes the proof of Theorem 6.1. □

We now define two subclass of $\mathcal{T}_p(n)$ as follows

$$\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho,\kappa)=\left\{f\in\mathcal{T}_p(n):\left|\frac{f(z)}{g(z)}-1\right|<\kappa\right\}$$

for some $g\in\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$, . Similarly

$$\widetilde{\mathcal{R}}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho,\kappa)=\left\{f\in\mathcal{T}_p(n):\left|\frac{f(z)}{g(z)}-1\right|<\kappa\right\}$$

for some $g\in\widetilde{\mathcal{R}}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho)$ $0<\kappa\leq 1,\ z\in\mathbb{U}$.

Theorem 6.2. *If $g\in\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$ then*

$$N_{n,\delta_7}^{+}(g)\subset\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho,\kappa).$$

where

$$\delta_7=(1+\beta(p+n-1))\kappa\left\{1-\frac{(p-\rho)|b|}{(n+(p-\rho)|b|)\eta_n(\alpha,e_1)}\right\}.$$

Proof. Suppose f defined by (1.18) belongs to the set $N_{n,\delta_7}^{+}(g)$, where g, is given by (5.11). Then

$$\sum_{k=n}^{\infty}(p+k)(1+\beta(p+k-1))|a_{p+k}-b_{p+k}|\leq\delta_7$$

which readily implies that

$$\sum_{k=n}^{\infty}|a_{p+k}-b_{p+k}|\leq\frac{\delta_7}{(1+\beta(p+n-1))(p+n)}.$$

Since $g\in\widetilde{\mathcal{S}}_{p,n}^{b,q,s}(e_1,\alpha,\beta;\rho)$, we have from Lemma 3.1

$$\sum_{k=n}^{\infty}\frac{(1+\beta(p+k-1))(k+(p-\rho)|b|)\,|\eta_k(\alpha,e_1)|}{(p-\rho)|b|(1+\beta(p+k-1))}b_{p+k}\leq 1.$$

$$\Rightarrow\sum_{k=n}^{\infty}b_{p+k}\leq\frac{(p-\rho)|b|(1+\beta(p+k-1))}{(1+\beta(p+n-1))(n+(p-\rho)|b|)\eta_n(\alpha,e_1)}.$$

Now we shall find

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right|&<\frac{\sum_{k=n}^{\infty}|a_{p+k}-b_{p+k}|}{1-\sum_{k=n}^{\infty}b_{p+k}}\\&\leq\frac{(n+(p-\rho)|b|)[\eta_n(\alpha,e_1)-1]\delta_7}{(1+\beta(p+n-1))\{(n+(p-\rho)|b|[\eta_n(\alpha,e_1)-1](p+n)-(p-\rho)|b|\}}\\&=\kappa\quad(z\in\mathbb{U}).\end{aligned}$$

and the proof of Theorem 6.2 is completed. □

Theorem 6.3. *If $g\in\widetilde{\mathcal{R}}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho)$ then*

$$N_{n,\delta_8}^{+}(g)\subset\widetilde{\mathcal{R}}_{p,n}^{b,q,s}(e_1,\alpha,\mu;\rho,\lambda),$$

where

$$\delta_8=\frac{(1+\beta(p+n-1))\lambda\{(p+\mu\,n)\eta_n(\alpha,e_1)-(p-\rho)|b|\}(p+n)}{(p+\mu\,n)\eta_n(\alpha,e_1)}.$$

7 Subordination results

In this section, we derive some subordination results for certain classes of functions in $\mathcal{A}_p$ involving the operator $\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)f(z)$.

Theorem 7.1. *Let $\gamma,\mu \in \mathbb{C}^*$ and q be univalent in $\mathbb{U}$ such that*

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -aRe\left(\frac{\gamma}{\mu}\right)\right\}. \tag{7.1}$$

If $f \in \mathcal{A}_p$ satisfies the subordination relation:

$$(1-\mu)\left(\frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1+1)f(z)}{z^p}\right)^{\gamma} + \mu\left(\frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1+1)f(z)}{z^p}\right)^{\gamma}\frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)f(z)}{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1+1)f(z)} \tag{7.2}$$
$$\prec q(z) + \frac{\mu}{a\gamma}zq'(z) \quad (z \in \mathbb{U}),$$

then

$$\left(\frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1+1)f(z)}{z^p}\right)^{\gamma} \prec q(z) \quad (z \in \mathbb{U}) \tag{7.3}$$

and the function q is the best dominant of (7.3).

