Palestine Journal of Mathematics

Vol 15(1)(2026) , 626–643

A NOTE ON RIGHT EIGENVALUES OF QUATERNIONIC MATRIX POLYNOMIALS

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Communicated by: Ayman Badawi

MSC 2010 Classifications: Primary 12E15; 34L15; Secondary 15A18; 15A66.

Keywords and phrases: Quaternionic matrices, Right eigenvalues, Matrix norms, Quaternionic matrix polynomial, Quaternionic monic matrix polynomial.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Acknowledgement: *The authors would like to thank Integral University, Lucknow, India, for providing the manuscript number IU/R & D/2025-MCN0003324 to present research work..*

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Abstract In this paper, upper and lower bounds for right eigenvalues of a quaternionic matrix polynomials are derived. Then, localization theorems for right eigenvalues of a quaternionic monic matrix polynomial are given. Finally, we give numerical examples to illustrate our results.

1 Introduction

The journey to understand and set limits for the eigenvalues of complex matrix polynomials began with the important work of Higham and Tisseur in 2003 [1]. Since then, many researchers have explored this area, focusing on finding the bounds and locations of these eigenvalues (see, for example, [2, 3, 4, 5, 6]). A key idea in this research is that for monic matrix polynomials, the eigenvalues of their block companion matrix are exactly the same as the eigenvalues of the polynomial itself. This fact helps a lot in finding upper and lower bounds by using different matrix norms, which measure the size or behavior of matrices. Besides complex matrices, quaternionic matrix polynomials have also become a topic of interest. These use quaternions—numbers with special rules, discovered by Sir William Hamilton in 1843. Quaternions are useful in many areas, such as video games, 3D graphics, quantum science, and system control (see, for example, [7, 8, 9, 10, 11]). The block companion matrix for a monic quaternionic matrix polynomial serves a similar purpose as in the complex case. It offers a structured approach for determining bounds on eigenvalues by employing matrix inequalities and norms. Early research in this area established initial upper and lower bounds for quaternionic eigenvalues (see [12, 13, 14]). Building on these foundational results, later studies, including those in (see [12, 15, 16, 17, 18]), have refined and extended these bounds. The current work continues this progression by introducing explicit bounds and locations for the right eigenvalues of quaternionic matrix polynomials through matrix norm-based techniques. Matrix norms are crucial tools in this analysis. They allow for deriving bounds that are both mathematically rigorous and practical for computation. By extending these methods to the quaternionic domain, researchers not only deepen our understanding of quaternionic eigenvalue problems but also offer new perspectives on complex matrix polynomials. The relationship between a matrix polynomial and its block companion matrix is particularly important because it enables the use of well-established techniques from matrix analysis to address more complex problems. The localization theorem has also proven invaluable in this field. It helps identify regions in the complex plane where eigenvalues are likely to be found. Combined with the right spectral radius inequality, this theorem provides a robust framework for exploring the spectral properties of quaternionic matrix polynomials. These methods

not only aid in approximating eigenvalue locations but also establish bounds that are essential for theoretical studies and practical applications. This line of research has relevance beyond pure mathematics. For instance, in computer graphics, quaternions are widely used to represent rotations. Understanding the spectral properties of quaternionic matrices can lead to more efficient algorithms for rendering and animation. Similarly, in control theory, analyzing the stability of systems involving quaternionic matrices often depends on eigenvalue bounds, enabling better system design and optimization.
The paper is organized as follows: Section 2 reviews some existing results from references [12] and [19]. Section 3 provides upper and lower bounds for right eigenvalues of quaternionic matrix polynomials as well as quaternionic monic matrix polynomials. In Section 4, we discuss the location of right eigenvalues of quaternionic monic matrix polynomials. Finally, Section 5 presents examples to illustrate our results.

2 Notation and preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The set of real quaternions is defined by

$$\mathbb{H} = \{q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. The conjugate of $q \in \mathbb{H}$ is $\overline{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and the modulus of q is $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. $\Im(a)$ denotes the imaginary part of $a \in \mathbb{C}$. The real part of a quaternion $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is defined as $\Re(q) = a_0$. The collection of all n-column vectors with elements in $\mathbb{H}$ is denoted by $\mathbb{H}^n$. For $x \in \mathcal{K}^n$, where $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the transpose of x is x^T. If $x = [x_1, \dots, x_n]^T$, the conjugate of x is defined as $\overline{x} = [\overline{x}_1, \dots, \overline{x}_n]^T$ and the conjugate transpose of x is defined as $x^H = [\overline{x}_1, \dots, \overline{x}_n]$. For $x, y \in \mathbb{H}^n$, the inner product is defined as $\langle x, y\rangle = y^H x$ and the norm of x is defined as $\|x\|_2 = \sqrt{\langle x, x\rangle}$. The sets of $m \times n$ real, complex, and quaternionic matrices are denoted by $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$, and $M_{m\times n}(\mathbb{H})$, respectively. When $m = n$, these sets are denoted by $M_n(\mathcal{K})$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $A \in M_{m\times n}(\mathcal{K})$, the conjugate, transpose, and conjugate transpose of A are defined as $\overline{A} = (\overline{a}_{ij})$, $A^T = (a_{ji}) \in M_{n\times m}(\mathcal{K})$, and $A^H = (\overline{A})^T \in M_{n\times m}(\mathcal{K})$, respectively. The set

$$[q] = \{r \in \mathbb{H} : r = \rho^{-1}\, q\, \rho \text{ for all } 0 \neq \rho \in \mathbb{H}\}$$

is called an equivalence class of $q \in \mathbb{H}$. We define the 2-matrix norm and Frobenius norm on $A \in M_n(\mathbb{H})$ by

$$||A||_2 = \sup_{x\neq 0}\left\{\frac{\|Ax\|_2}{\|x\|_2} : x \in \mathbb{H}^n\right\} = \|A^H\|_2 \text{ and } ||A||_F = \left[\text{trace}(A^H A)\right]^{1/2}, \text{ respectively.}$$

Definition 2.1. *Let $A \in M_n(\mathbb{H})$. Then the right eigenvalues of A is defined as follows*

$$\Lambda_r(A) \quad = \quad \{\lambda \in \mathbb{H} : Ay = y\lambda \text{ for some non-zero } y \in \mathbb{H}^n\}.$$

Definition 2.2. *Let $x \in \mathbb{H}^n$. Then x can be uniquely expressed as $x = x_1 + x_2\mathbf{j}$, where $x_1, x_2 \in \mathbb{C}^n$. Define the function $\xi : \mathbb{H}^n \to \mathbb{C}^{2n}$ by*

$$\xi_x = \begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix}.$$

This function ξ is an injective linear transformation from $\mathbb{H}^n$ to $\mathbb{C}^{2n}$. The vector ξ_x is called the complex adjoint vector of x.

Definition 2.3. *Let $A \in M_n(\mathbb{H})$. Then A can be uniquely expressed as $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in M_n(\mathbb{C})$. Define the function $\chi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ by*

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{bmatrix}.$$

The matrix χ_A is called the complex adjoint matrix of the quaternionic matrix A.

Definition 2.4. *Let $\mathbb{L}_m(M_{2n}(\mathbb{C}))$ be the space of complex matrix polynomials of degree $\leq m$. Then $L \in \mathbb{L}_m(M_{2n}(\mathbb{C}))$ is defined as*

$$P(\mu) = \sum_{i=0}^{m} \chi_{A_i} \mu^i,$$

where $A_i \in M_n(\mathbb{H})$, $\chi_{A_i} \in M_{2n}(\mathbb{H})$, and $\mu \in \mathbb{C}$.

Let $\mathbb{L}_m(M_n(\mathbb{H}))$ be the space of right matrix polynomials over the skew field of quaternions. Any element $L \in \mathbb{L}_m(M_n(\mathbb{H}))$ (degree $\leq m$) is defined as

$$L(\lambda) = \sum_{i=0}^{m} A_i \lambda^i, \tag{2.1}$$

where $A_i \in M_n(\mathbb{H})$, $0 \leq i \leq m$.

Definition 2.5. *[12, Definition 4.1] Let $L \in \mathbb{L}_m(M_n(\mathbb{H}))$ be as in (2.1), and $\mu \in \mathbb{H}$. Then μ is called a right eigenvalue of the polynomial L if*

$$A_0 x + A_1 x\mu + \cdots + A_m x \mu^m = 0$$

for some nonzero $x \in \mathbb{H}^n$, x is called the right eigenvector corresponding to right eigenvalue μ.

