

APPLICATION OF JORDAN-TYPE MATRIX OPERATOR IN THE THEORY OF SEQUENCE SPACES

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Abstract. This research introduces a novel regular matrix operator uniquely characterized by an arithmetic Jordan-type function. The study primarily investigates its domains in the sequence spaces of absolutely p -summable and bounded sequences, establishing the theoretical framework supporting these analyses. It further explores fundamental properties, inclusion relations, and the Schauder basis of these spaces, providing insights into their topological and functional structure. The identification of α -, β - and γ -duals enhances the theoretical contributions by offering a dual perspective on the studied spaces. Additionally, the classification of certain matrix classes aids in characterizing the operator's action across different mathematical settings. Finally, a detailed examination of a specific class of compact operators acting on the newly introduced sequence spaces highlights their significance and applicability.

1 Introduction

Let ω be the vector space containing all sequences of real numbers. For any subset $\Lambda \subset \omega$, we define it as a sequence space. Below, we present several fundamental sequence spaces that will be referenced throughout this paper:

- ℓ_p represents the space of sequences, for $1 \leq p < \infty$, whose p -th powers of absolute values form a convergent series.
- ℓ_∞ represents the space of all bounded sequences.
- c represents the space of all convergent sequences.
- c_0 represents the space of sequences that converge to zero.
- cs represents the space of sequences whose partial sums converge to a real number.
- bs represents the space of sequences whose partial sums are bounded.

These spaces are all Banach spaces, with the following norms:

- $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{s \in \mathbb{N}} |u_s|$,
- $\|u\|_{\ell_p} = \left(\sum_{s=1}^{\infty} |u_s|^p \right)^{\frac{1}{p}}$ for $1 \leq p < \infty$,

where \mathbb{N} denotes the set of natural numbers $\{1, 2, 3, \dots\}$.

Banach spaces where all coordinate functionals f_s , defined by $f_s(u) = u_s$, are continuous are referred to as BK-spaces. On the other hand, metric vector spaces where all coordinate functionals are continuous are known as FK-spaces.

A BK-space Λ is said to have AK-property if every element $u = (u_r) \in \Lambda \subset \omega$ can be uniquely expressed as $u = \sum_r u_r e^{(r)}$, where $e^{(r)}$ represents the sequence with 1 in the r -th position and 0 elsewhere. The AK-property is satisfied by the sequence spaces ℓ_p (for $1 \leq p < \infty$) and c_0 , but not by c or ℓ_∞ .

Let $D = (d_{rs})$ be an infinite matrix with real entries, where each D_r denotes the r -th row for $r \in \mathbb{N}$. The transformation of a sequence $u = (u_s) \in \omega$ under D is given by

$$(Du)_r = \sum_{s=1}^{\infty} d_{rs} u_s,$$

provided that the series converges for every $r \in \mathbb{N}$.

If $Du \in \Upsilon$, then D defines a mapping from Λ to Υ for all $u \in \Lambda$. The set of all such matrices is denoted as $(\Lambda : \Upsilon)$. A matrix D is said to be regular if Du converges to the same limit for every convergent sequence u . The necessary and sufficient conditions for a matrix $D = (d_{rs})$ to be regular are that $\sup_{r \in \mathbb{N}} \sum_s |d_{rs}| < \infty$, $\lim_{r \rightarrow \infty} \sum_s d_{rs} = 1$ and $\lim_{r \rightarrow \infty} d_{rs} = 0$.

The matrix domain of D within Λ is defined by

$$\Lambda_D = \{u \in \omega : Du \in \Lambda\}. \quad (1.1)$$

For $\Lambda, \Upsilon \in \omega$, the set of multipliers from Λ to Υ is

$$M(\Lambda : \Upsilon) = \{y = (y_s) \in \omega : (y_s u_s) \in \Upsilon \text{ for all } u = (u_s) \in \Lambda\}.$$

When Υ is ℓ_1 , cs , or bs , the corresponding multiplier spaces determine α -, β - and γ -duals of Λ , respectively,

$$\Lambda^\alpha = M(\Lambda : \ell_1), \quad \Lambda^\beta = M(\Lambda : cs), \quad \Lambda^\gamma = M(\Lambda : bs).$$

For a deeper understanding of the fundamental concepts discussed above, readers may refer to [1, 3, 9, 10, 11, 12, 13, 18, 19, 20, 26, 28, 29, 33, 34, 38, 40, 41] and standard references such as [4, 8, 30].

1.1 Jordan-type function $\mathfrak{J}_k^{(1)}$

Two of the most enchanting functions in number theory are the Euler-totient function φ and the Jordan-totient function \mathfrak{J}_k , which is a generalization of φ . In recent years, special functions have been widely utilized in the study of sequence spaces. Matrices obtained with the help of these arithmetic and multiplicative functions and sequence spaces created as the domain of these matrices and some properties were discussed in depth in the studies [16, 21, 22, 23, 24].

Another Jordan-type function $\mathfrak{J}_k^{(1)}$, other than the Jordan totient function mentioned above, was also defined by Andrica and Piticari [2].

For positive integer's set \mathbb{Z}_+ , a fixed $k \in \mathbb{Z}_+$ and $\mathbb{Z}_+^{k+1} = \underbrace{\mathbb{Z}_+ \times \dots \times \mathbb{Z}_+}_{k+1 \text{ times}}$, let us consider the set

$$N_k(s) = \{(\tau_1, \dots, \tau_k, s) \in \mathbb{Z}_+^{k+1} : 1 \leq \tau_1 \leq \dots \leq \tau_{k+1} \leq s \text{ and } \gcd(\tau_1, \dots, \tau_{k+1}, s) = 1\}.$$

The Jordan-type function $\mathfrak{J}_k^{(1)}$ is described as the cardinal number of the finite set $N_k(s)$. For the special case $k = 1$, the function $\mathfrak{J}_k^{(1)}$ is reduced to the well-known Euler totient function φ . To make an important point, the Jordan-type function $\mathfrak{J}_k^{(1)}$ is arithmetic just like the Euler totient and Jordan totient functions, but not multiplicative. The Gauss type formula for the function $\mathfrak{J}_k^{(1)}$ is given by

$$\sum_{s|r} \mathfrak{J}_k^{(1)}(s) = \binom{r+k-1}{k}, \text{ for } k \in \mathbb{N}. \quad (1.2)$$

The Möbius function μ is described by

$$\mu(r) := \begin{cases} (-1)^m, & \text{if } r \text{ is the product of } m \text{ distinct prime numbers,} \\ 0, & \text{if } r \text{ is divisible by the square of a prime,} \end{cases}$$

for all $r \in \mathbb{N}$ with $r > 1$ and $\mu(1) = 1$. From (1.2) and the well-known Möbius inversion formula, it is seen that

$$\mathfrak{J}_k^{(1)}(r) = \sum_{s|r} \mu\left(\frac{r}{s}\right) \binom{s+k-1}{k}$$

for all $k \in \mathbb{N}$. More comprehensive information about the Euler totient, Jordan totient and $\mathfrak{J}_k^{(1)}$ Jordan-type functions can be obtained from the studies [2, 17].

