

# Square Element Graph of a Near-ring

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**Abstract** *The notion of square element graph  $\mathbb{S}q(N)$  of a near-ring  $N$  has been introduced, which is defined as a simple undirected graph whose vertex set consists of all non-zero elements of  $N$ , and two vertices  $a$  and  $b$  are adjacent if and only if  $a + b = x^2$  or  $b + a = y^2$  for some  $x, y \in N - \{0\}$ . Among other results, the completeness of  $\mathbb{S}q(N)$  has been characterized. Also, necessary and sufficient conditions on the near-ring  $N$  have been obtained for  $\mathbb{S}q(N)$  to be planar, regular and Eulerian, respectively.*

## 1 Introduction

*In the last few decades, several graphs have been defined over various algebraic structures in several ways to study the interconnection between graph theoretic and algebraic properties, for instance, see [1, 4, 5, 6, 14, 15]. In particular, for a ring  $R$ , the graph theoretical structures of the set  $S = \{x^2 | x \in R - \{0\}\}$  have been determined by studying the square element graph over the ring  $R$ , the notion of which was introduced by R. Sen Gupta and M.K. Sen [12]. This study has been continued in [11, 13] and also extended in the setting of a semigroup [2]. We introduce here the square element graph of a near-ring  $N$  and denote it by  $\mathbb{S}q(N)$ . It is interesting to see that the set  $S$  is neither closed under addition nor closed under multiplication for a near-ring, unlike the special types of rings. It is also worth noticing that in a finite integral near-ring, each element has a unique square-root. Keeping these in mind, we establish several connections between algebraic properties of a finite integral near-ring  $N$  and graph theoretical properties of the graph  $\mathbb{S}q(N)$ . In Section 3,  $\mathbb{S}q(N)$  is defined and its various properties have been explored in the setting of integral near-rings. We obtain some conditions on a near-ring  $N$  so that  $\mathbb{S}q(N)$  becomes connected (cf. Theorem 3.3). Also, we obtain a necessary and sufficient condition on the near-ring  $N$  so that  $\mathbb{S}q(N)$  becomes complete (cf. Theorem 3.5). We also determine several graphical parameters such as diameter, girth, clique number, chromatic number, edge chromatic number, dominated edge chromatic number, Shannon capacity, degrees of the vertices, and number of edges of  $\mathbb{S}q(N)$ . Also, we check the perfectness of  $\mathbb{S}q(N)$  (cf. Theorem 3.11). In Section 4, we mainly focus on the planarity of  $\mathbb{S}q(N)$ . Here, we obtain a necessary and sufficient condition of planarity of  $\mathbb{S}q(N)$  for a special class of near-rings  $N$  (cf. Remark 4.1). We also obtain some conditions on the near-ring  $N$  so that  $\mathbb{S}q(N)$  becomes non-planar (cf. Remark 4.1, Theorem 4.2). The domination number of  $\mathbb{S}q(N)$  is also determined here (cf. Theorem 4.3). In the last section, we give our full attention to establish the relations between regularity and Eulerianity of  $\mathbb{S}q(N)$  for integral as well as Abelian near-rings  $N$  (cf. Remark 5.5, Theorem 5.15). Here, we obtain some necessary and sufficient conditions on the near-ring  $N$  so that  $\mathbb{S}q(N)$  becomes regular and Eulerian (cf. Theorems 5.1, 5.2, 5.11, 5.12, 5.13, 5.14, 5.16).*

## 2 Preliminaries

At first, we recall some definitions for their use in the sequel. We refer to Pilz [9] for the theory of near-rings. A near-ring is a set  $N$  together with two binary operations ‘+’ and ‘.’ such that (i)  $(N, +)$  is a group (not necessarily Abelian), (ii)  $(N, \cdot)$  is a semi-group and (iii) for all  $n_1, n_2, n_3 \in N : (n_1 + n_2)n_3 = n_1n_3 + n_2n_3$  (“right distributive law”).  $N$  is said to be an Abelian near-ring if  $(N, +)$  is Abelian. An element  $a$  of  $N$  is said to be a zero-divisor if there exists  $b (\neq 0)$  in  $N$  such that  $ab = 0$  or  $ba = 0$ .  $N$  is said to be integral if  $N$  has no non-zero zero-divisor.

For the graph-theoretical concepts, we refer to West[16], Bondy and Murty [3] and Malik, Sen and Ghosh [8]. Let  $G = (V(G), E(G))$  be a graph, where  $V(G)$  is the set of vertices of  $G$  and  $E(G)$  is the set of edges of  $G$ .  $G$  is said to be connected if there exists a path between any two distinct vertices of  $G$  and is said to be complete if any two vertices of  $G$  are adjacent. The distance  $d(x, y)$  between two vertices  $x$  and  $y$  is the length of a shortest path from  $x$  to  $y$ . The diameter of  $G$  is  $diam(G) = \max\{d(x, y) : x, y \in V(G)\}$  and the girth of  $G$  is the length of a smallest cycle in  $G$ . A cycle on three or more vertices is a simple subgraph of  $G$  whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are non-adjacent otherwise. The length of a cycle is the number of its edges. A cycle of length  $k$  is called a  $k$ -cycle. The degree of a vertex  $v$  in  $G$ , denoted by  $d_G(v)$ , is the number of edges of  $G$  incident with  $v$ .  $G$  is said to be  $k$ -regular if  $d_G(v) = k$  for all  $v \in V(G)$ . A clique of  $G$  is a non-empty subset  $U$  of  $V(G)$  such that the subgraph induced by  $U$  is a complete graph. The number of vertices in a largest clique is called the clique number of  $G$  and is denoted by  $\omega(G)$ . The chromatic number of  $G$ , denoted by  $\chi(G)$ , is the minimal number of colors needed to color the vertices so that no two adjacent vertices have the same color. The edge chromatic number of  $G$ , denoted by  $\chi'(G)$ , is the minimum number of colors needed to make the edge coloring of  $G$  such that any two edges that meet at a common vertex have different colors.  $G$  is called perfect if  $\omega(H) = \chi(H)$  for all induced subgraphs  $H$  of  $G$ . A stable set in  $G$  is a set of vertices so that no two of them are adjacent. The cardinality of a maximum stable set in  $G$  is called the stability number of  $G$  and is denoted by  $\alpha(G)$ . The Shannon capacity of  $G$  is a measure of the capacity for error-free transmission over a noisy channel whose associated graph is  $G$ . It is denoted by  $\Theta(G)$  and defined by  $\Theta(G) := \lim_{k \rightarrow \infty} \sqrt[k]{\alpha(G^k)}$ , where  $G^k$  is the strong product of  $k$ -copies of  $G$ . A dominating set in  $G$  is a subset  $S$  of  $V(G)$  such that each vertex of  $G$  either belongs to  $S$  or is adjacent to some element of  $S$ . The domination number  $\gamma(G)$  of  $G$  is the cardinality of a smallest dominating set. A subgraph  $G'$  of  $G$  is said to be a spanning subgraph if  $V(G') = V(G)$ .