Furthermore, for $\delta \geq 0$, represent the open ball

$$\tilde{D}(0, \delta) = \{\lambda \in \mathbb{H} : |\lambda| < \delta\}$$

and the closed ball

$$D(0, \delta) = \{\lambda \in \mathbb{H} : |\lambda| \leq \delta\}.$$

We now revisit the following results, which serve as a foundation for the development of our theory.

Lemma 2.6. *[19, Theorem 4.1] Let $A \in M_n(\mathbb{H})$. Then $\chi_A \in M_{2n}(\mathbb{C})$ and*

$$\underset{||x||_2 \neq 0}{max} \frac{||Ax||_2}{||x||_2} = \underset{||y||_2 \neq 0}{max} \frac{||\chi_A y||_2}{||y||_2}.$$

Theorem 2.7. *[12, Lemma 3.5] Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then $||A||_2 \leq ||A||_F$.*

3 Bounds for right eigenvalues of quaternionic matrix polynomials

If $L(\lambda)x = 0$, that is, $A_0 x + A_1 \lambda x + A_2 \lambda^2 x + \cdots + A_m \lambda^m x = 0$, then λ is not a right eigenvalue problem for quaternionic matrix polynomial $L(\lambda)$. For example

Example 3.1. Let choose 2×2 quaternionic matrices $A_0 = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{k} \end{bmatrix}$, and $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $x = \begin{bmatrix} \mathbf{j} \\ 0 \end{bmatrix}$ and $\lambda = \mathbf{i}$. Putting these values in (3.1), we get

$$A_0 x + A_1 \lambda x \quad = \quad 0. \tag{3.1}$$

However,

$$A_0 x + A_1 x \lambda \quad \neq \quad 0. \tag{3.2}$$

From (3.1) and (3.2), we observe that

$$A_0 x + A_1 \lambda x \neq A_0 x + A_1 x \lambda.$$

Thus, $\lambda = \mathbf{i}$ is not a right eigenvalue of $A_0 x + A_1 x \lambda$.

However, if $P(\mu)\xi_x = 0$, then μ is a right eigenvalue of quaternionic matrix polynomials.

3.1 Upper Bounds for right eigenvalues of quaternionic matrix polynomials

In this section, we provide multiple upper bounds for the right eigenvalues of quaternionic monic matrix polynomials in terms of 2-matrix norm and Frobenius norm.

Theorem 3.2. *Let $L(\lambda) = \sum_{i=0}^{m} A_i \lambda^i$ be a quaternionic matrix polynomial that satisfy the dominant property*

$$||A_m||_\alpha > ||A_i||_\alpha, \ \forall \, i = 0, 1, 2, \cdots, m-1.$$

Then, every right eigenvalues μ of $L(\lambda)$ satisfy the following inequality

$$|\mu| < 1 + ||A_m||_\alpha ||(A_m)^{-1}||_\alpha, \ \ \alpha \in \{2, F\}.$$

Proof. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. The conclusion is straightforward when $|\mu| \leq 1$. Therefore, we may assume $|\mu| > 1$. Then, we have

$$\begin{aligned}
||P(\mu)\xi_x||_\alpha &= ||(\chi_{A_0} + \chi_{A_1}\mu + \cdots + \chi_{A_m}\mu^m)\xi_x||_\alpha \\
||P(\mu)\xi_x||_\alpha &= |\mu|^m \left[||(\chi_{A_0}\mu^{-m} + \chi_{A_1}\mu^{1-m} + \cdots + \chi_{A_m})\xi_x||_\alpha\right] \\
||P(\mu)\xi_x||_\alpha &\geq |\mu|^m \left[||\chi_{A_m}\xi_x||_\alpha - ||\sum_{i=0}^{m-1} \frac{\chi_{A_i}\xi_x}{\mu^{m-1}}||_\alpha\right].
\end{aligned}$$

Since x is a unit vector, i.e., $||x||_\alpha = 1$, then $||\xi_x||_\alpha = 1$. Hence,

$$||P(\mu)\xi_x||_\alpha \geq |\mu|^m \left[||(\chi_{A_m})^{-1}||_\alpha^{-1} - ||\sum_{i=0}^{m-1} \frac{\chi_{A_i}}{\mu^{m-1}}||_\alpha\right].$$

Let $j = m - i$. When $i = 0$, then $j = m$ and when $i = m-1$, we have $j = 1$. The order of summation can be reversed without altering the result. i.e., $\sum_{j=m}^{1} \frac{1}{|\mu|^j} = \sum_{j=1}^{m} \frac{1}{|\mu|^j}$.

$$\begin{aligned}
||P(\mu)\xi_x||_\alpha &\geq |\mu|^m ||(\chi_{A_m})^{-1}||_\alpha^{-1} \left[1 - ||\chi_{A_m}||_\alpha ||(\chi_{A_m})^{-1}||_\alpha \sum_{j=1}^{m} \frac{1}{|\mu|^j}\right] \\
||P(\mu)\xi_x||_\alpha &> |\mu|^m ||(\chi_{A_m})^{-1}||_\alpha^{-1} \left[1 - ||\chi_{A_m}||_\alpha ||(\chi_{A_m})^{-1}||_\alpha \sum_{j=0}^{\infty} \frac{1}{|\mu|^j}\right].
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{1}{|\mu|^j} &= \frac{1}{|\mu| - 1} \\
||P(\mu)\xi_x||_\alpha &\geq |\mu|^m ||(\chi_{A_m})^{-1}||_\alpha^{-1} \left[1 - \frac{||\chi_{A_m}||_\alpha ||(\chi_{A_m})^{-1}||_\alpha}{|\mu| - 1}\right] \\
||P(\mu)\xi_x||_\alpha &\geq \frac{|\mu|^m ||(\chi_{A_m})^{-1}||_\alpha^{-1}}{|\mu| - 1} \left[|\mu| - 1 - ||\chi_{A_m}||_\alpha ||(\chi_{A_m})^{-1}||_\alpha\right].
\end{aligned}$$

If $\alpha = 2$, then by Lemma 2.6, we get

$$||P(\mu)\xi_x||_2 \geq \frac{|\mu|^m ||(A_m)^{-1}||_2^{-1}}{|\mu| - 1} \left[|\mu| - 1 - ||A_m||_2 ||(A_m)^{-1}||_2\right]. \tag{3.3}$$

If $\alpha = F$, then by Lemma 2.7, we have

$$\begin{aligned}
||P(\mu)\xi_x||_F &\geq ||P(\mu)\xi_x||_2 \\
||P(\mu)\xi_x||_\alpha &\geq \frac{|\mu|^m ||(A_m)^{-1}||_2^{-1}}{|\mu| - 1} \left[|\mu| - 1 - ||A_m||_2 ||(A_m)^{-1}||_2\right]. \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4), we obtain

$$||P(\mu)\xi_x||_\alpha \geq \frac{|\mu|^m ||(A_m)^{-1}||_\alpha^{-1}}{|\mu|-1} \left[|\mu| - 1 - ||A_m||_\alpha ||(A_m)^{-1}||_\alpha\right].$$

If $|\mu| > 1 + ||A_m||_\alpha ||(A_m)^{-1}||_\alpha$, then we arrive at $||P(\mu)\xi_x||_\alpha > 0$, which leads to a contradiction. Hence

$$|\mu| < 1 + ||A_m||_\alpha ||A_m^{-1}||_\alpha.$$

If $\mu_1, \mu_2, \cdots, \mu_n$ are complex right eigenvalues of $L(\lambda)$, then

$$\Lambda_r(L(\lambda)) = [\mu_1] \cup [\mu_2] \cup \cdots \cup [\mu_n].$$

Therefore

$$|\mu| = |\rho^{-1}\lambda\rho| < 1 + ||A_m||_\alpha ||(A_m)^{-1}||_\alpha.$$

□

Theorem 3.3. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$N = \max_{0 \leq i \leq m-1} ||A_i||_\alpha, \quad \alpha \in \{2, F\}.$$