The present study focuses on characterizing a regular matrix operator with the help of the Jordan-type function $\mathfrak{J}_k^{(1)}$, analyzing its domain in ℓ_p and ℓ_∞ , duals and matrix mappings, and identifying a specific class of compact operators on the newly defined sequence spaces using the Hausdorff measure of non-compactness (Hmnc).

2 Jordan-type matrix operator $\mathcal{J}^{(k,1)}$ and its sequence spaces

In this section, the regular Jordan-type matrix operator $\mathcal{J}^{(k,1)}$, which is constructed using the Jordan-type function $\mathfrak{J}_k^{(1)}$, is introduced. Subsequently, the normed sequence space $\ell_p(\mathcal{J}^{(k,1)})$ is defined, and it is demonstrated that this space is complete and linearly norm isomorphic to ℓ_p for $1 \leq p \leq \infty$. Furthermore, it is proven that $\ell_p(\mathcal{J}^{(k,1)})$ is not a Hilbert space, except in the case where $p = 2$, and certain inclusion relations are discussed. Finally, the existence of a Schauder basis is established at the end of this section.

We describe the matrix operator $\mathcal{J}^{(k,1)} = (j_{rs}^{(k,1)})_{r,s \in \mathbb{N}}$ with the help of the function $\mathfrak{J}_k^{(1)}$ as

$$j_{rs}^{(k,1)} := \begin{cases} \frac{\mathfrak{J}_k^{(1)}(s)}{\binom{r+k-1}{k}}, & \text{if } s \mid r, \\ 0, & \text{if } s \nmid r. \end{cases} \quad (k \in \mathbb{N}) \tag{2.1}$$

The matrix operator $\mathcal{J}^{(k,1)}$ can be expressed more clearly as

$$\mathcal{J}^{(k,1)} := \begin{bmatrix} \frac{\mathfrak{J}_k^{(1)}(1)}{\binom{k}{k}} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{\mathfrak{J}_k^{(1)}(1)}{\binom{k+1}{k}} & \frac{\mathfrak{J}_k^{(1)}(2)}{\binom{k+1}{k}} & 0 & 0 & 0 & 0 & \dots \\ \frac{\mathfrak{J}_k^{(1)}(1)}{\binom{k+2}{k}} & 0 & \frac{\mathfrak{J}_k^{(1)}(3)}{\binom{k+2}{k}} & 0 & 0 & 0 & \dots \\ \frac{\mathfrak{J}_k^{(1)}(1)}{\binom{k+3}{k}} & \frac{\mathfrak{J}_k^{(1)}(2)}{\binom{k+3}{k}} & 0 & \frac{\mathfrak{J}_k^{(1)}(4)}{\binom{k+3}{k}} & 0 & 0 & \dots \\ \frac{\mathfrak{J}_k^{(1)}(1)}{\binom{k+4}{k}} & 0 & 0 & 0 & \frac{\mathfrak{J}_k^{(1)}(5)}{\binom{k+4}{k}} & 0 & \dots \\ \frac{\mathfrak{J}_k^{(1)}(1)}{\binom{k+5}{k}} & \frac{\mathfrak{J}_k^{(1)}(2)}{\binom{k+5}{k}} & \frac{\mathfrak{J}_k^{(1)}(3)}{\binom{k+5}{k}} & 0 & 0 & \frac{\mathfrak{J}_k^{(1)}(6)}{\binom{k+5}{k}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $k = 1$, $\mathcal{J}^{(k,1)}$ is reduced to the Euler totient matrix operator studied in [21]. From its definition, it can be seen that $\mathcal{J}^{(k,1)}$ is regular. Moreover, the $\mathcal{J}^{(k,1)}$ -transform of any $u = (u_s) \in \omega$ is obtained as

$$\nu_r := (\mathcal{J}^{(k,1)}u)_r = \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s)u_s \quad (r \in \mathbb{N}). \tag{2.2}$$

The inverse $\mathcal{J}^{(k,1)^{-1}} = (j_{rs}^{(k,1)^{-1}})_{r,s \in \mathbb{N}}$ of the matrix operator $\mathcal{J}^{(k,1)}$ is computed as

$$j_{rs}^{(k,1)^{-1}} := \begin{cases} \frac{\mu\left(\frac{r}{s}\right)\binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)}, & \text{if } s \mid r, \\ 0, & \text{if } s \nmid r \end{cases} \tag{2.3}$$

for each $k \in \mathbb{N}$.

We define the sets $\ell_p(\mathcal{J}^{(k,1)})$ and $\ell_\infty(\mathcal{J}^{(k,1)})$ by

$$\ell_p(\mathcal{J}^{(k,1)}) = \left\{ u = (u_s) \in \omega : \sum_{r=1}^{\infty} \left| \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) u_s \right|^p < \infty \right\}$$

and

$$\ell_\infty(\mathcal{J}^{(k,1)}) = \left\{ u = (u_s) \in \omega : \sup_{r \in \mathbb{N}} \left| \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) u_s \right| < \infty \right\}$$

for $1 \leq p < \infty$. In that case, the sets $\ell_p(\mathcal{J}^{(k,1)})$ and $\ell_\infty(\mathcal{J}^{(k,1)})$ can be rewritten as $\ell_p(\mathcal{J}^{(k,1)}) = (\ell_p)_{\mathcal{J}^{(k,1)}}$ and $\ell_\infty(\mathcal{J}^{(k,1)}) = (\ell_\infty)_{\mathcal{J}^{(k,1)}}$, respectively, with the notation (1.1). Sequence spaces obtained by half Jordan-type functions will be called Jordan-type sequence spaces.

Unless stated otherwise, the assumption $1 \leq p < \infty$ will be adopted in the subsequent sections of this study.

Wilansky [39] demonstrated that if D is a triangular matrix and Λ is a BK-space, then the domain Λ_D is also a BK-space, with the norm given by $\|u\|_{\Lambda_D} = \|Du\|_{\Lambda}$. Therefore, the following result concerning the BK-spaceness is stated without proof.

Theorem 2.1. $\ell_p(\mathcal{J}^{(k,1)})$ and $\ell_\infty(\mathcal{J}^{(k,1)})$ are BK-spaces with

$$\|u\|_{\ell_p(\mathcal{J}^{(k,1)})} = \left(\sum_{r=1}^{\infty} \left| \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) u_s \right|^p \right)^{\frac{1}{p}}$$

and

$$\|u\|_{\ell_\infty(\mathcal{J}^{(k,1)})} = \sup_{r \in \mathbb{N}} \left| \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) u_s \right|,$$

respectively.

Theorem 2.2. $\ell_p(\mathcal{J}^{(k,1)})$ and $\ell_\infty(\mathcal{J}^{(k,1)})$ are linearly norm isomorphic to ℓ_p and ℓ_∞ , respectively.

Proof. It will be proven only for the spaces $\ell_p(\mathcal{J}^{(k,1)})$ and ℓ_p , since it can be shown similarly for the other spaces.

In the first step of proving our claim, the linearity of the function defined as $\mathcal{R} : \ell_p(\mathcal{J}^{(k,1)}) \rightarrow \ell_p$, $\mathcal{R}(u) = \mathcal{J}^{(k,1)}u$ is seen. Then, it is obtained the injectivity of \mathcal{R} from the expression $\mathcal{R}(u) = 0 \Rightarrow u = 0$.

