Conformal vector fields on almost α-Cosymplectic (κ, μ)-spaces

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Abstract. This paper investigates almost α-cosymplectic manifolds that admit a conformal vector field. It is shown that such manifolds are locally isometric to the product of a Kähler manifold and either an open interval or the unit circle S^1, and that the Reeb vector field ξ is Killing. Furthermore, we explore the existence of η-Ricci-Yamabe solitons associated with conformal vector fields on α-cosymplectic manifolds. In addition, it is demonstrated that if a three-dimensional almost α-cosymplectic (κ, μ)-space admits a homothetic vector field V, then the manifold either possesses constant scalar curvature or V is an infinitesimal contact transformation. Finally, example is constructed to illustrate and validate the theoretical results.

1 Introduction

In recent years, considerable attention has been devoted to the study of the geometry of almost cosymplectic manifolds, originally introduced by Goldberg and Yano in 1969 [7]. An almost cosymplectic manifold refers to a smooth $(2n+1)$-dimensional manifold equipped with a closed 1-form η and a closed 2-form Φ. It is known that if the almost contact structure is normal, the manifold is called cosymplectic. Furthermore, Endo [8] introduced the notion of almost cosymplectic (κ, μ)-spaces, characterized by a curvature tensor satisfying

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for all vector fields $X, Y \in \chi(M)$, where κ and μ are constants, and h is the $(1,1)$-tensor field defined by $h = \frac{1}{2}\mathcal{L}_\xi\phi$ where L_ξ is Lie-derivative along ξ. Kim [10] introduced the concept of an almost α-cosymplectic manifold, which refers to an almost contact manifold satisfying $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, where α is a real constant. Aktan et al. [12] conducted an in-depth study of almost α-cosymplectic (κ, μ, ν)-spaces and obtained several significant results. In [17] Yildirim and Aktan investigated holomorphically planar conformal vector fields on almost α-cosymplectic (κ, μ)-spaces.

In a contact manifold, the presence of a conformal vector field can be explained by some intrinsic properties of contact manifolds. For instance, it is well known [16] that a Riemannian manifold M is conformally flat if it admits a maximal group of conformal motions, consisting of $\frac{(n+1)(n+2)}{2}$-parameters. In [11] Okumura showed that for $n > 3$, a connected complete Sasakian manifold admitting a conformal motion is isometric to a sphere. Later, Sharma [14] extended this study to $N(K)$-contact metric manifolds admitting conformal motions. In [5] De at al. Studied conformal vector fields on almost Kenmotsu manifolds. Moreover, Sardar and De [13] characterized $f-$cosymplectic manifolds admitting conformal vector fields. In [1] Alias, Caminha, and Nascimento explored complete Kählerian manifolds endowed with non-trivial closed conformal vector fields.

A vector field V on a Riemannian manifold is said to satisfy the equation [5]:

$$\pounds_V g = 2\Omega g, \tag{1.1}$$

where Ω being a smooth function and $\pounds_V$ is the Lie-derivative, is called a conformal vector field. If V is not Killing, it is termed non-trivial. If Ω vanishes, then the conformal vector field V is named Killing. If Ω is constant, then the conformal vector field V is homothetic.

The Killing, conformal and homothetic vector fields have numerous applications in differential geometry as well as in mathematical physics.

If r, R, and Ric indicate the scalar curvature, the curvature tensor, and the Ricci tensor, respectivelly, then for conformal vector field V we have [16]:

$$(\pounds_V \nabla)(X, Y) = (X\Omega)Y - (Y\Omega)X - g(X, Y)D\Omega, \tag{1.2}$$

$$\begin{aligned}(\pounds_V R)(X, Y)Z &= g(\nabla_X D\Omega, Z)Y - g(\nabla_Y D\Omega, Z)X \\ &\quad + g(X, Z)\nabla_Y D\Omega - g(Y, Z)\nabla_X D\Omega,\end{aligned} \tag{1.3}$$

$$(\pounds_V Ric)(X, Y) = -(2n - 1)g(\nabla_X D\Omega, Y) - (\Delta\Omega)g(X, Y), \tag{1.4}$$

$$(\pounds_V r) = -4n(\Delta\Omega) - 2r\Omega, \tag{1.5}$$

for all vector fields X, Y, and Z on M, where $D\Omega$ and $\Delta\Omega = div D\Omega$ respectively denote the gradient and Laplacian of Ω.