## 3 Main Results

**Definition 3.1.** Let  $N^*$  be the set of all non-zero elements of a near-ring  $N$ . The square element graph of  $N$ , denoted by  $Sq(N)$ , is a undirected simple graph (without loops and multiple edges) with vertex set  $N^*$  and two vertices  $a$  and  $b$  ( $a \neq b$ ) are adjacent if and only if  $a + b = x^2$  or  $b + a = y^2$  for some  $x, y \in N^*$ .

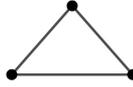
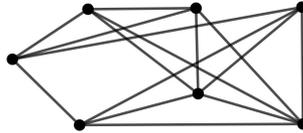
Let us look at some examples of  $Sq(N)$ .

**Example 3.2.** The graphs  $Sq(M_c(\mathbb{Z}_2))$ ,  $Sq(M_c(\mathbb{Z}_3))$ ,  $Sq(\mathbb{Z}_2 \times M_0(\mathbb{Z}_2))$  and  $Sq(\mathbb{Z}_2 \times M(\mathbb{Z}_2))$  are shown in Figures 1,2,3,4 respectively, where for any additive group  $G$  with identity 0,  $M(G) := \{f|f : G \rightarrow G \text{ is a function}\}$ ,  $M_c(G) := \{f \in M(G) : f(x) = c, \text{ a constant, for all } x \in G\}$  and  $M_0(G) := \{f \in M(G) : f(0) = 0\}$  are near-rings with pointwise addition and composition of functions.

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Figure 1.  $Sq(M_c(\mathbb{Z}_2))$

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**Figure 2.**  $\mathbb{S}q(M_c(\mathbb{Z}_3))$ **Figure 3.**  $\mathbb{S}q(\mathbb{Z}_2 \times M_0(\mathbb{Z}_2))$ **Figure 4.**  $\mathbb{S}q(\mathbb{Z}_2 \times M(\mathbb{Z}_2))$ 

**Theorem 3.3.** *If  $N$  is a finite integral near-ring such that  $|N| \neq 3$ , then  $\mathbb{S}q(N)$  is connected and  $\text{diam}(\mathbb{S}q(N)) \leq 2$ .*

*Proof.* Let  $N$  be a finite integral near-ring of order  $n$ .

Case I: Assume  $n = 2$ . Then it is clear that  $\mathbb{S}q(N)$  is a single vertex graph and hence, connected.

Case II: Suppose  $n \geq 4$ . Since  $N$  is a finite integral near-ring, each element of  $N$  has a unique square root (cf. Remark 1.112 pp. 40 [9]). Suppose that  $u$  and  $v$  are any two vertices of  $\mathbb{S}q(N)$ . If  $u + v \neq 0$ , then  $u$  and  $v$  are adjacent. Now, consider  $u + v = 0$ . Clearly,  $u$  and  $v$  are not adjacent as there does not exist  $a (\neq 0)$  in  $N$  such that  $a^2 = 0$ . Since  $|N| \geq 4$ , there exists  $v' \in V(\mathbb{S}q(N))$  such that  $v' \neq u, v$ . So,  $v' + v \neq 0$  which implies that  $v'$  and  $v$  are adjacent. Thus,  $u-v'-v$  is a path between  $u$  and  $v$ . Hence, the result is proved.  $\square$

**Remark 3.4.** For a near-ring  $N$  of order 3, it is clear that  $\mathbb{S}q(N)$  is a null graph because  $N$  has only two non-zero elements of additive order 3.

**Theorem 3.5.** *Let  $N$  be a finite integral near-ring. Then  $\mathbb{S}q(N)$  is complete if and only if  $\text{Char}(N) = 2$ .*

*Proof.* Let  $N$  be a finite integral near-ring and let  $\mathbb{S}q(N)$  be complete. If possible, suppose  $\text{Char}(N) \neq 2$ . Then there exists at least one  $v (\neq 0) \in N$  such that  $v \neq -v$ . So,  $v, -v \in V(\mathbb{S}q(N))$  and  $v + (-v) = 0$ . Therefore,  $v$  and  $-v$  are not adjacent, which is a contradiction. So,  $\text{Char}(N) = 2$ . Conversely, let  $\text{Char}(N) = 2$ . Suppose that  $u$  and  $v$  are two distinct vertices of  $\mathbb{S}q(N)$ . Since  $u \neq v$  and  $\text{Char}(N) = 2$ ,  $u + v \neq 0$ . This implies that  $u$  and  $v$  are adjacent (cf. Remark 1.112 pp. 40 [9]). This completes the proof.  $\square$

**Theorem 3.6.** *If  $N$  is a finite integral near-ring such that  $|N| \geq 6$ , then  $\mathbb{S}q(N)$  contains a 3-cycle and hence neither bipartite nor tree.*

*Proof.* Suppose that  $u$  is a non-zero element of  $N$ . Then  $u$  is adjacent to at least three vertices  $v, w, z$  (say) as  $|N| \geq 6$  and each element of  $N$  has a unique square root (cf. Remark 1.112 pp. 40 [9]). Now,  $w$  is adjacent to  $v$  or  $z$  or both. Without loss of generality, we assume that  $w$  is adjacent to  $v$ . Then  $u-v-w$  forms a 3-cycle. Hence,  $\mathbb{S}q(N)$  is neither a bipartite graph or a tree.  $\square$

The following corollary comes directly from the above theorem.

**Corollary 3.7.** *If  $N$  is a finite integral near-ring such that  $|N| \geq 6$ , then  $\text{girth}(\mathbb{S}q(N)) = 3$ .*

**Theorem 3.8.** *If  $N$  is a finite integral near-ring. Then the following results hold.*

(i) *If  $|N| = 2n + 1$  ( $n \in \mathbb{N}$ ), then  $\omega(\mathbb{S}q(N)) = n$ .*

(ii) *If  $|N| = 2(n + 1)$  ( $n \in \mathbb{N}_0$ ), then  $\omega(\mathbb{S}q(N)) = (n + 1) + \frac{1}{2} \times$  (the number of elements of order 2 – 1).*

*Proof.* (i) Let us consider  $|N| = 2n + 1$  ( $n \in \mathbb{N}$ ). Since the order of  $N$  is odd,  $N$  has no elements of order 2. Now, suppose that  $\{u_i, v_i : i = 1, 2, \dots, n\} \subseteq N$  is the set of non-zero elements of  $N$  such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, n$ . Since each element of  $N$  has a unique square root (cf. Remark 1.112 pp. 40 [9]), each subgraph induced by the vertex subsets  $\{u_1, u_2, \dots, u_n\}$  and  $\{v_1, v_2, \dots, v_n\}$  forms a complete graph. Clearly, the subgraph induced by the vertex set either  $\{u_1, u_2, \dots, u_n\}$  or  $\{v_1, v_2, \dots, v_n\}$  is a maximal complete graph. Hence, the clique number of  $\mathbb{S}q(N)$ ,  $\omega(\mathbb{S}q(N))$ , is  $n$ .