Then, every right eigenvalue of $L(\lambda)$ is lie in the open ball

$$D(0, r_3) = \{\lambda \in \mathbb{H} | |\lambda| < r_3\}, \quad r_3 = 1 + N.$$

Proof. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. The conclusion is evident when $|\mu| \leq 1$. Thus, we can take $|\mu| > 1$ as our assumption. Consequently, we get

$$\begin{aligned} ||P(\mu)\xi_x||_\alpha &= ||(\chi_{A_0} + \chi_{A_1}\mu + \cdots + \chi_I \mu^m)\xi_x||_\alpha \\ ||P(\mu)\xi_x||_\alpha &\geq |\mu|^m \left[||\chi_I \xi_x||_\alpha - ||\sum_{i=0}^{m-1} \frac{\chi_{A_i}\xi_x}{\mu^{m-i}}||_\alpha\right]. \end{aligned}$$

Since x is a unit vector, i.e., $||x||_\alpha = 1$, then $||\xi_x||_\alpha = 1$, $||\chi_I \xi_x||_\alpha \leq ||\chi_I||_\alpha ||\xi_x||_\alpha = 1$, and $||\chi_{A_i}\xi_x||_\alpha \leq ||\chi_{A_i}||_\alpha ||\xi_x||_\alpha = ||\chi_{A_i}||_\alpha$. We get

$$||P(\mu)\xi_x||_\alpha \geq |\mu|^m \left[1 - \sum_{i=0}^{m-1} \frac{||\chi_{A_i}||_\alpha}{|\mu|^{m-i}}\right].$$

If $\alpha = 2$, then by Lemma 2.6, we obtain

$$\begin{aligned} ||P(\mu)\xi_x||_2 &\geq |\mu|^m \left[1 - \sum_{i=0}^{m-1} \frac{||A_i||_2}{|\mu|^{m-i}}\right] \\ ||P(\mu)\xi_x||_\alpha &\geq |\mu|^m \left[1 - N\sum_{j=1}^{m} \frac{1}{|\mu|^j}\right] > |\mu|^m \left[1 - N\sum_{j=1}^{\infty} \frac{1}{|\mu|^j}\right] \\ ||P(\mu)\xi_x||_2 &\geq |\mu|^m \left[1 - \frac{N}{|\mu|-1}\right] = \frac{|\mu|^m}{|\mu|-1}\left[|\mu| - 1 - N\right]. \end{aligned} \tag{3.5}$$

If $\alpha = F$, then by Lemma 2.7, we have

$$||P(\mu)\xi_x||_F \geq ||P(\mu)\xi_x||_2 \geq \frac{|\mu|^m}{|\mu|-1}\left[|\mu| - 1 - N\right]. \tag{3.6}$$

From (3.5) and (3.6), we get

$$||P(\mu)\xi_x||_\alpha \quad \geq \quad \frac{|\mu|^m}{|\mu|-1}\left[|\mu|-1-N\right].$$

Thus, when $|\mu| \geq 1+N$, we find $||L(\mu)\xi_x||_\alpha > 0$, leading to a contradiction. Therefore, we have

$$|\mu| < 1+N.$$

□

Theorem 3.4. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$M \quad = \quad \max_{0\leq i\leq m-2} ||A_i||_\alpha.$$

Then, each right eigenvalue μ of $L(\lambda)$ is approximated by

$$|\mu| \quad \leq \quad \frac{1}{2}\left\{1 + ||A_{m-1}||_\alpha + \left[(1-||A_{m-1}||_\alpha)^2 + 4M\right]^{1/2}\right\}, \quad \alpha \in \{2, F\}.$$

Proof. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Let us suppose the contrary to derive a contradiction.

$$\begin{aligned} |\mu| \quad &> \quad \frac{1}{2}\left\{1 + ||A_{m-1}||_\alpha + \left[(1-||A_{m-1}||_\alpha)^2 + 4M\right]^{1/2}\right\} \\ 2|\mu| - 1 - ||A_{m-1}||_\alpha \quad &> \quad \left[(1-||A_{m-1}||_\alpha)^2 + 4M\right]^{1/2}. \end{aligned}$$

Squaring on both sides, we get

$$\{2|\mu| - (1 + ||A_{m-1}||_\alpha)\}^2 \quad > \quad (1-||A_{m-1}||_\alpha)^2 + 4M$$

$$(|\mu|-1)(|\mu| - ||A_{m-1}||_\alpha) - M \quad > \quad 0. \tag{3.7}$$

Multiplying (3.7) by $|\mu|^{m-1}$ and then dividing by $|\mu|-1$, we have

$$|\mu|^m - ||A_{m-1}||_\alpha|\mu|^{m-1} - M\,\frac{|\mu|^{m-1}}{|\mu|-1} \quad > \quad 0. \tag{3.8}$$

We know that,

$$\begin{aligned} M\,\frac{|\mu|^{m-1}}{|\mu|-1} \quad &> \quad M\,\frac{|\mu|^{m-1}-1}{|\mu|-1} = M(1 + |\mu| + \cdots + |\mu|^{m-2}) \\ M\,\frac{|\mu|^{m-1}}{|\mu|-1} \quad &\geq \quad ||(A_0 + A_1\mu + \cdots + A_{m-2}\mu^{m-2})x||_\alpha. \end{aligned} \tag{3.9}$$

Also,

$$|\mu|^m - ||A_{m-1}||_\alpha|\mu|^{m-1} \quad \leq \quad ||(I\mu^m + A_{m-1}\mu^{m-1})x||_\alpha. \tag{3.10}$$

By (3.8), we obtain

$$0 \quad < \quad |\mu|^m - ||A_{m-1}||_\alpha|\mu|^{m-1} - M\,\frac{|\mu|^{m-1}}{|\mu|-1}.$$

Putting (3.9) and (3.10) in (3.8), we get

$$\begin{aligned} 0 \quad &< \quad ||(I\mu^m + A_{m-1}\mu^{m-1})x||_\alpha - ||(A_0 + A_1\mu + \cdots + A_{m-2}\mu^{m-2})x||_\alpha \\ &\leq \quad ||(A_0 + A_1\mu + \cdots + A_{m-2}\mu^{m-2})x + (A_{m-1}\mu^{m-1} + I\mu^m)x||_\alpha. \end{aligned}$$

If $\alpha = 2$, then by Lemma 2.6, we have

$$0 \leq ||\chi_{A_0} + \chi_{A_1}\mu + \cdots + \chi_{A_{m-2}}\mu^{m-2} + \chi_{A_{m-1}}\mu^{m-1} + \chi_I \mu^m)\xi_x||_\alpha = ||P(\mu)\xi_x||_2.$$

If $\alpha = F$, then by Lemma 2.7 we obtain,

$$0 < ||P(\mu)\xi_x||_2 \leq ||P(\mu)\xi_x||_F.$$

Therefore, for

$$|\mu| > \frac{1}{2}\left\{1 + ||A_{m-1}||_\alpha + \left[(1 - ||A_{m-1}||_\alpha)^2 + 4M\right]^{1/2}\right\},$$

the inequality $||P(\mu)\xi_x||_\alpha > 0$ holds, resulting in a contradiction. Hence, we get

$$|\mu| \leq \frac{1}{2}\left\{1 + ||A_{m-1}||_\alpha + \left[(1 - ||A_{m-1}||_\alpha)^2 + 4M\right]^{1/2}\right\}.$$

□

Theorem 3.5. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$\beta = \max_{1\leq i\leq m} ||A_{m-i} - A_{m-i-1}||_\alpha, \text{ and } A_{-1} = 0.$$

Then, each right eigenvalue μ of $L(\lambda)$ is approximated by

$$|\mu| \leq \frac{1}{2}\left\{1 + ||I - A_{m-1}||_\alpha + \left[(1 - ||I - A_{m-1}||_\alpha)^2 + 4\beta\right]^{1/2}\right\}, \quad \alpha \in \{2, F\}.$$

Proof. Consider the quaternionic monic matrix polynomial

$$B(\lambda) = (1-\lambda)L(\lambda) = -I\lambda^{m+1} + \sum_{i=0}^{m}(A_{m-i} - A_{m-i-1})\lambda^{m-i}.$$

Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Let us assume the opposite to reach a contradiction.

$$\begin{aligned} |\mu| &> \frac{1}{2}\left\{1 + ||I - A_{m-1}||_\alpha + \left[(1 - ||I - A_{m-1}||_\alpha)^2 + 4\beta\right]^{1/2}\right\} \\ 2|\mu| - (1 + ||I - A_{m-1}||_\alpha) &> \left[(1 - ||I - A_{m-1}||_\alpha)^2 + 4\beta\right]^{1/2}. \end{aligned}$$