A vector field V satisfying the relation

$$\pounds_V \eta = \Omega\eta, \tag{1.6}$$

where Ω being a scalar function is named an infinitesimal contact transformation. It is named as infinitesimal strict contact transformation, if Ω vanishes identically.

A Riemannian manifold (M, g) is called nearly quasi-Einstein manifold if its Ricci tensor Ric satisfies [18]:

$$Ric = ag + bE,$$

where a, b are functions and E is a non-vanishing symmetric $(0, 2)$-tensor on M.

2 Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold equipped with a triple (ϕ, ξ, η), where ϕ is a tensor field of type $(1, 1)$ (i.e., an endomorphism of the tangent bundle), ξ is a vector field known as the Reeb vector field, and η is a 1-form. These structures satisfy the following fundamental relations:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\phi) = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

which characterize the almost contact structure on M.

It is known that it an almost contact manifold M admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ g(X, \xi) = \eta(X), \tag{2.2}$$

then g is termed the compatible metric with the structure. Consequently, the manifold (M, ϕ, ξ, η, g) is denoted as an almost contact metric manifold. An almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is termed normal when the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ denotes the Nijenheus tensor of ϕ. The fundamental 2-form Φ on M is defined as $\Phi(X, Y) = g(\phi X, Y)$ for any vector fields X, Y on M.

An almost cosymplectic manifold is characterized as an almost contact metric manifold where $d\eta = 0$, and $d\Phi = 2\alpha\eta \wedge \phi$, where α is any real number (refer to [12]). A normal almost

α-cosymplectic manifold aligns with the definition of an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic when $\alpha = 0$ *or* α-Kenmotsu ($\alpha \neq 0$), where $\alpha \in \mathbb{R}$.

Let M be an almost α-cosymplectic manifold with structure (ϕ, ξ, η, g), and we denote by the distribution orthogonal to ξ, that is $= ker\eta = \{X : \eta(X) = 0\}$.

Since the 1-form η is closed, we have $\mathcal{L}_\xi \eta = 0$ and $[X, \xi] \in$ for any $X \in$, the Levi-civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \phi \in$, which implies that $\nabla_\xi X \in$ for any $X \in$.

Now, the tensor field h denoted by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ is symmetric operator.

In addition, in an almost α-cosymplectic manifold satisfies the following equations [12]:

$$\nabla_X \xi = -\alpha \phi^2 X - \phi h X, \tag{2.3}$$

$$h\xi = 0, \ \ g(hX, Y) = g(X, hY), \ \ trace(h) = 0, \ \ \phi X + X\phi = 0, \tag{2.4}$$

$$(\nabla_X \eta)Y = \alpha[g(X, Y) - \eta(X)\eta(Y)] + g(\phi Y, hX), \tag{2.5}$$

$$\begin{aligned}R(X, Y)\xi &= \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\phi hY - \eta(Y)\phi hX] \\ &+ (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,\end{aligned} \tag{2.6}$$

$$Ric(\xi, \xi) = -(2n\alpha^2 + tr(h^2)), \tag{2.7}$$

where X, Y are arbitrary vector fields on M.

Remark.1 [3] If a f-cosymplectic manifold with $\xi(f) = 0$ is compact, then it becomes an α-cosymplectic manifold. In particular, if $f = 0$, then M is cosymplectic.

Remark.2 [2] A cosymplectic manifold is locally the product of a Kähler manifold and an interval *or* unit circle S^1.

3 Almost α-cosymplectic manifold admitting conformal vector fields:

In this section, we give a characterization of almost α-cosymplectic manifolds admitting conformal vector field. First we prove the following results.

Theorem 3.1. *If the Reeb vector field ξ of M^{2n+1} is a conformal vector field, then M is locally the product of Kähler manifold and an interval or unit circle S^1 and the Reeb vector field ξ is Killing.*

Proof. Let us assume that the Reeb vector field ξ is a conformal vector field on M. Then, equation (1.1) implies

$$(\mathcal{L}_\xi g)(X, Y) = 2\Omega g(X, Y), \tag{3.1}$$

which means that

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2\Omega g(X, Y). \tag{3.2}$$

Using (2.1) and (2.3) in (3.2), we obtain

$$\alpha g(X, Y) - \alpha\eta(X)\eta(Y) - g(X, \phi hY) = \Omega g(X, Y). \tag{3.3}$$

By putting $X = Y = \xi$ into the foregoing equation, we acquire that

$$\Omega = 0. \tag{3.4}$$

Applying (3.4) in (3.1), we obtain

$$(\mathcal{L}_\xi g)(X, Y) = 0.$$

Hence, the above equation implies that ξ is Killing. Thus, we acquire $h = 0$. Therefore, the manifold becomes a cosymplectic manifold. Hence from Remark 2, we can complete the proof.