(ii) Suppose that  $|N| = 2(n + 1)$  ( $n \in \mathbb{N}_0$ ). Since the order of  $N$  is even, the number of elements of order 2 in  $N$  is  $(2k + 1)$  for some  $k \geq 0$ . Assume that  $\{w_j : j = 1, 2, \dots, 2k + 1\}$  is the set of elements of order 2. Now, consider that  $\{u_i, v_j : i, j = 1, 2, \dots, n - k\} \subseteq N$  is the set of non-zero elements of  $N$  such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, n - k$ . Then, clearly, the subgraph induced by the vertex set either  $\{u_1, u_2, \dots, u_{n-k}\} \cup \{w_1, w_2, \dots, w_{2k+1}\}$  or  $\{v_1, v_2, \dots, v_{n-k}\} \cup \{w_1, w_2, \dots, w_{2k+1}\}$  is a maximal complete graph. Hence, the clique number of  $\mathbb{S}q(N)$ ,  $\omega(\mathbb{S}q(N))$ , is  $(n - k) + (2k + 1) = (n + 1) + k$ .  $\square$

Note that  $\chi(G) \geq \omega(G)$  for any graph  $G$ , where  $\chi(G)$  and  $\omega(G)$  denote the chromatic number and the clique number of  $\mathbb{S}q(N)$  respectively. So, for an integral near-ring  $N$ ,  $\chi(\mathbb{S}q(N)) \geq n$ , whenever  $|N| = 2n + 1$ , and  $\chi(\mathbb{S}q(N)) \geq (n + 1) + \frac{1}{2} \times$  (the number of elements of order 2 - 1), whenever  $|N| = 2(n + 1)$ . But the following result tells more.

**Theorem 3.9.** *If  $N$  is a finite integral near-ring. Then the following results hold.*

(i) *If  $|N| = 2n + 1$  ( $n \in \mathbb{N}$ ), then  $\chi(\mathbb{S}q(N)) = n$ .*

(ii) *If  $|N| = 2(n + 1)$  ( $n \in \mathbb{N}_0$ ), then  $\chi(\mathbb{S}q(N)) = (n + 1) + \frac{1}{2} \times$  (the number of elements of order 2 - 1).*

*Proof.* (i) Let  $|N| = 2n + 1$  ( $n \in \mathbb{N}$ ). Since the order of  $N$  is odd, there is no element of order 2 in  $N$ . Now, suppose that  $\{u_i, v_i : i = 1, 2, \dots, n\} \subseteq N$  is the set of non-zero elements of  $N$  such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, n$ . Clearly, each element of  $N$  has a unique square root (cf. Remark 1.112 pp. 40 [9]). Thus,  $u_1, u_2, \dots, u_n$  are adjacent to each other. Therefore, we color  $u_1, u_2, \dots, u_n$  by the distinct colors  $c_1, c_2, \dots, c_n$  respectively. Now,  $v_1$  is adjacent to all vertices other than  $u_1$ . So, we color  $v_1$  by  $c_1$ . Similarly,  $v_2, v_3, \dots, v_n$  can be colored by  $c_2, c_3, \dots, c_n$  respectively. This observation along with the fact  $\chi(\mathbb{S}q(N)) \geq n$  shows that  $\mathbb{S}q(N) = n$ .

(ii) Let  $|N| = 2(n + 1)$  ( $n \in \mathbb{N}_0$ ). Since the order of  $N$  is even, the number of elements of order 2 in  $N$  is  $(2k + 1)$  for some  $k \geq 0$ . Assume that  $\{w_j : j = 1, 2, \dots, 2k + 1\}$  is the set of elements of order 2. Now, suppose that  $\{u_i, v_j : i, j = 1, 2, \dots, n - k\} \subseteq N$  is the set of non-zero elements of  $N$  such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, n - k$ . Now, in a similar way of (i), we can color the vertices  $u_1, u_2, \dots, u_{n-k}, v_1, v_2, \dots, v_{n-k}$  by  $(n - k)$  distinct colors, namely  $c_1, c_2, \dots, c_{n-k}$ . Clearly,  $w_j$  is adjacent to all the other vertices for each  $j = 1, 2, \dots, 2k + 1$ . So,  $w_j$  is colored by  $c'_j$  such that  $c'_j \notin \{c_1, c_2, \dots, c_{n-k}\}$  for each  $j = 1, 2, \dots, 2k + 1$ . Again,  $\chi(\mathbb{S}q(N)) \geq (n + 1) + \frac{1}{2} \times$  (the number of elements of order 2 - 1). Hence, the minimum number of colors to color  $\mathbb{S}q(N)$  is  $(n - k) + (2k + 1) = (n + 1) + k$ . This completes the proof.  $\square$

The following corollary comes directly from the above theorem.

**Corollary 3.10.** *If  $N$  is a finite integral near-ring of order  $n$ . Then  $\lfloor \frac{n}{2} \rfloor \leq \chi(\mathbb{S}q(N)) \leq (n - 1)$ .*

Theorems 3.8 and 3.9 show that  $\chi(\mathbb{S}q(N)) = \omega(\mathbb{S}q(N))$  for a finite integral near-ring  $N$ . This evokes us to check whether  $\mathbb{S}q(N)$  is perfect or not. So, we obtain the following result.

**Theorem 3.11.** *If  $N$  is a finite integral near-ring, then  $\mathbb{S}q(N)$  is perfect.*

*Proof.* First, we will determine the clique number and the chromatic number of the complement of  $\mathbb{S}q(N)$ . Suppose that  $\{u_i, v_j : i, j = 1, 2, \dots, k\}$  is the set of non-zero elements of  $N$  such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, k$ , where  $k \geq 0$ . Then  $u_i$  is adjacent to  $v_i$  in the complement of  $\mathbb{S}q(N)$  for each  $i = 1, 2, \dots, k$  and no other vertices are adjacent in the complement of  $\mathbb{S}q(N)$ . Thus, any induced subgraph of the complement of  $\mathbb{S}q(N)$  is a null graph or a graph containing a maximal complete subgraph  $K_2$ . Therefore, the chromatic number

and the clique number are equal in any induced subgraph of the complement of  $\mathbb{S}q(N)$ . So, the complement of  $\mathbb{S}q(N)$  is perfect, and hence  $\mathbb{S}q(N)$  is perfect (cf. Theorem 14.12 pp. 374 [3]).  $\square$

**Remark 3.12.** We have already determined the clique number  $\omega(\mathbb{S}q(N))$  for a finite integral near-ring  $N$ . So, by Theorem 14.13 [3], we have the stability number  $\alpha(\mathbb{S}q(N)) \geq \frac{|N|}{\omega(\mathbb{S}q(N))}$ . But the following result tells us more about the stability number.