Squaring on both sides, we have

$$(|\mu| - 1)(|\mu| - ||I - A_{m-1}||_\alpha) - \beta > 0. \tag{3.11}$$

Multiplying (3.11) by $|\mu|^{m-1}$ and then dividing by $|\mu| - 1$, we obtain

$$|\mu|^m - ||I - A_{m-1}||_\alpha|\mu|^{m-1} - \beta\,\frac{|\mu|^{m-1}}{|\mu| - 1} > 0. \tag{3.12}$$

We know that,

$$\beta\,\frac{|\mu|^{m-1}}{|\mu| - 1} > \beta\,\frac{|\mu|^{m-1} - 1}{|\mu| - 1} = \beta(1 + |\mu| + \cdots + |\mu|^{m-2})$$

$$\beta\,\frac{|\mu|^{m-1}}{|\mu| - 1} \geq ||\{(A_{m-1} - A_{m-2}) + (A_{m-2} - A_{m-3})\mu + \cdots + (A_2 - A_1)\mu^{m-2}\}x||_\alpha. \tag{3.13}$$

Also,

$$|\mu|^m - ||I - A_{m-1}||_\alpha|\mu|^{m-1} \leq ||\{I\mu^m + (I - A_{m-1})\mu^{m-1}\}x||_\alpha. \tag{3.14}$$

By (3.12), we obtain

$$0 \quad < \quad |\mu|^m - ||I - A_{m-1}||_\alpha |\mu|^{m-1} - \beta \, \frac{|\mu|^{m-1}}{|\mu| - 1}.$$

Substituting (3.13) and (3.14) in (3.12), we get

$$0 < ||\{I\mu^m + (I - A_{m-1})\}x||_\alpha - ||\{(A_{m-1} - A_{m-2}) + (A_{m-2} - A_{m-3})\mu + \cdots + (A_2 - A_1)\mu^{m-2}\}x||_\alpha$$
$$\leq ||\{(A_{m-1} - A_{m-2}) + (A_{m-2} - A_{m-3})\mu + \cdots + (A_2 - A_1)\mu^{m-2} + (I - A_{m-1})\mu^{m-1} + I\mu^m\}x||_\alpha.$$

If $\alpha = 2$, then by Lemma 2.6, we have,

$$0 \leq ||\{(\chi_{A_{m-1}} - \chi_{A_{m-2}}) + (\chi_{A_{m-2}} - \chi_{A_{m-3}})\mu + \cdots + (\chi_I - \chi_{A_{m-1}})\mu^{m-1} + \chi_I \mu^m\}\xi_x||_2 = ||Q(\mu)\xi_x||_2.$$

If $\alpha = F$, then by Lemma 2.7, we obtain,

$$0 \quad < \quad ||Q(\mu)\xi_x||_2 \leq ||Q(\mu)\xi_x||_F.$$

It follows that if

$$|\mu| \quad > \quad \frac{1}{2}\left\{1 + ||I - A_{m-1}||_\alpha + \left[(1 - ||I - A_{m-1}||_\alpha)^2 + 4\beta\right]^{1/2}\right\},$$

then $||Q(\mu)\xi_x||_\alpha > 0$, which leads to a contradiction. Therefore, we have

$$|\mu| \quad \leq \quad \frac{1}{2}\left\{1 + ||I - A_{m-1}||_\alpha + \left[(1 - ||I - A_{m-1}||_\alpha)^2 + 4\beta\right]^{1/2}\right\}.$$

□

Theorem 3.6. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$\gamma = \max_{0 \leq i \leq m-1} ||A_{m-1}A_i - A_{i-1}||_\alpha, \quad \textit{and} \quad A_{-1} = 0.$$

Then, each right eigenvalue μ of $L(\lambda)$ satisfies

$$|\mu| \leq \frac{1}{2}\left(1 + \sqrt{1 + 4\gamma}\right), \quad \alpha \in \{2, F\}.$$

Proof. Consider the quaternionic monic matrix polynomial $C(\lambda) = (I\lambda - A_{m-1})L(\lambda)$. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Assume the opposite is true to find a contradiction.

$$|\mu| \quad > \quad \frac{1}{2}\left(1 + \sqrt{1 + 4\gamma}\right)$$
$$2|\mu| \quad > \quad 1 + \sqrt{1 + 4\gamma}$$
$$2|\mu| - 1 \quad > \quad \sqrt{1 + 4\gamma}.$$

Squaring on both sides, we have

$$|\mu|(|\mu| - 1) - \gamma \quad > \quad 0. \tag{3.15}$$

Multiplying (3.15) by $|\mu|^m$ and then dividing by $(|\mu| - 1)$, we obtain

$$|\mu|^{m+1} - \gamma \, \frac{|\mu|^m}{|\mu| - 1} \quad > \quad 0. \tag{3.16}$$

We know that,

$$\gamma \, \frac{|\mu|^m}{|\mu| - 1} \quad > \quad \gamma \, \frac{|\mu|^m - 1}{|\mu| - 1} = \gamma(1 + |\mu| + \cdots + |\mu|^{m-1})$$
$$\gamma \, \frac{|\mu|^m}{|\mu| - 1} \quad \geq \quad ||\left\{(A_{m-1}A_0 - A_{-1}) + \cdots + (A_{m-1}A_{m-1} - A_{m-2})\mu^{m-1}\right\} x||_\alpha. \tag{3.17}$$

By (3.16), we have

$$0 \quad < \quad |\mu|^{m+1} - \gamma \frac{|\mu|^m}{|\mu| - 1}.$$

Putting (3.17) in (3.16) we get

$$0 < ||\{I\mu^{m+1}\}x||_\alpha - ||\left\{(A_{m-1}A_0 - A_{-1}) + (A_{m-1}A_1 - A_0)\mu + \cdots + (A_{m-1}A_{m-1} - A_{m-2})\mu^{m-1}\right\}x||_\alpha$$

If $\alpha = 2$, then by Lemma 2.6, we have,

$$0 \quad \leq \quad ||\left\{\chi_{(A_{m-1}A_0 - A_{-1})} + \chi_{(A_{m-1}A_1 - A_0)}\mu + \cdots + \chi_{(A_{m-1}A_m - A_{m-1})}\mu^m + \chi_I \mu^{m+1}\right\}\xi_x||_2 = ||H(\mu)\xi_x||_2.$$

If $\alpha = F$, then by Lemma 2.7, we get,

$$0 \quad < \quad ||H(\mu)\xi_x||_2 \leq ||H(\mu)\xi_x||_F.$$

As a result, when

$$|\mu| \quad > \quad \frac{1}{2}\left(1 + \sqrt{1 + 4\gamma}\right),$$

we observe $||H(\mu)\xi_x||_\alpha > 0$, causing a contradiction. Hence, we have

$$|\mu| \quad \leq \quad \frac{1}{2}\left(1 + \sqrt{1 + 4\gamma}\right).$$

□

Theorem 3.7. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote,*

$$N_p \quad = \quad \left(\sum_{i=0}^{m-1} ||A_i||_\alpha^p\right)^{1/p},$$

$$\tilde{N}_p \quad = \quad \left(\sum_{i=1}^{m-1} ||A_i||_\alpha^p\right)^{1/p}.$$

Then, each right eigenvalue μ of $L(\lambda)$ satisfies

$$|\mu| \quad < \quad (1 + N_p^q)^{1/q}, \quad \alpha \in \{2, F\}.$$

Proof. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Let us consider the negation to uncover a contradiction.

$$|\mu| \quad \geq \quad (1 + N_p^q)^{1/q}.$$

We have,

$$||P(\mu)\xi_x||_\alpha \quad = \quad ||(\chi_{A_0} + \chi_{A_1}\mu + \cdots + \chi_{A_{m-1}}\mu^{m-1} + \chi_I\mu^m)\xi_x||_\alpha$$

$$||P(\mu)\xi_x||_\alpha \quad \geq \quad ||\chi_I\xi_x||_\alpha - ||\sum_{i=0}^{m-1}\mu^i\chi_{A_i}\xi_x||_\alpha.$$