$\square$

Theorem 3.2. *If a conformal vector field V in M is pointwise collinear with the Reeb vector field ξ, then M is locally the product of the Kähler manifold and an interval or unit circle S^1 and the Reeb vector field ξ is Killing.*

Proof. Suppose $V = b\xi$, where b is a smooth function on M.
Then, from (1.1), we get

$$(\pounds_{b\xi}g)(X,Y) = 2\Omega g(X,Y), \tag{3.5}$$

which implies

$$g(\nabla_X b\xi, Y) + g(X, \nabla_Y b\xi) = 2\Omega g(X,Y). \tag{3.6}$$

Using (2.3) into the preceding equation yields

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2b[\alpha g(X,Y) - \alpha\eta(X)\eta(Y) - g(X,\phi hY)] = 2\Omega g(X,Y), \tag{3.7}$$

contracting the foregoing equation gives

$$(\xi b) = (2n+1) - 2nb\alpha. \tag{3.8}$$

Replacing $X = \phi X$ in (3.7), we obtain

$$(\phi Xb)\eta(Y) + 2b[\alpha g(\phi X, Y) - g(\phi X, \phi hY)] = 2\Omega g(\phi X, Y). \tag{3.9}$$

By setting $Y = \xi$ in (3.9), we get

$$(\phi X)b = 0. \tag{3.10}$$

Using equation (3.10) in (3.9), we acquire

$$b[\alpha g(\phi X, Y) - g(\phi X, \phi hY)] = \Omega g(\phi X, Y). \tag{3.11}$$

Interchanging X and Y in (3.11), we get

$$b[-\alpha g(\phi X, Y) - g(\phi X, \phi hY)] = -\Omega g(\phi X, Y). \tag{3.12}$$

In view of the above two equations, we obtain

$$(b\alpha - \Omega)g(\phi X, Y) = 0, \tag{3.13}$$

which implies

$$b\alpha = \Omega. \tag{3.14}$$

Using (3.14) in (3.12), we infer

$$bg(\phi X, \phi hY) = 0,$$

which implies either $b = 0$ *or* $b \neq 0$.
Case(i): If $b = 0$, then (3.14) implies $\Omega = 0$. Hence, V is Killing, which means that ξ is Killing.
Case(ii): If $b \neq 0$, then $g(\phi X, \phi hY) = 0$, hence h=0, which means that ξ is Killing.
Hence, both cases imply ξ is Killing. Hence using Remark 1. This completes the proof.   □

4 η-Ricci-Yamabe soliton with conformal vector fields

The concept of an η-Ricci-Yamabe soliton comes as a generalization of Ricci-Yamabe soliton of type (α_1, β_1) defined as ([15]):

$$\pounds_V g + 2\alpha_1 Ric + (2\lambda_1 - \beta_1 r)g + 2\nu_1 \eta(X)\eta(Y) = 0, \tag{4.1}$$

where α_1, β_1, and ν_1 are constants. If $\nu_1 = 0$, then η-Ricci-Yamabe soliton reduces to a Ricci-Yamabe soliton, and for $\nu_1 \neq 0$, it is called proper η-Ricci-Yamabe soliton.
Again, let us note that η-Ricci-Yamabe soliton of type $(1,0)$ or type $(0,1)$ is η-Ricci soliton and η-Yamabe soliton, respestively.
In [4] De, at al. studied Ricci-Yamabe solitons on three dimensional Riemannian manifolds.

Theorem 4.1. *If an almost α-cosympectic manifold with constant scalar curvature admits an η-Ricci-Yamabe soliton with conformal vector, then either it is an Einstein manifold or it is locally the product of a Kähler manifold and an interval or unit circle S^1.*

Proof. Let the almost α-cosymplectic manifold M endowed an η-Ricci-Yamabe soliton admit conformal vector field.