**Theorem 3.13.** *If  $N$  is a finite integral near-ring, then the stability number  $\alpha(\mathbb{S}q(N))$  is either 1 or 2.*

*Proof.* Let  $N$  be a finite integral near-ring. Then an element of  $N$  is adjacent with all non-zero elements of  $N$  other than its additive inverse (cf. Remark 1.112 pp. 40 [9]). Clearly, the number of non-zero elements of  $N$  which is not its own additive inverse is  $2k$  for some  $k \geq 0$ .

Case I: If  $k = 0$ , then  $\mathbb{S}q(N)$  is complete. So, the maximal stable set is a singleton set, and hence the stability number is 1.

Case II: Assume  $k \geq 1$ . Suppose that  $v_1, v_2, \dots, v_k$  are the additive inverse of  $u_1, u_2, \dots, u_k$  respectively. Then each  $u_i$  is adjacent to all the vertices other than  $v_i$  for each  $i$ . So, for each  $i$ ,  $\{u_i, v_i\}$  is a maximal stable set. Hence, the stability number is 2. This completes the proof.  $\square$

**Corollary 3.14.** *If  $N$  is a finite integral near-ring, then the Shannon capacity  $\Theta(\mathbb{S}q(N))$  is either 1 or 2.*

*Proof.* Let  $N$  be a finite integral near-ring. Then from Theorem 3.11, we can say that  $\mathbb{S}q(N)$  is perfect. Thus, from the above result and the fact that the Shannon capacity of a perfect graph is equal to its stability number (cf. Corollary 14.17 pp. 376 [3]), the result is obvious.  $\square$

**Remark 3.15.** In Theorem 3.9, we have studied the chromatic number of  $\mathbb{S}q(N)$ . Now, we focus on the edge chromatic number and the dominated edge chromatic number (Definition 2.1[10]) of  $\mathbb{S}q(N)$ , denoted by  $\chi'(\mathbb{S}q(N))$  and  $\chi'_{dom}(\mathbb{S}q(N))$  respectively.

**Theorem 3.16.** *Let  $N$  be an integral near-ring of order  $n$  ( $\geq 3$ ). Then the following results hold.*

(i) *If  $n$  is odd, then  $\chi'(\mathbb{S}q(N)) = n - 2$  or  $n - 3$ .*

(ii) *If  $n$  is even, then  $\chi'(\mathbb{S}q(N)) = n - 1$  or  $n - 2$ .*

*Proof.* Suppose that  $\{w_i : i = 1, 2, \dots, r\} \subseteq N$  is the set of all non-zero elements of order 2 and  $\{u_i, v_j : i, j = 1, 2, \dots, k\} \subseteq N$  is the set of non-zero elements such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, k$ . Then  $w_i$  is adjacent to all the vertices of  $\mathbb{S}q(N)$  for each  $i = 1, 2, \dots, r$  and  $u_i$  is adjacent to all the other vertices except  $v_i$  for each  $i = 1, 2, \dots, k$  (cf. Remark 1.112 pp. 40 [9]). Now, we draw a new graph  $\mathbb{S}q(N)^*$  by adjoining the vertices  $u_i$  and  $v_i$  in  $\mathbb{S}q(N)$  for each  $i = 1, 2, \dots, k$ . Clearly,  $\mathbb{S}q(N)^*$  is  $K_{n-1}$ . Assume that  $\mathbb{S}q(N)$  has a proper C-edge coloring. Now, we color the edges  $u_i v_i$  in  $\mathbb{S}q(N)^*$  by a color  $c'$ , where  $c' \notin C$  for each  $i = 1, 2, \dots, k$ . Then  $\chi'(\mathbb{S}q(N)^*) \leq \chi'(\mathbb{S}q(N)) + 1$ . Also, it is clear that  $\chi'(\mathbb{S}q(N)) \leq \chi'(\mathbb{S}q(N)^*)$ . Therefore,  $\chi'(\mathbb{S}q(N)^*) = \chi'(\mathbb{S}q(N))$  or  $\chi'(\mathbb{S}q(N)) + 1$ . But  $\chi'(\mathbb{S}q(N)^*) = \chi'(K_{n-1})$ . Hence  $\chi'(\mathbb{S}q(N)) = \chi'(K_{n-1})$  or  $\chi'(K_{n-1}) - 1$ .

(i) Suppose that  $n$  is odd. Then  $\chi'(\mathbb{S}q(N)) = (n - 1) - 1 = n - 2$  or  $(n - 2) - 1 = n - 3$ .

(ii) Suppose that  $n$  is even. Then  $\chi'(\mathbb{S}q(N)) = n - 1$  or  $(n - 1) - 1 = n - 2$  (cf. Theorem 9.3.1 [8]).  $\square$

**Theorem 3.17.** *Let  $N$  be a finite integral near-ring such that  $|N| = n$  ( $n \geq 3$ ).*

(i) *If  $n$  is even, then the maximum degree of  $\mathbb{S}q(N)$  is  $n - 2$ .*

(ii) *If  $n$  is odd, then the maximum degree of  $\mathbb{S}q(N)$  is  $n - 3$ .*

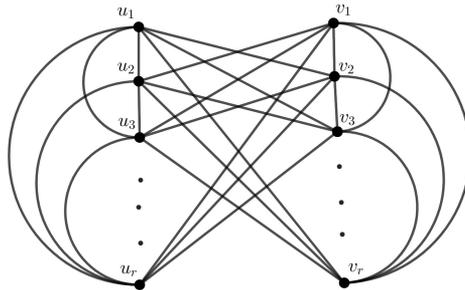
*Proof.* (i) Suppose that  $N$  is a finite integral near-ring of even degree. Then there exists at least one element  $w$  of order 2 in  $N$ . Clearly,  $w$  is adjacent to all the vertices of  $\mathbb{S}q(N)$  (cf. Remark 1.112 pp. 40 [9]) and hence the maximum degree of  $\mathbb{S}q(N)$  is  $n - 2$ .

(ii) Suppose that  $N$  is a finite integral near-ring of odd degree. Then there exists no element of order 2 in  $N$ . Let  $u, v$  be non-zero elements of  $N$  such that  $u + v = 0$ . Then  $u$  is adjacent to all the other vertices except  $v$ . Thus,  $u$  attains the maximum degree, and hence the maximum degree of  $\mathbb{S}q(N)$  is  $n - 3$ . This proves the result.  $\square$

**Remark 3.18.** In view of Remark 2.2 [10] and Theorem 3.17, we can conclude that the dominated edge chromatic number  $\chi'_{dom}(\mathbb{S}q(N)) \geq n - 2$ , for a finite integral near-ring  $N$ . Now, we determine a more specific bound on  $\chi'_{dom}(\mathbb{S}q(N))$ .