Since x is a unit vector, i.e., $||x||_\alpha = 1$, then $||\xi_x||_\alpha = 1$, $||\chi_{A_i}\xi_x||_\alpha \leq ||\chi_{A_i}||_\alpha$, and $||\chi_{A_0}\xi_x||_\alpha \leq ||\chi_{A_0}||_\alpha = ||(\chi_{A_0})^{-1}||_\alpha^{-1}$.

$$||P(\mu)\xi_x||_\alpha \quad \geq \quad |\mu| - \left(\sum_{i=0}^{m-1} ||\chi_{A_i}||_\alpha^p\right)^{1/p} \left(\sum_{i=0}^{m-1} |\mu|^{iq}\right)^{1/q}.$$

If $\alpha = 2$, then by Lemma 2.6, we get,

$$||P(\mu)\xi_x||_2 \geq |\mu| - \left(\sum_{i=0}^{m-1} ||A_i||_2^p\right)^{1/p} \left(\sum_{i=0}^{m-1} |\mu|^{iq}\right)^{1/q} = |\mu|^m \left[1 - N_p \left(\sum_{i=0}^{m-1} |\mu|^{(i-m)q}\right)^{1/q}\right]$$

$$||P(\mu)\xi_x||_2 > |\mu|^m \left[1 - N_p \left(\sum_{j=1}^{\infty} |\mu|^{(-j)q}\right)^{1/q}\right] = |\mu|^m \left[1 - N_p \frac{1}{(|\mu|^q - 1)^{1/q}}\right] \geq 0.$$

If $\alpha = F$, then by Lemma 2.7, we have,

$$0 < ||P(\mu)\xi_x||_2 \leq ||P(\mu)\xi_x||_F.$$

For $|\mu| \geq (1 + N_p^q)^{1/q}$, the condition $||P(\mu)\xi_x||_\alpha > 0$ is satisfied, contradicting our assumption. Hence $|\mu| < (1 + N_p^q)^{1/q}$ is the upper bound of $L(\lambda)$. □

3.2 Lower Bounds for right eigenvalues of quaternionic matrix polynomials

In this section, we introduce various lower bounds for the right eigenvalues of quaternionic monic matrix polynomials in terms of 2-matrix norm and Frobenius norm.

Theorem 3.8. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$K_i = (A_0)^{-1}A_i \text{ for all } i = 1, 2, \cdots, m-1.$$

Then, each right eigenvalue μ of $L(\lambda)$ is approximated by

$$|\mu| \geq \frac{2}{1 + ||K_1||_\alpha + \left[(1 - ||K_1||_\alpha)^2 + 4\tilde{M}\right]^{1/2}},$$

where $K_m = (A_0)^{-1}$, $\tilde{M} = \max_{2 \leq i \leq m} ||K_i||_\alpha$, and $\alpha \in \{2, F\}$.

Proof. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Assume the reverse to demonstrate a contradiction.

$$|\mu| < \frac{2}{1 + ||K_1||_\alpha + \left[(1 - ||K_1||_\alpha)^2 + 4\tilde{M}\right]^{1/2}}$$

$$\frac{1}{|\mu|} > \frac{1 + ||K_1||_\alpha + \left[(1 - ||K_1||_\alpha)^2 + 4\tilde{M}\right]^{1/2}}{2}$$

$$\frac{2}{|\mu|} > 1 + ||K_1||_\alpha + \left[(1 - ||K_1||_\alpha)^2 + 4\tilde{M}\right]^{1/2}$$

$$\frac{2}{|\mu|} - (1 + ||K_1||_\alpha) > \left[(1 - ||K_1||_\alpha)^2 + 4\tilde{M}\right]^{1/2}.$$

Squaring on both sides, we have

$$\left(\frac{2}{|\mu|} - (1 + ||K_1||_\alpha)\right)^2 > (1 - ||K_1||_\alpha)^2 + 4\tilde{M}$$

$$\left(\frac{1}{|\mu|} - 1\right)\left(\frac{1}{|\mu|} - ||K_1||_\alpha\right) - \tilde{M} > 0. \tag{3.18}$$

Multiplying (3.18) by $\left(\frac{1}{|\mu|}\right)^{m-1}$ and dividing by $\left(\frac{1}{|\mu|} - 1\right)$, we obtain

$$\left(\frac{1}{|\mu|}\right)^m - ||K_1||_\alpha \left(\frac{1}{|\mu|}\right)^{m-1} - \tilde{M} \frac{\left(\frac{1}{|\mu|}\right)^{m-1}}{\left(\frac{1}{|\mu|} - 1\right)} > 0. \tag{3.19}$$

We know that,

$$\tilde{M}\frac{\left(\frac{1}{|\mu|}\right)^{m-1}}{\left(\frac{1}{|\mu|}-1\right)} > \tilde{M}\frac{\left(\frac{1}{|\mu|}\right)^{m-1}-1}{\left(\frac{1}{|\mu|}-1\right)} = \tilde{M}\left(1+\frac{1}{|\mu|}+\cdots+\left(\frac{1}{|\mu|}\right)^{m-2}\right)$$

$$\tilde{M}\frac{\left(\frac{1}{|\mu|}\right)^{m-1}}{\left(\frac{1}{|\mu|}-1\right)} \geq ||\left(K_m+K_{m-1}\frac{1}{|\mu|}+\cdots+K_2\left(\frac{1}{|\mu|}\right)^{m-2}\right)x||_\alpha. \tag{3.20}$$

Also,

$$|\frac{1}{\mu}|^m-||K_1||_\alpha|\frac{1}{\mu}|^{m-1} \leq ||(I\frac{1}{\mu^m}+K_1\frac{1}{\mu^{m-1}})x||_\alpha. \tag{3.21}$$

By (3.19), we have

$$0 < \left(\frac{1}{|\mu|}\right)^m-||K_1||_\alpha\left(\frac{1}{|\mu|}\right)^{m-1}-\tilde{M}\frac{\left(\frac{1}{|\mu|}\right)^{m-1}}{\left(\frac{1}{|\mu|}-1\right)}.$$

Putting (3.20) and (3.21) in (3.19), we get

$$\begin{aligned} 0 &< ||(I\frac{1}{\mu^m}+K_1\frac{1}{\mu^{m-1}})x||_\alpha-||\left(K_m+K_{m-1}\frac{1}{\mu}+\cdots+K_2\left(\frac{1}{\mu}\right)^{m-2}\right)x||_\alpha \\ &\leq ||\left(K_m+K_{m-1}\frac{1}{\mu}+\cdots+K_2\frac{1}{\mu^{m-2}}+K_1\frac{1}{\mu^{m-1}}+I\frac{1}{\mu^m}\right)x||_\alpha. \end{aligned}$$

If $\alpha=2$, then by Lemma 2.6, we obtain,

$$0 < ||\left(\chi_{K_m}+\chi_{K_{m-1}}\frac{1}{\mu}+\cdots+\chi_{K_1}\frac{1}{\mu^{m-1}}+\chi_I\frac{1}{\mu^m}\right)\xi_x||_2=||P(\frac{1}{\mu})\xi_x||_2.$$

If $\alpha=F$, then by Lemma 2.7, we have,

$$0 < ||P(\frac{1}{\mu})\xi_x||_2 \leq ||P(\frac{1}{\mu})\xi_x||_F.$$

Consequently, if

$$|\mu| < \frac{2}{1+||K_1||_\alpha+\left[(1-||K_1||_\alpha)^2+4\tilde{M}\right]^{1/2}},$$

then $||P(\frac{1}{\mu})\xi_x||_\alpha>0$, which gives rise to a contradiction. Therefore,

$$|\mu| \geq \frac{2}{1+||K_1||_\alpha+\left[(1-||K_1||_\alpha)^2+4\tilde{M}\right]^{1/2}}.$$

□

Theorem 3.9. *Let $L(\lambda)=A_0+A_1\lambda+\cdots+A_{m-1}\lambda^{m-1}+I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$\tilde{\beta} = \max_{1\leq i\leq m}||K_i-K_{i+1}||_\alpha, \text{ and } K_{m+1}=0.$$

Then, each right eigenvalue μ of $L(\lambda)$ is approximated by

$$|\mu| \geq \frac{2}{1+||I-K_1||_\alpha+\left[(1-||I-K_1||_\alpha)^2+4\tilde{\beta}\right]^{1/2}}, \quad \alpha\in\{2,F\}.$$