Using (1.1) in (4.1), we get

$$2\Omega g(X, Y) + 2\alpha_1 Ric(X, Y) + (2\lambda - \beta_1 r)g(X, Y) + 2\nu_1 \eta(X)\eta(Y) = 0. \tag{4.2}$$

Setting $X = Y = \xi$ in (4.2) entails that

$$\Omega = \alpha_1(2n\alpha^2 + tr(h^2)) - (\lambda - \frac{\beta_1}{2}r) - \nu_1. \tag{4.3}$$

Contracting (4.2), we get

$$\Omega(2n + 1) = -\alpha_1 r - (\lambda - \frac{\beta_1}{2}r)(2n + 1) - \nu_1, \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\alpha_1(2n\alpha^2 + tr(h^2))(2n + 1) + \alpha_1 r = 2n\nu_1, \tag{4.5}$$

if we take r is constant and $h = 0$. Therefore, the manifold becomes an α-cosymplectic manifold. Also, (4.5) implies

$$\nu_1 = \frac{\alpha_1(2n\alpha^2 + tr(h^2))(2n + 1) + \alpha_1 r}{2n}. \tag{4.6}$$

Since r is constant, the equation (4.4) implies Ω is constant.
Hence from (1.5), we get

$$\Omega r = 0, \tag{4.7}$$

which means that $\Omega = 0$, Hence V is Killing. Therefore, (4.2) implies

$$\alpha_1 Ric(X, Y) = -(\lambda - \frac{\beta_1}{2}r)g(X, Y) - \nu_1 \eta(X)\eta(Y), \tag{4.8}$$

which is an η-Einstein manifold with constant co-efficients.
Now, taking covariant differentiation of (4.8), we get

$$\begin{aligned} \alpha_1(\nabla_Z Ric)(X, Y) &= -\nu_1[\alpha g(X, Y)\eta(Y) + \alpha\eta(X)g(Y, Z) - 2\alpha\eta(X)\eta(Y)\eta(Z) \\ &+ g(\phi X, hZ)\eta(Y) + g(\phi Y, hZ)\eta(X)]. \end{aligned} \tag{4.9}$$

Contracting the foregoing equation, we obtain

$$2n\alpha\nu_1\eta(Y) = 0, \tag{4.10}$$

which implies either $\nu_1 = 0$ or $\alpha = 0$.
Case(i): If $\nu_1 = 0$ then (4.8) implies

$$\alpha_1 Ric(X, Y) = -(\lambda - \frac{\beta_1}{2}r)g(X, Y),$$

Which shows that the manifold is an Einstein manifold.
Case(ii): If $\alpha = 0$ then the manifold becomes a cosymplectic manifold. Hence, from the Remark 2, the proof is completed. $\quad\square$

Theorem 4.2. *Let M be an almost α-cosymplectic manifold admitting a η-Ricci-Yamabe soliton. If the potential vector field V is pointwise collinear with ξ, then M is a nearly quasi Einstein manifold.*

Proof. Since M is a η-Ricci-Yamabe soliton whose potential vector field V is pointwise collinear with ξ, that is $V = b\xi$, then
we know that

$$(£_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X).$$

Using (2.1) and (2.3), above equation becomes

$$
\begin{aligned}
(£_V g)(X, Y) &= g(Db, X)\eta(Y) + g(Db, Y)\eta(X) + 2b\alpha g(X, Y) \\
&\quad - 2b\alpha\eta(X)\eta(Y) - 2bg(\phi hX, Y),
\end{aligned}
\tag{4.11}
$$

for any $X, Y \in \chi(M)$, where Db is the gradient of a smooth function b on M.
By combining (4.1) and (4.11), we have

$$
\begin{aligned}
Ric(X, Y) &= \frac{1}{2\alpha_1}[(-2b\alpha - 2\lambda + \beta_1 r)g(X, Y) - 2(b\alpha + \nu_1)\eta(X)\eta(Y) \\
&\quad - g(Db, X)\eta(Y) - g(Db, Y)\eta(X)].
\end{aligned}
\tag{4.12}
$$

If we denote the dual 1-form of Db by ϕ, then (4.12) reduces to

$$
\begin{aligned}
Ric(X, Y) &= \frac{1}{2\alpha_1}[(-2b\alpha - 2\lambda + \beta_1 r)g(X, Y) - 2(b\alpha + \nu_1)\eta(X)\eta(Y) \\
&\quad - \phi(X)\eta(Y) - \phi(X)\eta(X)].
\end{aligned}
\tag{4.13}
$$