Consider that $\nu = (\nu_s) \in \ell_p$ and $u = (u_s) \in \omega$ whose terms are

$$u_s = \sum_{l|s} \frac{\mu\left(\frac{s}{l}\right) \binom{l+k-1}{k}}{\mathfrak{J}_k^{(1)}(s)} \nu_l. \quad (2.4)$$

From the equality

$$\begin{aligned} \left(\mathcal{J}^{(k,1)}u \right)_r &= \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) u_s \\ &= \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) \sum_{l|s} \frac{\mu\left(\frac{s}{l}\right) \binom{l+k-1}{k}}{\mathfrak{J}_k^{(1)}(s)} \nu_l \\ &= \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \sum_{l|s} \mu\left(\frac{s}{l}\right) \binom{l+k-1}{k} \nu_l = \nu_r, \end{aligned}$$

it is seen that \mathcal{R} is surjective. Additionally, since the relation $\|u\|_{\ell_p(\mathcal{J}^{(k,1)})} = \|\mathcal{J}^{(k,1)}u\|_{\ell_p}$ holds, then \mathcal{R} keeps the norm. \square

Remark 2.3. It is well-known that the space ℓ_2 is a Hilbert space, which means it is a vector space endowed with an inner product $\langle \cdot, \cdot \rangle_{\ell_2}$ that induces the norm $\| \cdot \|_{\ell_2}$. Similarly, the space $\ell_2(\mathcal{J}^{(k,1)})$ is also an inner product space, with the inner product defined as

$$\langle x, y \rangle_{\ell_2(\mathcal{J}^{(k,1)})} = \langle \mathcal{J}^{(k,1)}x, \mathcal{J}^{(k,1)}y \rangle_{\ell_2}.$$

Theorem 2.4. *If $p \neq 2$, then $\ell_p(\mathcal{J}^{(k,1)})$ isn't a Hilbert space.*

Proof. If it is regarded that

$$x_r = \begin{cases} \sum_{s|r} \frac{\mu(\frac{r}{s}) \binom{s+j-1}{j}}{\mathfrak{J}_j^{(1)}(r)} & , \quad \text{if } r \text{ is even,} \\ \frac{\mu(r)}{\mathfrak{J}_j^{(1)}(r)} & , \quad \text{if } r \text{ is odd} \end{cases}$$

and

$$y_r = \begin{cases} \sum_{s|r} (-1)^{r-s} \frac{\mu(\frac{r}{s}) \binom{s+j-1}{j}}{\mathfrak{J}_j^{(1)}(r)} & , \quad \text{if } r \text{ is even,} \\ \frac{\mu(r)}{\mathfrak{J}_j^{(1)}(r)} & , \quad \text{if } r \text{ is odd} \end{cases}$$

for each $k \in \mathbb{N}$, in that case it is obtained that $\mathcal{J}^{(k,1)}x = (1, 1, 0, 0, \dots)$ and $\mathcal{J}^{(k,1)}y = (1, -1, 0, 0, \dots)$ and

$$\|x + y\|_{\ell_p(\mathcal{J}^{(k,1)})}^2 + \|x - y\|_{\ell_p(\mathcal{J}^{(k,1)})}^2 = 8 \neq 2^{2+\frac{2}{p}} = 2 \left(\|x\|_{\ell_p(\mathcal{J}^{(k,1)})}^2 + \|y\|_{\ell_p(\mathcal{J}^{(k,1)})}^2 \right),$$

for $p \neq 2$. Thus, $\| \cdot \|_{\ell_p(\mathcal{J}^{(k,1)})}$ doesn't hold the parallelogram equality and so $\ell_p(\mathcal{J}^{(k,1)})$ isn't a Hilbert space for $p \neq 2$. □

Theorem 2.5. *It holds that $\ell_\infty \subset \ell_\infty(\mathcal{J}^{(k,1)})$.*

Proof. Let $u = (u_s) \in \ell_\infty$ be a sequence. Based on the inequality,

$$\begin{aligned} \|u\|_{\ell_\infty(\mathcal{J}^{(k,1)})} &= \sup_{r \in \mathbb{N}} \left| \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) u_s \right| \\ &\leq \|u\|_\infty \sup_{r \in \mathbb{N}} \left| \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) \right| \\ &= \|u\|_\infty < \infty, \end{aligned}$$

it can be deduced that $u \in \ell_\infty(\mathcal{J}^{(k,1)})$, which is the desired result. □

Theorem 2.6. *The inclusion $\ell_p(\mathcal{J}^{(k,1)}) \subset \ell_\infty(\mathcal{J}^{(k,1)})$ is strict.*

Proof. It is immediately seen the inclusion part of the proof from the relation $\ell_p \subset \ell_\infty$. Consider that $u_s = \sum_{l|s} \frac{\mu(\frac{s}{l}) \binom{l+k-1}{k}}{\mathfrak{J}_k^{(1)}(s)} (-1)^l$. In that case, it is obtained that

$$\mathcal{J}^{(k,1)}u = \frac{1}{\binom{r+k-1}{k}} \sum_{s|r} \mathfrak{J}_k^{(1)}(s) \sum_{l|s} \frac{\mu(\frac{s}{l}) \binom{l+k-1}{k}}{\mathfrak{J}_k^{(1)}(s)} (-1)^l = (-1)^r \in \ell_\infty \setminus \ell_p,$$

which implies that $u \in \ell_\infty(\mathcal{J}^{(k,1)}) \setminus \ell_p(\mathcal{J}^{(k,1)})$. □

Theorem 2.7. *The inclusion $\ell_p(\mathcal{J}^{(k,1)}) \subset \ell_{\tilde{p}}(\mathcal{J}^{(k,1)})$ is strict for $1 \leq p < \tilde{p} < \infty$.*

Proof. Regard the sequence $u = (u_s) \in \ell_p(\mathcal{J}^{(k,1)})$ such that $\mathcal{J}^{(k,1)}u \in \ell_p$. Furthermore, it is known that $\ell_p \subset \ell_{\tilde{p}}$ for $1 \leq p < \tilde{p} < \infty$ and thus $\mathcal{J}^{(k,1)}u \in \ell_{\tilde{p}}$. Consequently, it can be wrote $u = (u_s) \in \ell_{\tilde{p}}(\mathcal{J}^{(k,1)})$.

The strictness of the inclusion becomes clear when $\tilde{v} = \mathcal{J}^{(k,1)}\tilde{u} \in \ell_{\tilde{p}} \setminus \ell_p$ is taken. □

Within the normed sequence space $(\Lambda, \|\cdot\|)$, a sequence (ϱ_s) in Λ is called a Schauder basis for Λ if every element $u \in \Lambda$ has a unique representation as an infinite series:

$$u = \sum_{s=1}^{\infty} \sigma_s \varrho_s,$$

where (σ_s) is a unique sequence of scalars. Furthermore, the partial sums of this series must converge to u in the norm of Λ :

$$\left\| u - \sum_{s=1}^r \sigma_s \varrho_s \right\| \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

This definition implies that each element $u \in \Lambda$ has a unique representation as an infinite linear combination of the basis elements (ϱ_s) , with the partial sums approaching u in the Λ -norm.