Proof. Consider the quaternionic monic matrix polynomial

$$B(\lambda) \quad = \quad (1-\lambda)L(\lambda) = -I\lambda^{m+1} + \sum_{i=0}^{m}(A_{m-i} - A_{m-i-1})\lambda^{m-i}.$$

Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Suppose the contrary statement to arrive at a contradiction.

$$\begin{aligned} |\mu| &< \frac{2}{1 + ||I-K_1||_\alpha + \left[(1-||I-K_1||_\alpha)^2 + 4\tilde{\beta}\right]^{1/2}} \\ \frac{2}{|\mu|} &> 1 + ||I-K_1||_\alpha + \left[(1-||I-K_1||_\alpha)^2 + 4\tilde{\beta}\right]^{1/2} \\ \frac{2}{|\mu|} - (1 + ||I-K_1||_\alpha) &> \left[(1-||I-K_1||_\alpha)^2 + 4\tilde{\beta}\right]^{1/2}. \end{aligned}$$

Squaring on both sides, we have

$$\left(|\frac{1}{\mu}| - 1\right)\left(|\frac{1}{\mu}| - ||I-K_1||_\alpha\right) - \tilde{\beta} \quad > \quad 0. \tag{3.22}$$

Multiplying (3.22) by $|\frac{1}{\mu}|^{m-1}$ and then dividing by $\left(|\frac{1}{\mu}| - 1\right)$, we get

$$|\frac{1}{\mu}|^m - ||I-K_1||_\alpha |\frac{1}{\mu}|^{m-1} - \tilde{\beta}\,\frac{|\frac{1}{\mu}|^{m-1}}{|\frac{1}{\mu}| - 1} \quad > \quad 0. \tag{3.23}$$

We know that,

$$\tilde{\beta}\,\frac{|\frac{1}{\mu}|^{m-1}}{|\frac{1}{\mu}| - 1} > \tilde{\beta}\,\frac{|\frac{1}{\mu}|^{m-1} - 1}{|\frac{1}{\mu}| - 1} = \tilde{\beta}\left(1 + |\frac{1}{\mu}| + \cdots + |\frac{1}{\mu}|^{m-2}\right)$$

$$\tilde{\beta}\,\frac{|\frac{1}{\mu}|^{m-1}}{|\frac{1}{\mu}| - 1} \geq ||\left\{(K_1-K_2)|\frac{1}{\mu}|^{m-2} + (K_2-K_3)|\frac{1}{\mu}|^{m-3} + \cdots + (K_m - K_{m+1}\right\}x||_\alpha. \tag{3.24}$$

Also,

$$|\frac{1}{\mu}|^m - ||I-K_1||_\alpha |\frac{1}{\mu}|^{m-1} \quad \leq \quad ||\left\{I\frac{1}{\mu} + (I-K_1)\frac{1}{\mu^{m-1}}\right\}x||_\alpha. \tag{3.25}$$

By (3.23), we obtain

$$0 \quad < \quad |\frac{1}{\mu}|^m - ||I-K_1||_\alpha |\frac{1}{\mu}|^{m-1} - \tilde{\beta}\,\frac{|\frac{1}{\mu}|^{m-1}}{|\frac{1}{\mu}| - 1}.$$

Substituting (3.24) and (3.25) in (3.23), we have

$$\begin{aligned} 0 &< ||\{I\frac{1}{\mu} + (I-K_1)\frac{1}{\mu^{m-1}}\}x||_\alpha - ||\{(K_1-K_2)|\frac{1}{\mu}|^{m-2} + (K_2-K_3)|\frac{1}{\mu}|^{m-3} + \cdots + (K_m - K_{m+1}\}x||_\alpha \\ &\leq ||\left\{I\frac{1}{\mu} + (I-K_1)\frac{1}{\mu^{m-1}} - (K_1-K_2)|\frac{1}{\mu}|^{m-2} + (K_2-K_3)|\frac{1}{\mu}|^{m-3} + \cdots + (K_m - K_{m+1}\right\}x||_\alpha. \end{aligned}$$

If $\alpha = 2$, then by Lemma 2.6, we get,

$$0 \leq || \left\{ \chi_I \frac{1}{\mu} + \chi_{(I-K_1)} \frac{1}{\mu^{m-1}} - \chi_{(K_1-K_2)} |\frac{1}{\mu}|^{m-2} + \chi_{(K_2-K_3)} |\frac{1}{\mu}|^{m-3} + \cdots + \chi_{(K_m-K_{m+1}} \right\} \xi_x ||_2$$
$$= ||Q(\frac{1}{\mu})\xi_x||_2.$$

If $\alpha = F$, then by Lemma 2.7, we have

$$0 \quad < \quad ||Q(\frac{1}{\mu})\xi_x||_2 \leq ||Q(\frac{1}{\mu})\xi_x||_F.$$

For

$$|\mu| \quad < \quad \frac{2}{1 + ||I - K_1||_\alpha + \left[(1 - ||I - K_1||_\alpha)^2 + 4\tilde{\beta}\right]^{1/2}},$$

the inequality $||Q(\frac{1}{\mu})\xi_x||_\alpha > 0$ holds true, leading to a contradiction. Hence, we obtain

$$|\mu| \quad \geq \quad \frac{2}{1 + ||I - K_1||_\alpha + \left[(1 - ||I - K_1||_\alpha)^2 + 4\tilde{\beta}\right]^{1/2}}.$$

□

Theorem 3.10. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$\tilde{\gamma} \quad = \quad \max_{1 \leq i \leq m} ||K_1K_i - K_{i+1}||_\alpha, \;\; \text{and} \;\; K_{m+1} = 0.$$

Then, each right eigenvalue μ of $L(\lambda)$ is bounded below by

$$|\mu| \quad \geq \quad \frac{2}{1 + \sqrt{1 + 4\tilde{\gamma}}}, \quad \alpha \in \{2, F\}.$$

Proof. Consider the quaternionic monic matrix polynomial $C(\lambda) = (I\lambda - A_{m-1})L(\lambda)$. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. Assume the opposite and prove it leads to a contradiction.

$$|\mu| \quad < \quad \frac{2}{1 + \sqrt{1 + 4\tilde{\gamma}}}$$
$$\frac{2}{|\mu|} \quad > \quad 1 + \sqrt{1 + 4\tilde{\gamma}}$$
$$\frac{2}{|\mu|} - 1 \quad > \quad \sqrt{1 + 4\tilde{\gamma}}.$$

Squaring on both sides, we get

$$|\frac{1}{\mu}| \left(|\frac{1}{\mu}| - 1 \right) - \tilde{\gamma} \quad > \quad 0. \tag{3.26}$$

Multiplying (3.26) by $|\frac{1}{\mu}|^m$ and then dividing by $\left(|\frac{1}{\mu}| - 1\right)$, we have

$$|\frac{1}{\mu}|^{m+1} - \tilde{\gamma} \, \frac{|\frac{1}{\mu}|^m}{|\frac{1}{\mu}| - 1} \quad > \quad 0. \tag{3.27}$$

We know that,

$$\tilde{\gamma}\,\frac{|\frac{1}{\mu}|^m}{|\frac{1}{\mu}|-1} \;>\; \tilde{\gamma}\,\frac{|\frac{1}{\mu}|^m-1}{|\frac{1}{\mu}|-1}=\tilde{\gamma}\left(1+|\frac{1}{\mu}|+\cdots+|\frac{1}{\mu}|^{m-1}\right)$$

$$\tilde{\gamma}\,\frac{|\frac{1}{\mu}|^m}{|\frac{1}{\mu}|-1} \;\geq\; \left|\left|\left\{(K_1K_m-K_{m+1})+\cdots+(K_1K_1-K_2)\frac{1}{\mu^{m-1}}\right\}x\right|\right|_\alpha. \tag{3.28}$$

By (3.27), we get

$$0 \;<\; |\frac{1}{\mu}|^{m+1}-\tilde{\gamma}\,\frac{|\frac{1}{\mu}|^m}{|\frac{1}{\mu}|-1}.$$

Putting (3.28) in (3.27), we obtain

$$\begin{aligned}0 \;&<\; \left|\left|\left\{I\frac{1}{\mu^{m+1}}\right\}x\right|\right|_\alpha-\left|\left|\left\{(K_1K_m-K_{m+1})+(K_1K_{m-1}-K_m)\frac{1}{\mu}+\cdots+(K_1K_1-K_2)\frac{1}{\mu^{m-1}}\right\}x\right|\right|_\alpha\\ &\leq\; \left|\left|\left\{I\frac{1}{\mu^{m+1}}+(K_1K_1-K_2)\frac{1}{\mu^{m-1}}+\cdots+(K_1K_m-K_{m+1})\right\}x\right|\right|_\alpha.\end{aligned}$$