If we take a non-vanishing symmetric $(0, 2)$-tensor E in (4.13) such that

$$
E(X, Y) = \frac{1}{2\alpha_1}[-2(b\alpha + \nu_1)\eta(X)\eta(Y) - \phi(X)\eta(Y) - \phi(Y)\eta(X)],
$$

then the equation (4.13) becomes

$$
Ric(X, Y) = \frac{1}{2\alpha_1}[(-2b\alpha - 2\lambda + \beta_1 r)g(X, Y)] + E(X, Y).
$$

This menas that M is a nearly quasi-Einstein manifold. □

As an immediate consequence of above theorem, we have the following corollary:

Corollary 4.3. *Let M be an almost α-cosymplectic manifold admitting a η-Ricci-Yamabe soliton. If the potential vector field V is the characteristic vector field ξ, then M is an η-Einstein manifold.*

4.1 Almost α-cosymplectic (κ, μ)-spces:

Lemma 4.4. *[12] Let M be an almost α-cosymplectic (κ, μ)-space. Then the following relations hold*

$$h^2 = (\kappa + \alpha^2)\phi^2, \quad for \ \ k \leq -\alpha^2, \tag{4.14}$$

$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(hY, X)\xi - \eta(Y)hX), \tag{4.15}$$

$$Ric(Z, \xi) = 2n\kappa\eta(Z), \ \ Q\xi = 2n\kappa\xi, \ \ Ric(\xi, \xi) = 2n\kappa, \tag{4.16}$$

for any $X, Y \in \chi(M)$, from (4.14), we find easily that $\kappa \leq 0$ and $\kappa = 0$ if and only if M is a cosymplectic manifold. Thus in the following we always suppose $\kappa < 0$.

Theorem 4.5. *If an almost α-cosymplectic (κ, μ)-space with constant scalar curvature admits an η-Ricci-Yamabe soliton with conformal vector, then either it is an Einstein manifold or it is locally the product of a Kähler manifold and an interval or unit circle S^1.*

Proof. We first assume that the almost α-cosymplectic (κ, μ)-spaces admits an η-Ricci -Yamabe soliton with conformal vector field.

Employing equation (1.1) in equation (4.1), we obtain

$$2\Omega g(X, Y) + 2\alpha_1 S(X, Y) + (2\lambda - \beta_1 r)g(X, Y) + 2\nu_1 \eta(X)\eta(Y) = 0. \tag{4.17}$$

Setting $X = Y = \xi$ in (4.17), we get

$$\Omega = -2n\alpha_1\kappa - (\lambda - \frac{\beta_1}{2}r) - \nu_1. \tag{4.18}$$

Contracting equation (4.17) yields

$$\Omega(2n + 1) = -\alpha_1 r - (\lambda - \frac{\beta_1}{2}r)(2n + 1) - \nu_1. \tag{4.19}$$

From (4.18) and (4.19), we obtain

$$-2n\alpha_1\kappa(2n + 1) + \alpha_1 r = 2n\nu_1. \tag{4.20}$$

Also, (4.20) implies

$$\nu_1 = \frac{\alpha_1 r - 2n\alpha_1\kappa(2n + 1)}{2n}.$$

Since r is constant, then (4.19) implies Ω is constant.

Hence, from (1.5), we get

$$\Omega r = 0, \tag{4.21}$$

which means that $\Omega = 0$. Hence V is Killing. Therefore, (4.17) implies

$$\alpha_1 Ric(X, Y) = -(\lambda - \frac{\beta_1}{2}r)g(X, Y) - \nu_1\eta(X)\eta(Y), \tag{4.22}$$

which is an η-Einstein manifold with constant co-efficients.

Now, by taking the covariant differentiation of equation (4.22), we obtain

$$\begin{aligned}\alpha_1(\nabla_Z Ric)(X, Y) &= -\nu_1[\alpha g(X, Y)\eta(Y) + \alpha\eta(X)g(Y, Z) - 2\alpha\eta(X)\eta(Y)\eta(Z) \\ &+ g(\phi X, hZ)\eta(Y) + g(\phi Y, hZ)\eta(X)].\end{aligned} \tag{4.23}$$

Contracting the foregoing equation, we get

$$2n\alpha\nu_1\eta(Y) = 0, \tag{4.24}$$

which implies either $\nu_1 = 0$ *or* $\alpha = 0$.

Case(i): If $\nu_1 = 0$ then (4.22) implies

$$\alpha_1 S(X, Y) = -(\lambda - \frac{\beta_1}{2}r)g(X, Y),$$

which shows M is an Einstein manifold.