Moreover, the basis for $\ell_p(\mathcal{J}^{(k,1)})$ is given by the inverse image of the canonical basis $(e^{(r)})_{r \in \mathbb{N}}$ of ℓ_p . This follows from the fact that the mapping $\mathcal{R} : \ell_p(\mathcal{J}^{(k,1)}) \rightarrow \ell_p$ is an isometric isomorphism, ensuring that the basis structure is preserved under \mathcal{R} .

For this reason, we present the following theorem without proof.

Theorem 2.8. *Let us examine the sequences $\sigma_s = (\mathcal{J}^{(k,1)}u)_s$ and $\varrho^{(k)} = (\varrho_r^{(k)}) \in \ell_p(\mathcal{J}^{(k,1)})$ described as*

$$\varrho_r^{(k)} := \begin{cases} \frac{\mu(\frac{r}{s}) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} & , \quad \text{if } s \mid r, \\ 0 & , \quad \text{if } s \nmid r. \end{cases}$$

Then, the set $\varrho^{(k)}$ is a basis of $\ell_p(\mathcal{J}^{(k,1)})$ and the unique representation of each $u \in \ell_p(\mathcal{J}^{(k,1)})$ is stated as $u = \sum_s \sigma_s \varrho^{(s)}$.

3 Dual spaces of the sequence spaces induced by $\mathcal{J}^{(k,1)}$

In this section, we will introduce the expression $[\ell_p(\mathcal{J}^{(k,1)})]^\eta$, where η can take values from the set $\{\alpha, \beta, \gamma\}$, and p is a parameter such that $1 \leq p \leq \infty$.

To properly identify the duals, we will consider the conditions defined in equations (3.1)

through (3.12).

$$\sup_{s \in \mathbb{N}} \sum_r |d_{rs}| < \infty, \tag{3.1}$$

$$\sup_{s \in \mathbb{N}} \sum_r |d_{rs}|^p < \infty, \tag{3.2}$$

$$\sup_{r,s \in \mathbb{N}} |d_{rs}| < \infty, \tag{3.3}$$

$$\lim_{r \rightarrow \infty} d_{rs} \text{ exists, } (\forall s \in \mathbb{N}) \tag{3.4}$$

$$\lim_{r \rightarrow \infty} d_{rs} = 0, \tag{3.5}$$

$$\sup_{E \in \mathcal{F}} \sum_s \left| \sum_{r \in E} d_{rs} \right|^q < \infty, \tag{3.6}$$

$$\sup_{r \in \mathbb{N}} \sum_s |d_{rs}|^q < \infty, \tag{3.7}$$

$$\sup_{E \in \mathcal{F}} \sum_r \left| \sum_{s \in E} d_{rs} \right| < \infty, \tag{3.8}$$

$$\sup_{E \in \mathcal{F}} \sum_r \left| \sum_{s \in E} d_{rs} \right|^p < \infty, \tag{3.9}$$

$$\sup_{r \in \mathbb{N}} \sum_s |d_{rs}| < \infty, \tag{3.10}$$

$$\lim_{r \rightarrow \infty} \sum_s |d_{rs}| = \sum_s \left| \lim_{r \rightarrow \infty} d_{rs} \right|, \tag{3.11}$$

$$\lim_{r \rightarrow \infty} \sum_s |d_{rs}| = 0. \tag{3.12}$$

Here, \mathcal{F} denotes the collection of all finite subsets of \mathbb{N} , $1 < p < \infty$ and $q = \frac{p}{p-1}$. Now, a table derived from the study [37] is presented to illustrate the conditions of certain classical matrix classes.

Table 1. Characterizations of $(\Lambda : \Psi)$

$(\Lambda \downarrow; \Psi \rightarrow)$	c_0	c	l_∞	l_1	l_p
c_0	•	•	(3.10)	(3.8)	(3.9)
c	•	•	(3.10)	(3.8)	(3.9)
l_∞	(3.12)	(3.4), (3.11)	(3.10)	(3.8)	(3.9)
l_1	(3.3), (3.5)	(3.3), (3.4)	(3.3)	(3.1)	(3.2)
l_p	(3.5), (3.7)	(3.4), (3.7)	(3.7)	(3.6)	•

The symbol“•” denote conditions of classes that is unspecified and irrelevant to the focus of this study.

Let us consider the sets $\varkappa_1 - \varkappa_7$ which will be benefited to specify the duals:

$$\begin{aligned}
\mathcal{X}_1 &= \left\{ \mu = (\mu_s) \in \omega : \sup_{E \in \mathcal{F}} \sum_s \left| \sum_{r \in E} \frac{\mu \binom{r}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} \mu_r \right|^q < \infty \right\}, \\
\mathcal{X}_2 &= \left\{ \mu = (\mu_s) \in \omega : \sup_{s \in \mathbb{N}} \sum_r \left| \frac{\mu \binom{r}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} \mu_r \right| < \infty \right\}, \\
\mathcal{X}_3 &= \left\{ \mu = (\mu_s) \in \omega : \sup_{E \in \mathcal{F}} \sum_r \left| \sum_{s \in E} \frac{\mu \binom{r}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} \mu_r \right| < \infty \right\}, \\
\mathcal{X}_4 &= \left\{ \mu = (\mu_s) \in \omega : \lim_{r \rightarrow \infty} \sum_{l=s, s|l}^r \frac{\mu \binom{l}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} \mu_l \text{ exists for each } s \in \mathbb{N} \right\}, \\
\mathcal{X}_5 &= \left\{ \mu = (\mu_s) \in \omega : \sup_{r \in \mathbb{N}} \sum_s \left| \sum_{l=s, s|l}^r \frac{\mu \binom{l}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} \mu_l \right|^q < \infty \right\}, \\
\mathcal{X}_6 &= \left\{ \mu = (\mu_s) \in \omega : \sup_{r, s \in \mathbb{N}} \left| \sum_{l=s, s|l}^r \frac{\mu \binom{l}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} \mu_l \right| < \infty \right\}, \\
\mathcal{X}_7 &= \left\{ \mu = (\mu_s) \in \omega : \lim_{r \rightarrow \infty} \sum_s \left| \sum_{l=s, s|l}^r \frac{\mu \binom{l}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} \mu_l \right| = \sum_s \left| \sum_{l=s, s|l}^{\infty} \frac{\mu \binom{l}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} \mu_l \right| \right\}.
\end{aligned}$$

Theorem 3.1. *The following equalities hold:*

- (i) $[\ell_p(\mathcal{J}^{(k,1)})]^\alpha = \mathcal{X}_1, \quad (1 < p < \infty),$
- (ii) $[\ell_1(\mathcal{J}^{(k,1)})]^\alpha = \mathcal{X}_2,$
- (iii) $[\ell_\infty(\mathcal{J}^{(k,1)})]^\alpha = \mathcal{X}_3.$