Proof. Letting

$$h(z) = \left(\frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1+1)f(z)}{z^p}\right)^{\gamma} \quad (z \in \mathbb{U}), \tag{7.4}$$

differentiating (7.4) logarithmically and using the identity (1.7) in the resulting expression, we get

$$h(z) + \frac{zh'(z)}{a\gamma} = \frac{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1)f(z)}{\mathcal{Q}_{p,q,s}^{n,\alpha}(e_1+1)f(z)} \quad (z \in \mathbb{U}),$$

so that by using (7.4) again, the above equation yields

$$h(z) + \frac{\mu}{a\gamma}zh'(z) \prec q(z) + \frac{\mu}{a\gamma}zq'(z) \quad (z \in \mathbb{U}). \tag{7.5}$$

Thus, by applying Lemma 2.3 to the subordination condition (7.5) with $\widetilde{\beta} = 1$ and $\widetilde{\gamma} = \mu/a\gamma$, we get the the desired assertion (7.3). □

Remark 7.2. If, we let $q(z) = (1+Az)/(1+Bz)$ $(-1 \le B < A \le 1; z \in \mathbb{U})$ in Theorem 7.1, the condition (7.1) reduces to

$$\text{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -a\text{Re}\left(\frac{\gamma}{\mu}\right)\right\} \quad (z \in \mathbb{U}). \tag{7.6}$$

It is easy to verify that the function $\psi(z) = (1-Bz)/(1+Bz)$ $(z \in \mathbb{U})$ is convex(univalent) in $\mathbb{U}$. Since $\psi(\overline{z}) = \overline{\psi(z)}$ for all $z \in \mathbb{U}$, the image of $\mathbb{U}$ under the function ψ is a convex domain and symmetrical with respect to the real axis. Thus,

$$\inf\left\{\text{Re}\left(\frac{1-Bz}{1+Bz}\right) : z \in \mathbb{U}\right\} = \frac{1-|B|}{1+|B|} > 0,$$

from which, it follows that the inequality (7.6) is equivalent to

$$a\text{Re}\left(\frac{\gamma}{\mu}\right) \ge \frac{|B|-1}{|B|+1}.$$

Thus in view of the above remark, if we let $q(z) = (1+Az)/(1+Bz)$, then we obtain the following result.

Corollary 7.3. *Let $-1 \leq B < A \leq 1$ and*

$$\frac{1-|B|}{1+|B|} \geq \max\left\{0, -a\mathit{Re}\left(\frac{\gamma}{\mu}\right)\right\}.$$

If $f \in \mathcal{A}_p$ satisfies

$$(1-\mu)\left(\frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1+1)f(z)}{z^p}\right)^{\gamma} + \mu\left(\frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1+1)f(z)}{z^p}\right)^{\gamma} \frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)}{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1+1)f(z)}$$
$$\prec \frac{1+Az}{1+Bz} + \frac{\mu}{a\gamma}\frac{(A-B)z}{(1+Bz)^2} \quad (z \in \mathbb{U}),$$

then

$$\left(\frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1+1)f(z)}{z^p}\right)^{\gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

and the function $(1+Az)/(1+Bz)$ is the best dominant.

Theorem 7.4. *Let $\mu \in \mathbb{C}^*$ and $\gamma \in \mathbb{C}$. Let q be univalent in $\mathbb{U}$ with $q(0) = 1$, $q(z) \neq 0$ in $\mathbb{U}$ and satisfies*

$$\mathit{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{7.7}$$

If $f \in \mathcal{A}_p$ satisfies

$$\frac{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}{(1-\beta+p\beta)z^p} \neq 0 \quad (z \in \mathbb{U}), \tag{7.8}$$

and

$$1 + \mu\gamma\left\{\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} - p\right\} \tag{7.9}$$
$$\prec 1 + \gamma\frac{zq'(z)}{q(z)} \quad (z \in \mathbb{U}),$$

then

$$\left\{\frac{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}{(1-\beta+p\beta)z^p}\right\}^{\mu} \prec q(z) \quad (z \in \mathbb{U}) \tag{7.10}$$

and the function q is the best dominant of (7.10).

Proof. Consider the function h defined by

$$h(z) = \left\{\frac{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}{(1-\beta+p\beta)z^p}\right\}^{\mu} \quad (z \in \mathbb{U}). \tag{7.11}$$

In view of (7.8), the function h is analytic in $\mathbb{U}$ and $h(0) = 1$. Differentiating both the sides of (7.11) logarithmically, we get

$$\frac{zh'(z)}{h(z)} = \mu\left\{\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} - p\right\} \quad (z \in \mathbb{U}). \tag{7.12}$$

By setting

$$\theta(w) = 1 \ (w \in \mathbb{C}) \quad \text{and} \quad \phi(w) = \frac{\gamma}{w} \ (w \in \mathbb{C}^*),$$

it is easily observed that θ is analytic in $\mathbb{C}$ and $\phi(w) \neq 0$ in $\mathbb{C}^*$. Further, if we let

$$Q(z) = zq'(z)\phi(q(z)) = \gamma\frac{zq'(z)}{q(z)} \quad \text{and} \quad g(z) = \theta(q(z)) + Q(z) = 1 + \gamma\frac{zq'(z)}{q(z)},$$

then by (7.7), the function Q is univalent starlike in $\mathbb{U}$. Also, by (7.7)

$$\operatorname{Re}\left\{\frac{zg'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Using (7.12) in (7.9), we get

$$1 + \gamma\frac{zh'(z)}{h(z)} \prec 1 + \gamma\frac{zq'(z)}{q(z)} \quad (z \in \mathbb{U}),$$

which is equivalent to

$$\theta(h(z)) + zh'(z)\phi(h(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathbb{U}).$$