If $\alpha=2$, then by Lemma 2.6, we have,

$$0 \;\leq\; \left|\left|\left\{\chi_I\frac{1}{\mu^{m+1}}+\chi_{(K_1K_1-K_2)}\frac{1}{\mu^{m-1}}+\cdots+\chi_{(K_1K_m-K_{m+1})}\right\}\xi_x\right|\right|_2=||H(\frac{1}{\mu})\xi_x||_2.$$

If $\alpha=F$, then by Lemma 2.6, we get,

$$0 \;<\; ||H(\frac{1}{\mu})\xi_x||_2\leq ||H(\frac{1}{\mu})\xi_x||_F.$$

When

$$|\mu| \;<\; \frac{2}{1+\sqrt{1+4\tilde{\gamma}}},$$

the condition $0<||H(\frac{1}{\mu})\xi_x||_\alpha$ is fulfilled, creating a contradiction. Therefore, we have

$$|\mu| \;\geq\; \frac{2}{1+\sqrt{1+4\tilde{\gamma}}}.$$

□

4 Location of right eigenvalues of quaternionic monic matrix polynomials

In this section, we find the location of right eigenvalues for quaternionic monic matrix polynomials.

Theorem 4.1. Let $L(\lambda)=A_0+A_1\lambda+\cdots+A_{m-1}\lambda^{m-1}+I\lambda^m$ be a quaternionic monic matrix polynomial. Then, every right eigenvalues of $L(\lambda)$ lie in the closed ball

$$D(0,r_1)=\{\lambda\in\mathbb{H}|\ |\lambda|\leq r_1\},$$

where $r_1=\max\{1,\delta\}$ and $\delta\neq 1$ is the positive root of the equation

$$\lambda^{m+1}-(1+N)\lambda^m+N=0.$$

We use the definition of N, from Theorem 3.3.

Proof. Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. When $|\mu| \leq 1$, the conclusion becomes clear. Hence, assuming $|\mu| > 1$, we obtain the following.

$$\begin{aligned} ||P(\mu)\xi_x||_\alpha &= ||(\chi_{A_0} + \chi_{A_1}\mu + \cdots + \chi_I\mu^m)\xi_x||_\alpha \\ ||P(\mu)\xi_x||_\alpha &\geq ||\chi_I\mu^m\xi_x||_\alpha - ||\sum_{i=0}^{m-1}\chi_{A_i}\mu^i\xi_x||_\alpha \\ ||P(\mu)\xi_x||_\alpha &\geq |\mu|^m||\chi_I\xi_x||_\alpha - \sum_{i=0}^{m-1}|\mu|^i||\chi_{A_i}\xi_x||_\alpha. \end{aligned}$$

Since x is a unit vector, i.e., $||x||_\alpha = 1$, then $||\xi_x||_\alpha = 1$, $||\chi_I\xi_x||_\alpha \leq ||\chi_I||_\alpha||\xi_x||_\alpha = 1$, and $||\chi_{A_i}\xi_x||_\alpha \leq ||\chi_{A_i}||_\alpha||\xi_x||_\alpha = ||\chi_{A_i}||_\alpha$. Hence, we get

$$||P(\mu)\xi_x||_\alpha \geq |\mu|^m||\chi_I||_\alpha - \sum_{i=0}^{m-1}|\mu|^i||\chi_{A_i}||_\alpha.$$

If $\alpha = 2$, then by Lemma 2.6, we have

$$\begin{aligned} ||P(\mu)\xi_x||_2 &\geq |\mu|^m - \sum_{i=0}^{m-1}||A_i||_2|\mu|^i \\ ||P(\mu)\xi_x||_\alpha &\geq |\mu|^m - N\sum_{i=0}^{m-1}|\mu|^i = |\mu|^m - N\,\frac{|\mu|^m - 1}{|\mu| - 1} = \frac{1}{|\mu| - 1}[|\mu|^{m+1} - (1+N)|\mu|^m + N]. \end{aligned}$$

If $\alpha = F$, then by Lemma 2.7, we get

$$||P(\mu)\xi_x||_F \geq ||P(\mu)\xi_x||_2 \geq \frac{1}{|\mu| - 1}[|\mu|^{m+1} - (1+N)|\mu|^m + N].$$

Therefore, we obtain

$$||P(\mu)\xi_x||_\alpha \geq \frac{1}{|\mu| - 1}[|\mu|^{m+1} - (1+N)|\mu|^m + N].$$

Note that the polynomial

$$f(\lambda) = \lambda^{m+1} - (1+N)\lambda^m + N$$

has exactly two positive real roots, 1 and $\delta \neq 1$, as determined by Descartes' Rule of signs and $f(0) > 0$. It follows that $|f(\lambda)| > 0$ holds for all $\lambda > \max\{\delta, 1\}$. Therefore, for $|\mu| > r_1$, we arrive at $||L(\mu)\xi_x||_\alpha > 0$, which leads to a contradiction. □

We investigate the location of right eigenvalues for quaternionic monic matrix polynomials in terms of norm of matrix difference.

Theorem 4.2. *Let $L(\lambda) = A_0 + A_1\lambda + \cdots + A_{m-1}\lambda^{m-1} + I\lambda^m$ be a quaternionic monic matrix polynomial. Denote,*

$$\tilde{N} = \max_{0\leq i\leq m}||A_{m-i} - A_{m-i-1}||_\alpha, \quad A_{-1} = 0 \ \text{ and } \ \alpha \in \{2, F\}.$$

Then, all right eigenvalues of $L(\lambda)$ are contained in the open ball $D(0, r_4)$, where $r_4 = 1 + \tilde{N}$.

Proof. Consider the quaternionic monic matrix polynomial

$$B(\lambda) = (1-\lambda)L(\lambda) = -I\lambda^{m+1} + \sum_{i=0}^{m}(A_{m-i} - A_{m-i-1})\lambda^{m-i}.$$

Let μ be a complex right eigenvalue of $L(\lambda)$ and let $x \in \mathbb{H}^n$ be a unit vector corresponding to μ. The conclusion is obvious under the condition $|\mu| \leq 1$. Accordingly, let us consider $|\mu| > 1$. Then, we derive

$$\begin{aligned} ||Q(\mu)\chi_x||_\alpha &= ||(\chi_I \mu^{m+1} + \sum_{i=0}^{m} (\chi_{A_{m-i}} - \chi_{A_{m-i-1}})\mu^{m-i})\xi_x||_\alpha \\ ||Q(\mu)\chi_x||_\alpha &\geq |\mu|^{m+1} \left[||\chi_I \xi_x||_\alpha - ||\sum_{i=0}^{m} \frac{(\chi_{A_{m-i}} - \chi_{A_{m-i-1}})\xi_x}{\mu^{i+1}}||_\alpha \right]. \end{aligned}$$

Since x is a unit vector, i.e., $||x||_\alpha = 1$, then $||\chi_x||_\alpha = 1$, $||\chi_I \xi_x||_\alpha \leq ||\chi_I||_\alpha ||\xi_x||_\alpha = 1$, and $||\chi_{A_i} \xi_x||_\alpha \leq ||\chi_{A_i}||_\alpha ||\xi_x||_\alpha = ||\chi_{A_i}||_\alpha$. Thus,

$$||Q(\mu)\xi_x||_\alpha \geq |\mu|^{m+1} \left[1 - \sum_{i=0}^{m} \frac{||(\chi_{A_{m-i}} - \chi_{A_{m-i-1}})||_\alpha}{|\mu|^{i+1}} \right].$$

If $\alpha = 2$, then by Lemma 2.6, we get

$$\begin{aligned} ||Q(\mu)\xi_x||_2 &\geq |\mu|^{m+1} \left[1 - \sum_{i=0}^{m} \frac{||(\chi_{A_{m-i}} - \chi_{A_{m-i-1}})||_2}{|\mu|^{i+1}} \right] \\ ||Q(\mu)\xi_x||_2 &\geq |\mu|^{m+1} \left[1 - \tilde{N} \sum_{i=0}^{m} \frac{1}{|\mu|^{i+1}} \right] \\ ||Q(\mu)\xi_x||_2 &> |\mu|^{m+1} \left[1 - \tilde{N} \sum_{i=0}^{\infty} \frac{1}{|\mu|^{i+1}} \right] = |\mu|^{m+1} \left[1 - \frac{\tilde{N}}{|\mu| - 1} \right] = \frac{|\mu|^{m+1}}{|\mu| - 1} \left[|\mu| - 1 - \tilde{N} \right]. \end{aligned}$$