Proof. (i) By considering (2.2), for $u \in \ell_p(\mathcal{J}^{(k,1)})$ it is reached

$$\begin{aligned}
\mu_r u_r &= \mu_r \sum_{s|r} \frac{\mu \binom{r}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} \nu_s \\
&= \sum_{s|r} \frac{\mu \binom{r}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} \mu_r \nu_s = (G\nu)_r,
\end{aligned} \tag{3.13}$$

in which the infinite matrix $G = (g_{rs})$ is described as

$$g_{rs} := \begin{cases} \frac{\mu \binom{r}{s} \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(r)} \mu_r, & \text{if } s | r, \\ 0, & \text{if } s \nmid r. \end{cases}$$

Hence, by the relation (3.13), it is obtained that $\mu u = (\mu_r u_r) \in \ell_1$ when $u \in \ell_p(\mathcal{J}^{(k,1)})$ if and only if $G\nu \in \ell_1$ when $\nu \in \ell_p$. Then, $\mu \in [\ell_p(\mathcal{J}^{(k,1)})]^\alpha$ if and only if $G \in (\ell_p : \ell_1)$. Thus, from Table 1, it is seen that $[\ell_p(\mathcal{J}^{(k,1)})]^\alpha = \mathcal{X}_1$ for $1 < p < \infty$, which is the desired result.

- (ii) The proof follows similarly to the first part by using the conditions of the class $(\ell_1 : \ell_1)$ from Table 1. Thus, it is omitted.

(iii) The proof follows in the same way as in the first part, with the aid of the conditions of the class $(\ell_\infty : \ell_1)$ from Table 1. Hence, it is also omitted. □

Theorem 3.2. *The following equalities hold:*

- (i) $[\ell_p(\mathcal{J}^{(k,1)})]^\beta = \varkappa_4 \cap \varkappa_5, (1 < p < \infty)$
- (ii) $[\ell_1(\mathcal{J}^{(k,1)})]^\beta = \varkappa_4 \cap \varkappa_6,$
- (iii) $[\ell_\infty(\mathcal{J}^{(k,1)})]^\beta = \varkappa_4 \cap \varkappa_7.$

Proof. (i) Let us choose two sequences $\mu = (\mu_s) \in \omega$ and $u \in \ell_p(\mathcal{J}^{(k,1)})$ satisfying (2.4). Then, it follows that

$$\begin{aligned} \psi_r &= \sum_{s=1}^r \mu_s u_s \\ &= \sum_{s=1}^r \mu_s \left(\sum_{t|s} \frac{\mu\left(\frac{s}{t}\right) \binom{t+k-1}{k}}{\mathfrak{J}_k^{(1)}(s)} \nu_t \right) \\ &= \sum_{s=1}^r \left(\sum_{t=s, s|t} \frac{\mu\left(\frac{t}{s}\right) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(t)} \mu_t \right) \nu_s \\ &= (O\nu)_r, \end{aligned}$$

where the infinite matrix $O = (o_{rs})$ is defined as

$$o_{rs} := \begin{cases} \sum_{t=s, s|t} \frac{\mu\left(\frac{t}{s}\right) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(t)} \mu_t, & 1 \leq s \leq r, \\ 0, & s > r. \end{cases} \tag{3.14}$$

Thus, $\mu u \in cs$ when $u = (u_s) \in \ell_p(\mathcal{J}^{(k,1)})$ if and only if $\psi = (\psi_r) \in c$ when $\nu \in \ell_p$. In that case, $\mu \in [\ell_p(\mathcal{J}^{(k,1)})]^\beta$ if and only if $O \in (\ell_p : c)$ for $1 < p < \infty$. Hence, in view of the conditions of $(\ell_p : c)$ from Table 1, it is concluded that

$$[\ell_p(\mathcal{J}^{(k,1)})]^\beta = \varkappa_4 \cap \varkappa_5.$$

(ii) The proof follows similarly to the first part by using the conditions of the class $(\ell_1 : c)$ from Table 1. Thus, it is omitted.

(iii) The proof follows in the same way as in the first part, with the aid of the conditions of the class $(\ell_\infty : c)$ from Table 1. Hence, it is also omitted. □

Theorem 3.3. *The following equalities hold:*

- (i) $[\ell_p(\mathcal{J}^{(k,1)})]^\gamma = \varkappa_5, (1 < p < \infty)$
- (ii) $[\ell_1(\mathcal{J}^{(k,1)})]^\gamma = \varkappa_6,$
- (iii) $[\ell_\infty(\mathcal{J}^{(k,1)})]^\gamma = \varkappa_5$ with $q = 1.$

Proof. It can be derived using a similar approach as in the proof of Theorem 3.2, by considering the classes $(\ell_p : \ell_\infty)$, $(\ell_1 : \ell_\infty)$, and $(\ell_\infty : \ell_\infty)$ together, as presented in Table 1. □

4 Matrix mappings on the sequence spaces induced by $\mathcal{J}^{(k,1)}$

In this section, we will introduce the matrix classes related to the spaces just described. The theorem we are about to present will form the foundation of this section and allow us to examine the relationships between these spaces in more detail.

Theorem 4.1. Consider that $\Psi \subset \omega$, and define the infinite matrices $\Theta^{(r)} = (\theta_{ts}^{(r)})$ and $\Theta = (\theta_{rs})$ as follows:

$$\theta_{ts}^{(r)} := \begin{cases} \sum_{l=s, s|l}^t \frac{\mu\left(\frac{l}{s}\right) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} d_{rl}, & 1 \leq s \leq t, \\ 0, & s > t. \end{cases} \quad (4.1)$$

and

$$\theta_{rs} = \sum_{l=s, s|l}^{\infty} \frac{\mu\left(\frac{l}{s}\right) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} d_{rl} \quad (4.2)$$

for all $t, r, s \in \mathbb{N}$. Then, $D = (d_{rs}) \in (\ell_p(\mathcal{J}^{(k,1)}) : \Psi)$ if and only if $\Theta^{(r)} \in (\ell_p : c)$ and $\Theta \in (\ell_p : \Psi)$ for $1 \leq p \leq \infty$.

Proof. Let us consider $D = (d_{rs}) \in (\ell_p(\mathcal{J}^{(k,1)}) : \Psi)$ and $u \in \ell_p(\mathcal{J}^{(k,1)})$. In that case,

$$\begin{aligned} \sum_{s=1}^t d_{rs} u_s &= \sum_{s=1}^t d_{rs} \left(\sum_{l|s} \frac{\mu\left(\frac{s}{l}\right) \binom{l+k-1}{k}}{\mathfrak{J}_k^{(1)}(s)} \nu_l \right) \\ &= \sum_{s=1}^t \left(\sum_{l=s, s|l}^t \frac{\mu\left(\frac{l}{s}\right) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} d_{rl} \right) \nu_s \\ &= \sum_{s=1}^t \theta_{ts}^{(r)} \nu_s \end{aligned} \quad (4.3)$$

for all $t, r \in \mathbb{N}$. Since Du exists, it must be $\Theta^{(r)} \in (\ell_p : c)$. By passing limit for $t \rightarrow \infty$ in the relation (4.3), it is reached that $Du = \Theta\nu$. Since $Du \in \Psi$, in that case $\Theta\nu \in \Psi$ and so $\Theta \in (\ell_p : \Psi)$.