**Theorem 3.19.** *Let  $N$  be a finite integral near-ring. If the order of  $N$  is  $n$  and  $k$  is the number of elements of order 2 in  $N$ , then the size of  $\mathbb{S}q(N)$ ,  $m(\mathbb{S}q(N)) = \frac{k}{2}(2n - k - 3) + \frac{1}{2}(n - k - 1)(n - k - 3)$ .*

*Proof.* Suppose that  $W = \{w_i : i = 1, 2, \dots, k\} \subseteq N$  is the set of all non-zero elements of order 2 and  $V' = \{u_i, v_j : i, j = 1, 2, \dots, r\} \subseteq N$  is the set of non-zero elements such that  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, \dots, r$ . In view of Remark 1.112 [9], we can say that  $w_i$  is adjacent to all the vertices for each  $i = 1, 2, \dots, k$  and  $u_i$  is adjacent to all the vertices except  $v_i$  for each  $i = 1, 2, \dots, r$ . Thus,  $|E(\mathbb{S}q(N))| =$  the number of edges at least one of whose end vertices belongs to  $W$  + the number of edges in the subgraph induced by the vertex subset  $V' = \{((n - 1) - 1) + ((n - 1) - 2) + \dots + ((n - 1) - k)\}$  + the number of edges in the complete subgraph induced by  $\{u_i : i = 1, 2, \dots, r\}$  + the number of edges in the complete subgraph induced by  $\{v_i : i = 1, 2, \dots, r\}$  +  $r(r - 1) = \{k(n - 1) - (1 + 2 + \dots + k)\} + \frac{r(r-1)}{2} + \frac{r(r-1)}{2} + r(r - 1) = k(n - 1) - \frac{k(k+1)}{2} + 2r(r - 1) = \frac{k}{2}(2n - k - 3) + \frac{1}{2}(n - k - 1)(n - k - 3)$  as  $k + 2r = n - 1$ . Hence, the theorem is proved.  $\square$



**Figure 5.** Subgraph of  $\mathbb{S}q(N)$  induced by  $V'$

**Corollary 3.20.** *Let  $N$  be a finite integral near-ring of order  $n$ . If  $n$  is odd, then  $m(\mathbb{S}q(N)) = \frac{1}{2}(n - 1)(n - 3)$ .*

*Proof.* Since the order of  $N$  is odd, it is clear that the number of elements of order 2 of  $N$  is 0 which implies  $k = 0$ . Hence, from Theorem 3.19,  $m(\mathbb{S}q(N)) = \frac{1}{2}(n - 1)(n - 3)$ .  $\square$

**Theorem 3.21.** *Let  $N$  be an integral near-ring of order  $n$ . Then the following results hold.*  
 (i) *If  $n$  is odd, then the dominated edge chromatic number,  $\chi'_{dom}(\mathbb{S}q(N)) \geq \lceil \frac{1}{4}(n - 1)(n - 3) \rceil$ .*  
 (ii) *If  $n$  is even, then the dominated edge chromatic number,  $\chi'_{dom}(\mathbb{S}q(N)) \geq \lceil \frac{1}{2} \{ \frac{k}{2}(2n - k - 3) + \frac{1}{2}(n - k - 1)(n - k - 3) \} \rceil$ , where  $k$  is the number of elements of order 2 in  $N$ .*

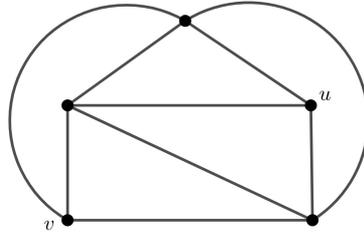
*Proof.* This follows immediately due to the fact that  $\chi'_{dom}(G) \geq \lceil \frac{m}{2} \rceil$  (cf. Remark 2.3 [10]).  $\square$

### 4 Planarity and Domination Number

Now, we study the planarity of  $\mathbb{S}q(N)$  for a finite integral near-ring  $N$ . It is clear that  $\mathbb{S}q(N)$  is planar when  $|N| \leq 5$ . So, we concentrate on a finite integral near-ring with  $|N| \geq 6$ .

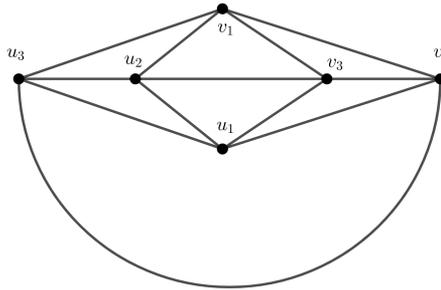
Suppose that  $|N| = 6$ . If  $\mathbb{S}q(N)$  is planar, then  $\mathbb{S}q(N)$  cannot be a complete graph. This implies that there exist at least two vertices that are not adjacent. Thus, we can conclude that  $N$  contains at least one non-zero element of order not equal to 2 due to the fact that each element of  $N$  has a unique square root (cf. Remark 1.112 pp. 40 [9]). Again, if  $N$  contains a non-zero element  $u$  of order not equal to 2, then there exists  $v \in N$  such that  $u + v = 0$ , which implies

$u$  and  $v$  are not adjacent. Clearly, for any integral near-ring of order 6,  $\mathbb{S}q(N)$  is a spanning subgraph of the graph in Figure 6 that is planar.



**Figure 6.** Supergraph of  $\mathbb{S}q(N)$  corresponding to an integral near-ring  $N$  of order 6

Now, suppose that  $|N| = 7$ . If  $N$  contains no element of order 2, then  $\{u_i, v_i : i = 1, 2, 3\}$  is the vertex set of  $\mathbb{S}q(N)$ , where  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, 3$ . Clearly,  $u_i$  is adjacent to all the other vertices except  $v_i$ , for each  $i = 1, 2, 3$ . So, from Figure 7, it is clear that  $\mathbb{S}q(N)$  is planar.



**Figure 7.**  $\mathbb{S}q(N)$  corresponding to an integral near-ring  $N$  of order 7

The above discussion leads to the following remark.

**Remark 4.1.** For an integral near-ring  $N$ , if  $|N| = 6$ , then  $\mathbb{S}q(N)$  is planar if and only if  $N$  contains at least one non-zero element of order not equal to 2 and if  $|N| = 7$  and  $N$  contains no element of order 2, then  $\mathbb{S}q(N)$  is planar.

Now, we discuss the planarity of the square graph of an integral near-ring  $N$  with  $|N| \geq 8$  in the following result.

**Theorem 4.2.** If  $N$  is an integral near-ring with  $|N| \geq 8$ , then  $\mathbb{S}q(N)$  is not planar.

*Proof.* Let us first consider  $|N| = 8$ . Then  $N$  contains  $2n + 1$  number of elements of order 2, where  $0 \leq n \leq 3$ .

Case I: Suppose that  $n = 0$  and  $\{w, u_i, v_i : i = 1, 2, 3\}$  is the vertex set of  $\mathbb{S}q(N)$ , where  $w$  is the unique element of order 2 and  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2, 3$ . Then  $K_{3,3}$  is a spanning subgraph of the vertex subset  $\{u_1, v_1, u_2, v_2, u_3, w\}$  with partitions  $\{u_1, v_1, w\}$  and  $\{u_2, v_2, u_3\}$ .