Case(ii): If $\alpha = 0$ then the manifold becomes a cosymplectic manifold. Therefore, referring to Remark 2, the proof is completed.

$\square$

4.2 Homothetic vector fields on three-dimensional almost α-cosymplectic (κ, μ)-spaces

Lemma 4.6. *For a three-dimensional almost α-cosymplectic (κ, μ)-spaces M^3, we have*

$$QY = (3\kappa - \frac{r}{2})\eta(Y)\xi + (\frac{r}{2} - \kappa)Y + \mu hY, \tag{4.25}$$

$$Ric(X, Y) = (3\kappa - \frac{r}{2})\eta(Y)\eta(X) + (\frac{r}{2} - \kappa)g(X, Y) + \mu g(hY, X), \tag{4.26}$$

where r is the scalar curature of M.

Proof. It is well-known that the curvature tensor of any three-dimensional Riemannian manifold is written as

$$
\begin{aligned}
R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + Ric(Y,Z)X - Ric(X,Z)Y \quad (4.27)\\
&\quad - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

Setting $Z = \xi$ in foregoing equation and using (4.15) and (4.16), we get

$$
\kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) = \eta(Y)QX - \eta(X)QY + (2n\kappa - \frac{r}{2})[\eta(Y)X - \eta(X)Y].
$$

Moreover, by taking $X = \xi$ and using (4.16) again, we obtain (4.25).   □

Theorem 4.7. *If a three-dimensional α-cosymplectic (κ, μ)-space admits a homothetic vector field V, then either the manifold is of constant sectional curvature or, V is an infinitesimal contact transformation.*

Proof. Let the vector field V in M^3 is homothetic. Then

$$
(\pounds_V g)(X,Y) = 2\Omega g(X,Y), \tag{4.28}
$$

where Ω is constant. From (1.4) and (1.5), we get

$$
(\pounds_V Ric)(X,Y) = 0 \ and \ \pounds_V r = -2r\Omega. \tag{4.29}
$$

Definition of Lie-derivative infers that

$$
(\pounds_V \eta)X = (\pounds_V \eta)(X) - \eta(\pounds_V X). \tag{4.30}
$$

Equation (4.28) and (4.30) together implies

$$
\eta(\pounds_V \xi) = -\Omega \quad (\pounds_V \eta)\xi = \Omega. \tag{4.31}
$$

Now, we take Lie-derivative of the equation (4.26) along the homothetic vector field V entails that

$$
\begin{aligned}
(\pounds_V Ric)(X,Y) &= (3\kappa - \frac{r}{2})((\pounds_V \eta)(X)\eta(Y) + \eta(X)(\pounds_V \eta)(Y)) \quad (4.32)\\
&\quad + (3(V\kappa) - \frac{\pounds_V r}{2})\eta(X)\eta(Y) + (\frac{r}{2} - \kappa)(\pounds_V g)(X,Y)\\
&\quad + (\frac{\pounds_V r}{2} - V\kappa)g(X,Y) + (V\mu)g(X,hY)\\
&\quad + \mu[(\pounds_V g)(hY,X) + g((\pounds_V h)Y,X)].
\end{aligned}
$$

Using (4.28) and (4.30) in (4.32), we provide

$$
\begin{aligned}
(3\kappa &- \frac{r}{2})((\pounds_V \eta)(X)\eta(Y) + \eta(X)(\pounds_V \eta)(Y)) + (3(V\kappa) - \frac{\pounds_V r}{2})\eta(X)\eta(Y) \quad (4.33)\\
&+ (\frac{r}{2} - \kappa)2\Omega g(X,Y) - (r\Omega + V\kappa)g(X,Y) + (V\mu)g(X,hY)\\
&+ \mu[(\pounds_V g)(hY,X) + g((\pounds_V h)Y,X)] = 0.
\end{aligned}
$$

By substituting $Y = \xi$ into equation (4.33) and utilizing equation (4.31), we obtain

$$
\begin{aligned}
(3\kappa &- \frac{r}{2})((\pounds_V \eta)(X) + \eta(X)\Omega) + (3(V\kappa) + r\Omega)\eta(X) \quad (4.34)\\
&+ (\frac{r}{2} - \kappa)2\Omega\eta(X) - (r\Omega + V\kappa)\eta(X) - \mu g(h(\pounds_V \xi), X) = 0.
\end{aligned}
$$