Conversely, let us suppose that $\Theta^{(r)} \in (\ell_p : c)$ and $\Theta \in (\ell_p : \Psi)$. Then, it is seen that $(\theta_{rs})_{s \in \mathbb{N}} \in \ell_p^\beta$ which gives $(d_{rs})_{s \in \mathbb{N}} \in (\ell_p(\mathcal{J}^{(k,1)}))^\beta$ for all $r \in \mathbb{N}$. Hence, Du exists for all $u \in \ell_p(\mathcal{J}^{(k,1)})$. Therefore, it is reached from the relation (4.3) for $t \rightarrow \infty$ that $Du = \Theta\nu$ and this implies that $D \in (\ell_p(\mathcal{J}^{(k,1)}) : \Psi)$, which is desired result. \square

Corollary 4.2. Let the infinite matrices $\Theta^{(r)} = (\theta_{ts}^{(r)})$ and $\Theta = (\theta_{rs})$ be defined by the equations (4.1) and (4.2), respectively. Under these definitions, the necessary and sufficient conditions for the space $(\ell_p(\mathcal{J}^{(k,1)}) : \Psi)$, where $1 \leq p \leq \infty$, can be determined by referring to the results presented in Table 2.

Theorem 4.3. Let the infinite matrices $\tilde{D} = (\tilde{d}_{rs})$ and $D = (d_{rs})$ be related by the following equation:

$$\tilde{d}_{rs} = \sum_{t|r} \frac{\mathfrak{J}_k^{(1)}(t)}{\binom{r+k-1}{k}} d_{tk}. \quad (4.4)$$

In this case, $D \in (\Lambda : \Psi(\mathcal{J}^{(k,1)}))$ if and only if $\tilde{D} \in (\Lambda : \Psi)$, where $\Lambda \in \{\ell_1, \ell_p, \ell_\infty, c, c_0\}$ and $\Psi \in \{\ell_1, \ell_p, \ell_\infty\}$.

Table 2. Characterizations of the classes $(\ell_p(\mathcal{J}^{(k,1)}) : \Psi)$, where $1 \leq p \leq \infty$.

$(\ell_p(\mathcal{J}^{(k,1)})) \downarrow: \Psi \rightarrow$	c_0	c	ℓ_1	ℓ_p	ℓ_∞
$\ell_1(\mathcal{J}^{(k,1)})$	(3.3) ^r , (3.4) ^r (3.3), (3.5)	(3.3) ^r , (3.4) ^r (3.3), (3.4)	(3.3) ^r , (3.4) ^r (3.1)	(3.3) ^r , (3.4) ^r (3.2)	(3.3) ^r , (3.4) ^r (3.3)
$\ell_p(\mathcal{J}^{(k,1)})$	(3.4) ^r , (3.7) ^r (3.5), (3.7)	(3.4) ^r , (3.7) ^r (3.4), (3.7)	(3.4) ^r , (3.7) ^r (3.6)	•	(3.4) ^r , (3.7) ^r (3.7)
$\ell_\infty(\mathcal{J}^{(k,1)})$	(3.4) ^r , (3.11) ^r (3.12)	(3.4) ^r , (3.11) ^r (3.4), (3.11)	(3.4) ^r , (3.11) ^r (3.8)	(3.4) ^r , (3.11) ^r (3.9)	(3.4) ^r , (3.11) ^r (3.10)

The notations $(\lambda)^r$ and (λ) signify that the criterion (λ) is met using the matrices $\Theta^{(r)}$ and Θ , respectively, within the interval $3.1 \leq \lambda \leq 3.12$.

Proof. Let the infinite matrices \tilde{D} and D be described by the relation (4.4), with $\Lambda \in \{\ell_1, \ell_p, \ell_\infty, c, c_0\}$ and $\Psi \in \{\ell_1, \ell_p, \ell_\infty\}$. For any $u = (u_s) \in \Lambda$, we obtain

$$\sum_{s=1}^\infty \tilde{d}_{rs} u_s = \sum_{t|r} \frac{\mathfrak{J}_k^{(1)}(t)}{\binom{r+k-1}{k}} \sum_{s=1}^\infty d_{ts} u_s.$$

This means that $\tilde{D}_r(u) = \mathcal{J}_r^{(k,1)}(Du)$ for all $r \in \mathbb{N}$, which implies that $Du \in \Psi(\mathcal{J}^{(k,1)})$ if and only if $\tilde{D}u \in \Psi$ for every $u \in \Lambda$. Therefore, we conclude that $D \in (\Lambda : \Psi(\mathcal{J}^{(k,1)}))$ if and only if $\tilde{D} \in (\Lambda : \Psi)$. □

Corollary 4.4. Let $\tilde{D} = (\tilde{d}_{rs})$ and $D = (d_{rs})$ be defined by the relation (4.4). Then, the conditions for $(\Lambda : \Psi(\mathcal{J}^{(k,1)}))$ are presented in Table 3, where Λ belongs to $\{\ell_1, \ell_p, \ell_\infty, c, c_0\}$ and Ψ belongs to $\{\ell_1, \ell_p, \ell_\infty\}$.

Table 3. Characterizations of $(\Lambda : \Psi(\mathcal{J}^{(k,1)}))$

$(\Lambda \downarrow: \Psi(\mathcal{J}^{(k,1)})) \rightarrow$	$\ell_1(\mathcal{J}^{(k,1)})$	$\ell_p(\mathcal{J}^{(k,1)})$	$\ell_\infty(\mathcal{J}^{(k,1)})$
ℓ_1	(3.1)	(3.2)	(3.3)
ℓ_p	(3.6)	•	(3.7)
ℓ_∞	(3.8)	(3.9)	(3.10)
c	(3.8)	(3.9)	(3.10)
c_0	(3.8)	(3.9)	(3.10)

Conditions hold with the matrix $\tilde{D} = (\tilde{d}_{rs})$

5 Compact operators on the sequence spaces induced by $\mathcal{J}^{(k,1)}$

For a normed space Λ , let \mathcal{D}_Λ denote the unit sphere in Λ . The notation

$$\|u\|_\Lambda^\diamond = \sup_{x \in \mathcal{D}_\Lambda} \left| \sum_s u_s x_s \right|$$

is employed for a BK-space $\Lambda \supset \Omega$ and $u = (u_s) \in \omega$, assuming the convergence of the series, where Ω represents the space of all finite sequences. In this context, we have $u \in \Lambda^\beta$.

Lemma 5.1. [27] *The following results are true:*

- (i) $\ell_\infty^\beta = \ell_1$ and $\|u\|_{\ell_\infty}^\diamond = \|u\|_{\ell_1}$, ($\forall u \in \ell_1$).
- (ii) $\ell_1^\beta = \ell_\infty$ and $\|u\|_{\ell_1}^\diamond = \|u\|_{\ell_\infty}$, ($\forall u \in \ell_\infty$).
- (iii) $\ell_p^\beta = \ell_q$ and $\|u\|_{\ell_p}^\diamond = \|u\|_{\ell_q}$, ($\forall u \in \ell_1$).

The notation $\mathfrak{B}(\Lambda : \Psi)$ represents the set of all bounded (continuous) linear operators from Λ to Ψ .