Case II: Suppose that  $n = 1$  and  $\{w_i, u_j, v_j : i = 1, 2, 3; j = 1, 2\}$  is the vertex set of  $\mathbb{S}q(N)$ , where  $w_i : i = 1, 2, 3$  are the elements of order 2 and  $v_i$  is the additive inverse of  $u_i$  for each  $i = 1, 2$ . Then  $K_{3,3}$  is a spanning subgraph of the vertex subset  $\{w_1, w_2, w_3, u_1, v_1, u_2\}$  with partitions  $\{u_1, v_1, w_1\}$  and  $\{u_2, w_2, w_3\}$ .

Case III: Suppose that  $n = 2$  or 3 and  $\{w_i : i = 1, 2, 3, 4, 5\}$  is the vertex subset of  $\mathbb{S}q(N)$ , where  $w_i : i = 1, 2, 3, 4, 5$  are the elements of order 2. Then, clearly,  $\{w_i : i = 1, 2, 3, 4, 5\}$  forms  $K_5$ . Thus, in any case,  $\mathbb{S}q(N)$  contains either  $K_{3,3}$  or  $K_5$  and therefore,  $\mathbb{S}q(N)$  is not planar with  $|N| = 8$ . Now, suppose  $|N| \geq 9$ . Then  $\mathbb{S}q(N)$  has at least eight vertices. Let  $u$  be a vertex in

$\mathbb{S}q(N)$ . If the order of  $u$  is 2, then, in view of Remark 1.112 [9], we have  $deg(u) = |9| - 2 \geq 7$ . Now, suppose that the order of  $u$  is not equal to 2. Then there exists  $v \in N$  such that  $u + v = 0$ . This implies that  $u$  is adjacent to all vertices other than  $v$ . Therefore,  $deg(u) = |N| - 3 \geq 6$ . Thus,  $\mathbb{S}q(N)$  always contains a vertex of degree  $\geq 6$ . Hence, from Corollary 9.1.5 [8], we can conclude that  $\mathbb{S}q(N)$  is not planar.  $\square$

We know that  $diam(\mathbb{S}q(N)) = 1$  when  $\mathbb{S}q(N)$  is complete. Consider that  $\mathbb{S}q(N)$  is not complete. Then from Theorem 3.3, it is clear that  $diam(\mathbb{S}q(N)) = 2$ , for a finite integral near-ring of any order except of order 3. Thus, in view of Remarks 4.1 and 1[7], we can conclude that  $\gamma(\mathbb{S}q(N)) \leq 3$ , for a finite integral near-ring of order up to 7 except of order 3 with some suitable conditions mentioned in Remark 4.1. But we cannot get any leads about the domination number of a finite integral near-ring of order  $\geq 8$ . To overcome this problem, we obtain the following result.

**Theorem 4.3.** *Let  $N$  be a finite integral near-ring of order  $n$ . Then the following results hold.*

- (i) *If  $n$  is even, then the domination number,  $\gamma(\mathbb{S}q(N))$ , is 1.*
- (ii) *If  $n$  is odd, then the domination number,  $\gamma(\mathbb{S}q(N))$ , is 2.*

*Proof.* (i) Suppose that  $n$  is even. Then there exists at least one non-zero element  $w$  (say) of order 2. This implies that  $w$  is adjacent to all the vertices in  $\mathbb{S}q(N)$  (cf. Remark 1.112 pp. 40 [9]). Thus,  $\{w\}$  is a dominating set with minimum cardinality and hence, the domination number,  $\gamma(\mathbb{S}q(N))$ , is 1.

(ii) Suppose that  $n$  is odd. Then  $N$  contains no elements of order 2. Let  $u$  be a non-zero element of  $N$ . Then there exists  $v (\neq u)$  such that  $u + v = 0$ . This implies that  $u$  is adjacent to all the vertices except  $v$ . Thus,  $\{u, v\}$  is a dominating set with minimum cardinality and hence, the domination number,  $\gamma(\mathbb{S}q(N))$ , is 2. This proves the result.  $\square$

## 5 Regularity and Eulerianity

**Theorem 5.1.** *Let  $N$  be a finite integral near-ring of even order. Then  $\mathbb{S}q(N)$  is regular if and only if every non-zero element of  $N$  is of order 2.*

*Proof.* Let  $N$  be a finite integral near-ring of even order and let  $\mathbb{S}q(N)$  be regular. Since the order of  $N$  is even, there exists at least one element  $w$  of order 2 and  $deg(w) = |N| - 2$ . If possible, suppose  $N$  contains a non-zero element  $u$  of order not equal to 2. Then there exists an element  $v$  of  $N$  such that  $u + v = 0$ . Thus, in view of Remark 1.112[9], it is clear that  $u$  is adjacent to all the other vertices except  $v$ . Therefore,  $deg(u) = |N| - 3$ , which contradicts the fact that  $\mathbb{S}q(N)$  is regular. So,  $N$  contains no non-zero elements of order not equal to 2. Conversely, suppose that  $N$  contains no non-zero elements of order not equal to 2. Then, clearly, all non-zero elements of  $N$  are of order 2 and each vertex has degree  $|N| - 2$ . This completes the proof.  $\square$

**Theorem 5.2.** *Let  $N$  be a finite integral near-ring of even order. Then  $\mathbb{S}q(N)$  is Eulerian if and only if every non-zero element of  $N$  is of order 2.*

*Proof.* Let  $N$  be a finite integral near-ring of even order. Then, from Theorem 3.3, it is clear that  $\mathbb{S}q(N)$  is connected. Now, suppose that  $\mathbb{S}q(N)$  is Eulerian. If possible, suppose  $N$  contains a non-zero element  $u$  of order not equal to 2. Then, clearly,  $deg(u) = |N| - 3 =$  an odd number. This contradicts the fact that  $\mathbb{S}q(N)$  is Eulerian (cf. Theorem 3.5[3]). Conversely, suppose that every non-zero element of  $N$  is of order 2. Then the degree of each vertex is  $|N| - 2 =$  an even number. Hence,  $\mathbb{S}q(N)$  is Eulerian.  $\square$

**Theorem 5.3.** *If  $N$  is a finite integral near-ring such that  $|N| = 2n + 1$ , then  $deg(v) = 2(n - 1)$  for each vertex  $v$  in  $\mathbb{S}q(N)$ .*

*Proof.* Let  $N$  be a finite integral near-ring such that  $|N| = 2n + 1$ . Then  $N$  does not contain any element of order 2. Suppose that  $v$  is any vertex in  $\mathbb{S}q(N)$ . Then  $v$  is adjacent to all vertices other than its additive inverse (cf. Remark 1.112 pp. 40 [9]). So,  $deg(v) = |V(\mathbb{S}q(N))| - |\{v, -v\}| = 2n - 2 = 2(n - 1)$ .  $\square$

The following corollary comes directly from the above theorem.