Therefore, we have

$$||Q(\mu)\xi_x||_2 \geq \frac{|\mu|^{m+1}}{|\mu| - 1} \left[|\mu| - 1 - \tilde{N} \right].$$

If $\alpha = F$, then by Lemma 2.7. We obtain

$$||Q(\mu)\xi_x||_F \geq ||Q(\mu)\xi_x||_2 \geq \frac{|\mu|^{m+1}}{|\mu| - 1} \left[|\mu| - 1 - \tilde{N} \right].$$

By applying Theorem 3.3 to the polynomial $B(\lambda)$ and noting that each eigenvalue of $L(\lambda)$ is also an eigenvalue of $B(\lambda)$, we have the conclusion. Therefore, we have

$$||Q(\mu)\xi_x||_\alpha \geq \frac{|\mu|^{m+1}}{|\mu| - 1} \left[|\mu| - 1 - \tilde{N} \right].$$

Hence, for $|\mu| \geq 1+\tilde{N}$, we get $||Q(\mu)\xi_x||_\alpha > 0$, yielding a contradiction. Therefore, $|\mu| < 1+\tilde{N}$.

□

5 Numerical examples

In this section, we give some numerical examples to illustrate our results.

Example 5.1. Consider $L \in \mathbb{L}_2(M_2(\mathbb{H}))$ of the form

$$L(\lambda) = A_2\lambda^2 + A_1\lambda + A_0,$$

where

$$A_0 = \begin{bmatrix} 1+\mathbf{i} & 1+\mathbf{i} \\ 1 & 1-\mathbf{i} \end{bmatrix}; A_1 = \begin{bmatrix} 2+\mathbf{i} & 2-\mathbf{i} \\ 2 & 1-\mathbf{i} \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} 11+2\mathbf{i}+3\mathbf{j}+\mathbf{k} & 7+3\mathbf{i}-\mathbf{j}+8\mathbf{k} \\ 1-8\mathbf{i}-3\mathbf{j}-7\mathbf{k} & 9+5\mathbf{i}+\mathbf{j}-\mathbf{k} \end{bmatrix}.$$

Therefore, we have

$$||A_0||_2 = 2.5887,\ ||A_1||_2 = 3.8555,\ ||A_2||_2 = 20.572,\ ||A_0||_F = 3.7417,$$

$$||A_1||_F = 5.6569,\ ||A_2||_F = 31.2730,\ ||(A_2)^{-1}||_F = 0.1874,\ ||(A_2)^{-1}||_2 = 0.1233.$$

Substituting these values in $||A_m||_\alpha > ||A_i||_\alpha, \forall\, \mathbf{i} = 0, 1, \cdots, m-1$, where $\alpha = \{2, F\}$. Then, we have right spectrum of $L(\lambda)$ as

$$\Lambda_r(L(\lambda)) = [0.0534 - 0.3324\mathbf{i}] \cup [0.0534 + 0.3324\mathbf{i}] \cup [-0.1816 - 0.2995\mathbf{i}] \cup [-0.1816 + 0.2995\mathbf{i}]$$

$$\cup [0.2683 + 0.0892\mathbf{i}] \cup [0.2683 - 0.0892\mathbf{i}] \cup [-0.2487 - 0.0529\mathbf{i}] \cup [-0.2487 + 0.0529\mathbf{i}].$$

$|\mu_1| = 0.3367$, $|\mu_2| = 0.3503$, $|\mu_3| = 0.2827$, and $|\mu_4| = 0.2543$. By putting these values, Theorem 3.2 is verified.

Example 5.2. Consider the quaternionic monic matrix polynomial

$$L(\lambda) \;=\; \lambda^2 + A_1\lambda + A_0,$$

where

$$A_0 = \begin{bmatrix} 1+\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{j}+\mathbf{k} \\ 7-\mathbf{k} & 2+\mathbf{j} \end{bmatrix}, \text{ and } A_1 = \begin{bmatrix} 1 & \mathbf{i} \\ \mathbf{j} & \mathbf{k} \end{bmatrix}.$$

Therefore, we have

$$C_p = \begin{bmatrix} 0 & I_2 \\ -A_0 & -A_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1+\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{j}+\mathbf{k} & 1 & \mathbf{i} \\ 7-\mathbf{k} & 2+\mathbf{j} & \mathbf{j} & \mathbf{k} \end{bmatrix}.$$

Now the complex adjoint matrix of $A_0, A_1,$ and C_p are as follows

$$\chi_{A_0} = \begin{bmatrix} 1+\mathbf{i} & 0 & -1+\mathbf{i} & 1+\mathbf{i} \\ 7 & 2 & -\mathbf{i} & 1 \\ 1+\mathbf{i} & -1+\mathbf{i} & 1-\mathbf{i} & 0 \\ -\mathbf{i} & -1 & 7 & 2 \end{bmatrix},\ \chi_{A_1} = \begin{bmatrix} 1 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 1 & -\mathbf{i} \\ 0 & 0 & 1 & -\mathbf{i} \\ -1 & \mathbf{i} & 1 & 0 \end{bmatrix},$$

and $$\chi_{C_p} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1+\mathbf{i} & 0 & 1 & \mathbf{i} & -1+\mathbf{i} & 1+\mathbf{i} & 0 & 0 \\ 7 & 2 & 0 & 0 & -\mathbf{i} & 1 & 1 & \mathbf{i} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1+\mathbf{i} & -1+\mathbf{i} & 0 & 0 & 1-\mathbf{i} & 0 & 1 & -\mathbf{i} \\ -\mathbf{i} & -1 & -1 & \mathbf{i} & 7 & 2 & 0 & 0 \end{bmatrix}.$$

Then, we have right spectrum of $L(\lambda)$ as

$$\Lambda_r(L) = [2.5902 + 1.0966\mathbf{i}] \cup [-1.8931 + 0.7999\mathbf{i}] \cup [-0.3093 + 1.2446\mathbf{i}] \cup [0.6122 + 1.1712\mathbf{i}].$$

Also, we obtain

$$||A_0||_2 = 7.7061,\ ||A_1||_2 = 2,\ ||A_0||_F = 11.0454,\ ||A_1||_F = 3,\ \text{and} \max_{\lambda_i \in \Lambda_r(L)} |\lambda_i| = 2.8128.$$

$$|\mu_1| = 2.8128,\ |\mu_2| = 2.0552,\ |\mu_3| = 1.2825,\ \text{and } |\mu_4| = 1.3216.$$

Putting these values into Theorem 3.3, and Theorem 3.4, verifies both.

$$\text{Let } K_i = (A_0)^{-1}A_i, \quad \forall i = 1, 2, \cdots, m-1.$$

$$(\chi_{A_0})^{-1} = \begin{bmatrix} -0.0417 & 0.1250 - 0.0417\mathbf{i} & 0.2083 + 0.0833\mathbf{i} & -0.0417 + 0.0417\mathbf{i} \\ -0.2083 + 0.0417\mathbf{i} & 0.0833 + 0.1667\mathbf{i} & -0.6250 - 0.2917\mathbf{i} & 0.0833 \\ -0.2083 + 0.0833\mathbf{i} & 0.0417 + 0.0417\mathbf{i} & -0.0417 & 0.1250 + 0.0417\mathbf{i} \\ 0.6250 - 0.2917\mathbf{i} & -0.0833 & -0.2083 - 0.0417\mathbf{i} & 0.0833 - 0.1667\mathbf{i} \end{bmatrix}$$

$$||\chi_{K_1}||_2 = ||(\chi_{A_0})^{-1}\chi_{A_1}||_2 = 1.1438.$$

Hence, $||K_1||_2 = ||(A_0)^{-1}A_1||_2 = 1.1438$, and $||K_1||_F = ||(A_0)^{-1}A_1||_F = 1.4860$. By substituting these values, Theorem 3.8 is verified.

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Received:2025-04-18
Accepted: 2025-09-08