Lemma 5.2. [27] Suppose that Λ and Ψ are BK-spaces. Then, for each $D \in (\Lambda : \Psi)$, there exists a linear map $\mathcal{L}_D \in \mathfrak{B}(\Lambda : \Psi)$ such that $\mathcal{L}_D(u) = Du$ for all $u \in \Lambda$.

Lemma 5.3. [27] Assume that $\Lambda \supset \Omega$ is a BK-space. If $D \in (\Lambda : \Psi)$, then $\|\mathcal{L}_D\| = \|D\|_{(\Lambda:\Psi)} = \sup_{r \in \mathbb{N}} \|D_r\|_{\Lambda}^{\diamond} < \infty$ for $\Psi \in \{c_0, c, \ell_{\infty}\}$.

Let us examine a metric space Λ and a bounded subset $A \subset \Lambda$. The Hmonc of A is denoted by $\chi(A)$, which is defined as

$$\chi(A) = \inf \left\{ \epsilon > 0 : A \subset \bigcup_{t=1}^r A(u_t, v_t), u_t \in \Lambda, v_t < \epsilon, r \in \mathbb{N} \setminus \{0\} \right\},$$

where $1 \leq t \leq r$ and $A(u_t, v_t)$ represents the open ball centered at u_t with radius v_t . For more detailed information about Hmonc, refer to the study [27].

Lemma 5.4. [35] Let $A \subset \ell_p$ be a bounded set, and consider the operator $\lambda_v : \ell_p \rightarrow \ell_p$ defined as $\lambda_v(u) = (u_0, u_1, u_2, \dots, u_v, 0, 0, \dots)$ for every $u = (u_s) \in \ell_p$, where $1 \leq p < \infty$ and $v \in \mathbb{N}$. Let I denote the identity map on ℓ_p . Then, the following holds:

$$\chi(A) = \lim_{v \rightarrow \infty} \left(\sup_{u \in A} \|(I - \lambda_v)(u)\|_{\ell_p} \right).$$

For the Banach spaces Λ and Ψ , a linear operator $\mathcal{L} : \Lambda \rightarrow \Psi$ is said to be compact if its domain covers the entire space Λ and the image $\mathcal{L}(A)$ is totally bounded in Ψ for every sequence $u = (u_s) \in \ell_{\infty} \cap \Lambda$. Equivalently, \mathcal{L} is compact if, for any sequence $u = (u_k) \in \ell_{\infty} \cap \Lambda$, the sequence $(\mathcal{L}(u))$ has a convergent subsequence in Ψ .

The quantity $\|\mathcal{L}\|_{\chi}$, known as the Hmonc of \mathcal{L} , is given by

$$\|\mathcal{L}\|_{\chi} = \chi(\mathcal{L}(\mathcal{D}_{\Lambda})).$$

There exists a fundamental relationship between compact operators and Hmonc, which states that

$$\mathcal{L} \text{ is compact if and only if } \|\mathcal{L}\|_{\chi} = 0.$$

For further exploration of sequence space theory and the role of Hmonc in identifying compact operators between BK-spaces, readers may refer to [31, 32, 5, 25, 6, 7, 14, 15].

The subsequent results are presented for $x = (x_s)$ and $y = (y_s) \in \omega$, which satisfy the equation

$$y_s = \sum_{l=s, s|l}^{\infty} \frac{\mu\left(\frac{l}{s}\right) \binom{s+k-1}{k}}{\mathfrak{J}_k^{(1)}(l)} x_l, \quad \text{for } s \in \mathbb{N}. \quad (5.1)$$

Lemma 5.5. Suppose that $x = (x_s) \in [\ell_p(\mathcal{J}^{(k,1)})]^{\beta}$ for $1 \leq p \leq \infty$. Then, $y = (y_s) \in \ell_q$ and

$$\sum_s x_s u_s = \sum_s y_s \nu_s \quad (5.2)$$

for all sequences $u = (u_s) \in \ell_p(\mathcal{J}^{(k,1)})$.

Lemma 5.6. Let $y = (y_s)$ be the sequence described by the relation (5.1). Then:

- (i) $\|x\|_{\ell_{\infty}(\mathcal{J}^{(k,1)})}^{\diamond} = \sum_s |y_s| < \infty$ for all $x = (x_s) \in [\ell_{\infty}(\mathcal{J}^{(k,1)})]^{\beta}$.
- (ii) $\|x\|_{\ell_1(\mathcal{J}^{(k,1)})}^{\diamond} = \sup_s |y_s| < \infty$ for all $x = (x_s) \in [\ell_1(\mathcal{J}^{(k,1)})]^{\beta}$.
- (iii) $\|x\|_{\ell_p(\mathcal{J}^{(k,1)})}^{\diamond} = (\sum_s |y_s|^q)^{\frac{1}{q}} < \infty$ for all $x = (x_s) \in [\ell_p(\mathcal{J}^{(k,1)})]^{\beta}$ and $1 < p < \infty$.

Proof. Since the proof for the other parts follows a similar approach, we will prove the theorem only for the first part.

(i) This follows from Lemma 5.5, where $y = (y_s) \in \ell_1$ and the relation (5.1) holds for $x = (x_s) \in (\ell_\infty(\mathcal{J}^{(k,1)}))^\beta$ and for all $u = (u_s) \in \ell_\infty(\mathcal{J}^{(k,1)})$. Since $\|u\|_{\ell_\infty(\mathcal{J}^{(k,1)})} = \|\nu\|_{\ell_\infty}$, it follows that " $u \in \mathcal{D}_{\ell_\infty(\mathcal{J}^{(k,1)})}$ if and only if $\nu \in \mathcal{D}_{\ell_\infty}$ ". Thus, we can write the equality

$$\|x\|_{\ell_\infty(\mathcal{J}^{(k,1)})}^\diamond = \sup_{u \in \mathcal{D}_{\ell_\infty(\mathcal{J}^{(k,1)})}} \left| \sum_s x_s u_s \right| = \sup_{\nu \in \mathcal{D}_{\ell_\infty}} \left| \sum_s y_s \nu_s \right| = \|y\|_{\ell_\infty}^\diamond.$$

By Lemma 5.1, we obtain that

$$\|x\|_{\ell_\infty(\mathcal{J}^{(k,1)})}^\diamond = \|y\|_{\ell_\infty}^\diamond = \|y\|_{\ell_1} = \sum_s |y_s| < \infty.$$

□

Lemma 5.7. [31] *Let Λ be a BK-space containing Ω . Then, the following properties hold:*

(i) *If $D \in (\Lambda : \ell_\infty)$, then the inequality*

$$0 \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \|D_r\|_\Lambda^\diamond$$

is satisfied, and \mathcal{L}_D is compact whenever $\lim_r \|D_r\|_\Lambda^\diamond = 0$.

(ii) *If $D \in (\Lambda : c_0)$, then we have the exact equality*

$$\|\mathcal{L}_D\|_\chi = \limsup_r \|D_r\|_\Lambda^\diamond,$$

and \mathcal{L}_D is compact if and only if $\lim_r \|D_r\|_\Lambda^\diamond = 0$.