By setting $X = \xi$ in (4.34) and using (4.31), we get

$$
V\kappa = -2\kappa\Omega. \tag{4.35}
$$

From the above two equations, we obtain

$$(3\kappa - \frac{r}{2})[(\pounds_V \eta)(X) - \Omega\eta(X)] + \Omega g(h(\pounds_V \xi), X) = 0. \tag{4.36}$$

On the other hand $g(\xi, \xi) = 1$ implies $\pounds_V \xi = -\Omega\xi$, then above equation becomes

$$(3\kappa - \frac{r}{2})[(\pounds_V \eta)(X) - \Omega\eta(X)] = 0,$$

which implies either $(3\kappa - \frac{r}{2}) = 0$ or $(3\kappa - \frac{r}{2}) \neq 0$.
Case(i): If $(3\kappa - \frac{r}{2}) = 0$, which means $r = 6\kappa$, here $\kappa < 0$, hence (4.26) implies

$$Ric(X, Y) = 2\kappa g(X, Y) + \mu g(X, hY). \tag{4.37}$$

In 3-dimension, using (4.37) in (4.27) with $h = 0$, we get
$R(X, Y)Z = \kappa[g(Y, Z)X - g(X, Z)Y]$,
which means that the manifold is of constant sectional curvature κ, $\kappa < 0$.
Case(ii): If $(3\kappa - \frac{r}{2}) \neq 0$, then $(L_V \eta)X = \Omega\eta(X)$. Hence V is an infinitesimal contact trans formation.
This complete the proof.

$\square$

It is well known that any homothetic vector field on a compact manifold without boundary is a Killing vector field [6]. Consequently, from the preceding theorem, we deduce the following result.

Corollary 4.8. *If a compact 3-dimensional almost α-cosymplectic (κ, μ)-space without boundary admits a homothetic vector field V, then either the manifold is of constant sectional curvature κ, $\kappa < 0$ or V is an infinitesimal contact transformation.*

Example 4.9. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, e_3 be a linearly independent frame field on M, given by,

$$e_1 = e^{\alpha z}\frac{\partial}{\partial x}, \quad e_2 = e^{\alpha z}\frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by,

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let η be the 1-form by $\eta(X) = g(X, e_3)$ for any $X \in \chi(M)$, and ϕ be the $(1, 1)$-tensor field defined by,

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

By direct computations, we can easily to see that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi),$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus $M^3(\phi, \xi, \eta, g)$ is a 3-dimensional almost α-cosymplectic (κ, μ)-space. From the Lie-operatory, we have the non-zero compnents

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1, \quad [e_2, e_3] = \alpha e_2.$$

Furthermore, by ∇, we denote the Levi-civita connection on M, by using Koszul's formula, we can calculate, easily

$$\nabla_{e_1} e_1 = -\alpha e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0,$$
$$\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_3} e_2 = 0,$$
$$\nabla_{e_1} e_3 = \alpha e_1, \quad \nabla_{e_2} e_3 = \alpha e_2, \quad \nabla_{e_3} e_3 = 0.$$

Comparing the above relation with

$$\nabla_X e_3 = -\alpha\phi^2(X) - \phi hX,$$

We can observe

$$he_1 = 0, \quad he_2 = 0, \quad he_3 = 0.$$

We obtain the components of the curvature tensor as follows:

$$R(e_1, e_2)e_1 = \alpha^2 e_2, \quad R(e_1, e_3)e_1 = \alpha^2 e_3, \quad R(e_2, e_3)e_1 = 0,$$
$$R(e_1, e_2)e_2 = -\alpha^2 e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = \alpha^2 e_3,$$
$$R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = \alpha^2 e_1, \quad R(e_2, e_3)e_3 = -\alpha^2 e_2.$$

From the above relations, we have

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2\alpha^2.$$

Hence $r = -6\alpha^2$.

Here, M^3 is an 3-dimensional almost α-cosymplectic (κ, μ)-space then curvature tensor R is given by

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

Hence,

$$R(e_1, e_3)e_3 = \kappa e_1,$$
$$R(e_2, e_3)e_3 = \kappa e_2,$$
$$R(e_3, e_3)e_3 = 0,$$

which imply that $\kappa = -\alpha^2$, therefore $r = 6\kappa$, $\quad \kappa < 0,$ $\quad$ this verifies theorem (4.4).

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