**Corollary 5.4.** *If  $N$  is a finite integral near-ring such that  $|N| = 2n + 1$ , then  $\mathbb{S}q(N)$  is  $k$ -regular graph, where  $k = 2(n - 1)$ .*

**Remark 5.5.** It is very clear from Theorems 5.1 and 5.2 that the notion of regularity and Eulerianity coincide for the graph  $\mathbb{S}q(N)$  in the setting of a finite integral near-ring  $N$  of even order.

**Theorem 5.6.** *Let  $N$  be a finite integral near-ring. If the order of  $N$  is an odd integer other than 3, then  $\mathbb{S}q(N)$  is Eulerian.*

*Proof.* Suppose that  $|N| = 2n + 1$  ( $n \neq 1$ ). Then, from Theorem 3.3 and Corollary 5.4, we have  $\mathbb{S}q(N)$  is connected and  $2(n - 1)$ -regular, respectively. Hence,  $\mathbb{S}q(N)$  is Eulerian (cf. Theorem 3.5 [3]).  $\square$

Now, we study the regularity and the Eulerianity of  $\mathbb{S}q(N)$  for a near-ring  $N$  which may not be an integral near-ring.

**Definition 5.7.** For  $|N| = n$ ,  $n > 1$ ,  $S^*(N)$  denotes the number of elements of the form  $x^2$ , where  $x \in N - \{0\}$ .

**Theorem 5.8.** *For a finite near-ring  $N$ , the minimum degree of a vertex  $v$  in  $\mathbb{S}q(N)$  is  $2S^*(N) - |C(-v)| - 2$ , where  $C(-v)$  denotes the centralizer of  $(-v)$  in  $(N, +)$ .*

*Proof.* Let  $v$  be any vertex in  $V(N)$ . Now,  $v$  is adjacent to  $u^2 - v$  or  $-v + u^2$  for each  $u \in N - \{0\}$ , unless  $u^2 - v$  and  $-v + u^2$  are equal to 0 or  $v$ . Hence, the neighborhood of  $v$  is the set of vertices  $S_v = \{u^2 - v | u \in N - \{0\}, \text{ and } u^2 - v \neq 0, v\} \cup \{-v + u^2 | u \in N - \{0\}, \text{ and } -v + u^2 \neq 0, v\}$ . We have,  $|S_v| = |\{u^2 - v | u \in N - \{0\}, \text{ and } u^2 - v \neq 0, v\}| + |\{-v + u^2 | u \in N - \{0\}, \text{ and } -v + u^2 \neq 0, v\}| - |\{u^2 - v = -v + u^2 | u \in N - \{0\}, \text{ and } u^2 - v \neq 0, v\}|$ . Let  $S_1 = \{u^2 - v | u \in N - \{0\}\}$  and  $S_2 = \{-v + u^2 | u \in N - \{0\}\}$ . Clearly,  $|S_1| = |S_2| = S^*(N)$  and  $|\{u^2 - v = -v + u^2 | u \in N - \{0\}\}| \leq |C(-v)|$ . Now,  $u^2 - v = 0$  or  $-v + u^2 = 0$  implies  $u^2 = v$ , and  $u^2 - v = v$  or  $-v + u^2 = v$  implies  $u^2 = 2v$ . So, there are three possibilities:

Case I: If both  $v$  and  $2v$  are squares, then  $\deg(v) = |S_v| \geq (S^*(N) - 2) + (S^*(N) - 2) - (|C(-v)| - 2) = 2S^*(N) - |C(-v)| - 2$ .

Case II: If exactly one of  $v$  and  $2v$  is a square, then  $\deg(v) = |S_v| \geq (S^*(N) - 1) + (S^*(N) - 1) - (|C(-v)| - 1) = 2S^*(N) - |C(-v)| - 1$ .

Case III: If none of  $v$  and  $2v$  is a square, then  $\deg(v) = |S_v| \geq S^*(N) + S^*(N) - |C(-v)| = 2S^*(N) - |C(-v)|$ .

Thus, in any case,  $\deg(v) \geq 2S^*(N) - |C(-v)| - 2$ , for any vertex  $v$  in  $\mathbb{S}q(N)$ .  $\square$

**Theorem 5.9.** *Let  $N$  be a finite Abelian near-ring. Then for any vertex  $v$  in  $\mathbb{S}q(N)$ ,  $\deg(v)$  is always one of the following three numbers:  $S^*(N)$ ,  $S^*(N) - 1$ ,  $S^*(N) - 2$ .*

*Proof.* Let  $N$  be a finite Abelian near-ring and let  $v$  be any vertex in  $\mathbb{S}q(N)$ . Clearly,  $u^2 - v = -v + u^2$  for any vertex  $u \in N - \{0\}$ . Now, from the proof of Theorem 5.8, we have  $\deg(v) = |S_v| = |\{u^2 - v | u \in N - \{0\}, \text{ and } u^2 - v \neq 0, v\}|$ . Here,  $u^2 - v = 0$  implies  $u^2 = v$  and  $u^2 - v = v$  implies  $u^2 = 2v$ . Now, we have the following three cases:

Case I: If both  $v$  and  $2v$  are squares, then  $\deg(v) = |S_v| = S^*(N) - 2$ .

Case II: If exactly one of  $v$  and  $2v$  is a square, then  $\deg(v) = |S_v| = S^*(N) - 1$ .

Case III: If none of  $v$  and  $2v$  is a square, then  $\deg(v) = |S_v| = S^*(N)$ .  $\square$

**Proposition 5.10.** *If  $N$  is a finite near-ring, then either  $\mathbb{S}q(N)$  is complete or there exists at least one non-square element in  $N$ .*

*Proof.* Let  $N$  be a finite near-ring. Clearly,  $S^*(N) = |\{u^2 | u \in N - \{0\}\}| \leq |N|$ . If  $S^*(N) = |N|$ , then  $\mathbb{S}q(N)$  is complete. If  $S^*(N) < |N|$ , then there exists an element  $v \in N$  such that  $v \neq n^2$ , for any  $n \in N - \{0\}$ . Hence, the result is proved.  $\square$

**Theorem 5.11.** *Let  $N$  be a finite Abelian near-ring with  $|N| \geq 4$ . Then  $\mathbb{S}q(N)$  is complete ( $(|N| - 2)$ -regular) if and only if  $S^*(N) = |N|$ .*

*Proof.* Let us first consider that  $\mathbb{S}q(N)$  is complete. Then  $\mathbb{S}q(N)$  is  $r$ -regular. Now, from Theorem 5.9, we have  $r = S^*(N), S^*(N) - 1$ , or  $S^*(N) - 2$ . If possible, let  $r = S^*(N) - 1$ . Since  $\mathbb{S}q(N)$  is complete for any vertex  $v$ ,  $deg(v) = |N| - 2$ . So  $S^*(N) - 1 = |N| - 2$ , whence  $S^*(N) = |N| - 1$ . Then there exists  $y \in N$  such that  $y \notin S^*(N)$ . Since  $|N| \geq 4$ , there exist  $u, v \in N - \{0, y\}$  such that  $u \neq v$  and  $u, v \in S^*(N)$ . Since  $(N, +)$  is a group, there exist unique  $x_1, x_2 \in N$  such that  $u + x_1 = y$  and  $v + x_2 = y$ . Clearly,  $x_1, x_2 \neq 0$ . Now, if  $u \neq x_1$ , then  $u$  and  $x_1$  are not adjacent, which contradicts the fact that  $\mathbb{S}q(N)$  is complete. So,  $u = x_1 \Rightarrow 2x_1 = y$ . Similarly,  $2x_2 = y$ . Therefore,  $x_1 = x_2 \Rightarrow u = v$ , which is a contradiction. Thus,  $r \neq S^*(N) - 1$ . In a similar way, we can show that  $r \neq S^*(N)$ . Hence,  $r = S^*(N) - 2 \Rightarrow |N| - 2 = S^*(N) - 2 \Rightarrow S^*(N) = |N|$ . The converse part is transparent. This completes the proof.  $\square$