Thus, by making use of Lemma 2.4, we deduce that

$$h(z) \prec q(z) \quad (z \in \mathbb{U})$$

and the function q is the best dominant. This completes the proof of Theorem 7.4. □

Theorem 7.5. *Let $\mu \in \mathbb{C}^*$ and $\eta \in \mathbb{C}$. Let q be a univalent function in $\mathbb{U}$ with $q(0) = 1$ and*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -Re(\eta)\} \quad (z \in \mathbb{U}). \tag{7.13}$$

If $f \in \mathcal{A}_p$ satisfies (7.8) *and*

$$\Psi(z) \prec \eta q(z) + \gamma\, zq'(z) \quad (z \in \mathbb{U}), \tag{7.14}$$

where

$$\Psi(z) = \left\{\frac{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z))'}{(1-\beta+p\beta)z^p}\right\}^{\mu} \times \tag{7.15}$$
$$\left\{\eta + \mu\gamma\left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{I}^{\lambda}_p(a,c)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} - p\right)\right\} \quad (z \in \mathbb{U}),$$

then

$$\left\{\frac{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z))'}{(1-\beta+p\beta)z^p}\right\}^{\mu} \prec q(z) \quad (z \in \mathbb{U}), \tag{7.16}$$

and the function q is the best dominant of (7.16).

Proof. The proof of this theorem being much similar to that of Theorem 7.4, we give the main steps only. We consider the function h, given by (7.11). Then by (7.12), we have

$$zh'(z) = \mu h(z)\left\{\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)' + \beta z^2\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)''}{(1-\beta)\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z) + \beta z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'} - p\right\} \quad (z \in \mathbb{U}). \tag{7.17}$$

Taking $\theta(w) = \eta w, \phi(w) = \gamma\ (w \in \mathbb{C}), Q(z) = zq'(z)\phi(q(z)) = \gamma zq'(z)$ and

$$g(z) = \theta(q(z)) + Q(z) = \eta q(z) + \gamma zq'(z),$$

we find from (7.13) that Q is univalent starlike in $\mathbb{U}$. Also, by the hypothesis (7.13)

$$\operatorname{Re}\left\{\frac{zg'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\eta + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Furthermore, by substituting the expression for h from (7.11) and $zh'(z)$ from (7.17), we have

$$\theta(h(z)) + zh'(z)\phi(h(z)) = \eta h(z) + \gamma zh'(z) = \Psi(z) \quad (z \in \mathbb{U}).$$

Thus, the hypothesis (7.14) reduces to

$$\theta(h(z)) + zh'(z)\phi(h(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathbb{U})$$

and an application of Lemma 2.4 gives the required assertion, This completes the proof of Theorem 7.5. □

Noting that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}=Re\left\{\frac{1-Bz}{1+Bz}\right\}>\frac{1-|B|}{1+|B|}\quad(z\in\mathbb{U}),$$

and taking $\beta=0,\gamma=1,q(z)=(1+Az)/(1+Bz)$ *in Theorem 7.5, we have*

Corollary 7.6. *Let* $\mu\in\mathbb{C}^*$ *and* $\eta=(|B|-1)/(|B|+1)$. *If* $f\in\mathcal{A}_p$ *satisfies* $\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)/z^p\neq 0$ *in* $\mathbb{U}$ *and*

$$\left\{\frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)}{z^p}\right\}^{\mu}\left\{\eta+\mu\left(\frac{z\left(\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)\right)'}{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)}-p\right)\right\}$$
$$\prec\frac{\eta(1+Az)}{1+Bz}+\frac{(A-B)z}{(1+Bz)^2}\quad(z\in\mathbb{U}),$$

then

$$\left\{\frac{\mathcal{Q}^{n,\alpha}_{p,q,s}(e_1)f(z)}{z^p}\right\}^{\mu}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U})$$

and the function $(1+Az)/(1+Bz)$ *is the best dominant.*

8 Conclusion remarks

The new subclasses $\widetilde{\mathcal{S}}p,q,s^{\alpha,\beta,n}(e_1,\rho)$ *and* $\widetilde{\mathcal{R}}p,n^{b,q,s}(e_1,\alpha,\mu;\rho)$ *of multivalent functions were studied with respect to various properties. Some other properties related to univalent functions may be generalized from the above study.*

Compliance with Ethical Standards:

The author(s) declare(s) that there is no conflict of interest. This article does not contain any studies involving human participants or animals performed by any of the authors.

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