**Theorem 5.12.** *Let  $N$  be a finite Abelian near-ring  $N$  with identity 1 such that  $N$  contains at least one non-zero element of order 2. Then  $\mathbb{S}q(N)$  is  $(|N| - 3)$ -regular if and only if for each vertex  $v$ , exactly one of  $v$  and  $2v$  is a square in  $N$ .*

*Proof.* Let us first consider that  $\mathbb{S}q(N)$  is  $(|N| - 3)$ -regular and let  $v$  be any vertex in  $\mathbb{S}q(N)$ . Then, from Theorem 5.9, we have  $deg(v) = S^*(N), S^*(N) - 1$ , or  $S^*(N) - 2$ . It is clear from Case-III in the proof of Theorem 5.9,  $deg(v) = deg(1) \neq S^*(N)$  as 1 is square. Now, suppose that  $deg(v) = S^*(N) - 2$ . Then, from Case-I in the proof of Theorem 5.9, we can say that  $v$  and  $2v$  are squares. So  $|N| - 3 = S^*(N) - 2$  whence  $S^*(N) = |N| - 1$ . This implies that there exists  $y$  in  $N$  such that  $y$  is not a square. Clearly,  $y = 0$ . Otherwise, if  $y \neq 0$ , then  $y \in V(\mathbb{S}q(N))$  and  $deg(y) = S^*(N) - 2$  which happens if  $y$  and  $2y$  are squares, a contradiction. Let  $u$  be a non-zero element of order 2. Since  $deg(u) = |N| - 3$ , there exists a non-zero element  $w$  different from  $u$  such that  $u + w = w + u = 0$ , which contradicts the fact that  $u$  is an element of order 2. Therefore,  $deg(v) = S^*(N) - 1$  which happens if exactly one of  $v$  and  $2v$  is a square. The converse part is obvious. Hence, the theorem is proved.  $\square$

**Theorem 5.13.** *Let  $N$  be a finite Abelian near-ring  $N$  with identity 1. Then  $\mathbb{S}q(N)$  is  $r$ -regular, where  $r \leq (|N| - 4)$ , if and only if for each vertex  $v$ , exactly one of  $v$  and  $2v$  is a square in  $N$ .*

*Proof.* Let  $\mathbb{S}q(N)$  be  $r$ -regular, where  $r \leq (|N| - 4)$ . Then, from Theorem 5.9, we have  $r = S^*(N), S^*(N) - 1$ , or  $S^*(N) - 2$ . Clearly, from Case-III of the proof of Theorem 5.9,  $r \neq S^*(N)$  as 1 is a square. Now, suppose that  $r = S^*(N) - 2$ . Then  $S^*(N) - 2 \leq |N| - 4$  whence  $S^*(N) \leq |N| - 2$ . This implies that there exist two distinct elements  $y$  and  $y'$  in  $N$  such that  $y, y'$  are not squares. Using the same argument used in the proof of Theorem 5.12, we have  $y = y' = 0$ , a contradiction. Thus, if the graph has to be regular, then  $r = S^*(N) - 1$ , which happens if exactly one of  $v$  and  $2v$  is a square for each vertex  $v$  in  $\mathbb{S}q(N)$ . The converse is transparent. This completes the proof.  $\square$

**Theorem 5.14.** *Let  $N$  be a finite Abelian near-ring and  $\mathbb{S}q(N)$  be connected. Suppose that  $S^*(N)$  is odd. Then  $\mathbb{S}q(N)$  is Eulerian if and only if for each vertex  $v$ , exactly one of  $v$  and  $2v$  is a square.*

*Proof.* Let us first consider that  $\mathbb{S}q(N)$  is Eulerian. Then each vertex in  $\mathbb{S}q(N)$  has an even degree (cf. Theorem 3.5[3]). Since  $S^*(N)$  is odd, no vertex has degree  $S^*(N)$  or  $S^*(N) - 2$ . So, by Theorem 5.9, each vertex has degree  $S^*(N) - 1$ . Thus, for each vertex  $v$ , exactly one of  $v$  and  $2v$  is a square. Conversely, suppose that exactly one of  $v$  and  $2v$  is a square for each vertex  $v$ . This implies that for each vertex  $v$ ,  $deg(v) = S^*(N) - 1$ , which is even. Hence,  $\mathbb{S}q(N)$  is Eulerian. This completes the proof.  $\square$

Now, combining Theorem 5.13 and Theorem 5.14 we obtain a nice connection between regularity and Eulerianity of the graph  $\mathbb{S}q(N)$  which is evident in the next result.

**Theorem 5.15.** *Let  $N$  be a finite Abelian near-ring  $N$  with identity 1 such that  $\mathbb{S}q(N)$  is connected and  $S^*(N)$  is odd. Then  $\mathbb{S}q(N)$  is Eulerian if and only if  $\mathbb{S}q(N)$  is  $r$ -regular, where  $r \leq (|N| - 4)$ .*

**Theorem 5.16.** *Let  $N$  be a finite Abelian near-ring with identity 1 and let  $\mathbb{S}q(N)$  be regular and connected. Suppose that  $S^*(N)$  is even. Then  $\mathbb{S}q(N)$  is Eulerian if and only if for each vertex  $v$ , both  $v$  and  $2v$  are squares.*

*Proof.* Suppose that  $\mathbb{S}q(N)$  is Eulerian. Clearly, the degree of each vertex is even (cf. Theorem 3.5[3]). Since  $S^*(N)$  is even,  $\deg(v) = S^*(N)$  or  $S^*(N) - 2$  for each vertex  $v$ . But  $N$  contains 1, a square element, which implies for each vertex  $v$ ,  $\deg(v) = S^*(N) - 2$  i.e. both  $v$  and  $2v$  are squares. Conversely, suppose that for each vertex  $v$ , both  $v$  and  $2v$  are squares which happens if  $\deg(v) = S^*(N) - 2$  for each vertex  $v$ . Since  $S^*(N)$  is even,  $S^*(N) - 2$  is even. So,  $\mathbb{S}q(N)$  is connected and each vertex has an even degree. This implies that  $\mathbb{S}q(N)$  is Eulerian.  $\square$

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