(iii) *If $D \in (\Lambda : \ell_1)$, then*

$$\limsup_t \sup_{E \in \mathcal{F}_t} \left\| \sum_{n \in E} D_r \right\|_\Lambda^\diamond \leq \|\mathcal{L}_D\|_\chi \leq 4 \cdot \limsup_t \sup_{E \in \mathcal{F}_t} \left\| \sum_{n \in E} D_r \right\|_\Lambda^\diamond.$$

Moreover, \mathcal{L}_D is compact if and only if

$$\limsup_t \sup_{E \in \mathcal{F}_t} \left\| \sum_{n \in E} D_r \right\|_\Lambda^\diamond = 0.$$

Here, \mathcal{F} denotes the collection of all finite subsets of \mathbb{N} , while \mathcal{F}_t represents the subfamily of \mathcal{F} consisting of subsets of \mathbb{N} whose elements are greater than t .

In the sequel of the study, it is used the matrices $\Theta = (\theta_{rs})$ and $D = (d_{rs})$ connected with the relation (4.2) with the supposition of the series' convergence.

Lemma 5.8. *Suppose that $\Psi \subset \omega$ and let $D = (d_{rs})$ be an infinite matrix. If D belongs to the matrix class $(\ell_p(\mathcal{J}^{(k,1)}) : \Psi)$, then it follows that $\Theta \in (\ell_p : \Psi)$, and the relation $Du = \Theta\nu$ holds for every sequence $u \in \ell_p(\mathcal{J}^{(k,1)})$, where $1 \leq p \leq \infty$.*

Proof. The result follows directly from Lemma 5.5. □

Theorem 5.9. *Let $1 < p < \infty$. In that case:*

(i) *If $D \in (\ell_p(\mathcal{J}^{(k,1)}) : \ell_\infty)$, then*

$$0 \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}}$$

and \mathcal{L}_D is compact if

$$\lim_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(ii) If $D \in (\ell_p(\mathcal{J}^{(k,1)}) : c_0)$, then

$$\|\mathcal{L}_D\|_{\mathcal{X}} = \limsup_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}}$$

and \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(iii) If $D \in (\ell_p(\mathcal{J}^{(k,1)}) : \ell_1)$, then

$$\lim_t \|D\|_{(\ell_p(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)} \leq \|\mathcal{L}_D\|_{\mathcal{X}} \leq 4 \cdot \lim_t \|D\|_{(\ell_p(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)}$$

and \mathcal{L}_D is compact if and only if

$$\lim_t \|D\|_{(\ell_p(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)} = 0,$$

where $\|D\|_{(\ell_p(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)} = \sup_{E \in \mathcal{F}_t} \left(\sum_s |\sum_{n \in E} \theta_{rs}|^q \right)^{\frac{1}{q}}$ for every $t \in \mathbb{N}$.

Proof. We proceed by considering the following cases:

(i) Let $D \in (\ell_p(\mathcal{J}^{(k,1)}) : \ell_\infty)$ and $u = (u_s) \in \ell_p(\mathcal{J}^{(k,1)})$. Since the series $\sum_s b_{rs} u_s$ converges for each $r \in \mathbb{N}$, it follows that $D_r \in (\ell_p(\mathcal{J}^{(k,1)}))^\beta$. By Lemma 5.6-(iii), we obtain

$$\|D_r\|_{\ell_p(\mathcal{J}^{(k,1)})}^\diamond = \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}}.$$

Thus, applying Lemma 5.7-(i), we conclude that

$$0 \leq \|\mathcal{L}_D\|_{\mathcal{X}} \leq \limsup_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}},$$

and \mathcal{L}_D is compact if

$$\lim_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(ii) Let $D \in (\ell_p(\mathcal{J}^{(k,1)}) : c_0)$. Since

$$\|D_r\|_{\ell_p(\mathcal{J}^{(k,1)})}^\diamond = \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}},$$

applying Lemma 5.7-(ii), we obtain

$$\|\mathcal{L}_D\|_{\mathcal{X}} = \limsup_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}}.$$

Moreover, \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sum_s |\theta_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(iii) Let $D \in (\ell_p(\mathcal{J}^{(k,1)}) : \ell_1)$. From Lemma 5.6, we see that

$$\left\| \sum_{n \in E} D_r \right\|_{\ell_p(\mathcal{J}^{(k,1)})}^\diamond = \left\| \sum_{n \in E} \Theta_r \right\|_{\ell_q}^\diamond.$$

Taking Lemma 5.7-(iii) into account, we conclude that

$$\lim_t \left(\sup_{E \in \mathcal{F}_t} \sum_s \left| \sum_{n \in E} \theta_{rs} \right|^q \right)^{\frac{1}{q}} \leq \|\mathcal{L}_D\|_\chi \leq 4 \lim_t \left(\sup_{E \in \mathcal{F}_t} \sum_s \left| \sum_{n \in E} \theta_{rs} \right|^q \right)^{\frac{1}{q}},$$

and \mathcal{L}_D is compact if and only if

$$\lim_t \left(\sup_{E \in \mathcal{F}_t} \sum_s \left| \sum_{n \in E} \theta_{rs} \right|^q \right)^{\frac{1}{q}} = 0.$$

□

Theorem 5.10. *The following assertions hold:*

(i) *If $D \in (\ell_\infty(\mathcal{J}^{(k,1)}) : \ell_\infty)$, then it follows that*

$$0 \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \sum_s |\theta_{rs}|.$$

Moreover, \mathcal{L}_D is compact if

$$\lim_r \sum_s |\theta_{rs}| = 0.$$

(ii) *If $D \in (\ell_\infty(\mathcal{J}^{(k,1)}) : c_0)$, then we obtain*

$$\|\mathcal{L}_D\|_\chi = \limsup_r \sum_s |\theta_{rs}|.$$

Furthermore, \mathcal{L}_D is compact if and only if

$$\lim_r \sum_s |\theta_{rs}| = 0.$$

(iii) *If $D \in (\ell_\infty(\mathcal{J}^{(k,1)}) : \ell_1)$, then it holds that*

$$\lim_t \|D\|_{(\ell_\infty(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)} \leq \|\mathcal{L}_D\|_\chi \leq 4 \cdot \lim_t \|D\|_{(\ell_\infty(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)}.$$

Additionally, \mathcal{L}_D is compact if and only if

$$\lim_t \|D\|_{(\ell_\infty(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)} = 0,$$

where the term $\|D\|_{(\ell_\infty(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)}$ is defined as

$$\|D\|_{(\ell_\infty(\mathcal{J}^{(k,1)}):\ell_1)}^{(t)} = \sup_{E \in \mathcal{F}_t} \sum_s \left| \sum_{n \in E} \theta_{rs} \right|,$$

for all $t \in \mathbb{N}$.

Proof. A similar reasoning as in Theorem 5.9 applies, so the proof is omitted.

□

Theorem 5.11. (i) If $D \in (\ell_1(\mathcal{J}^{(k,1)}) : \ell_\infty)$, then

$$0 \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \left(\sup_s |\theta_{rs}| \right)$$

and \mathcal{L}_D is compact if

$$\lim_r \left(\sup_s |\theta_{rs}| \right) = 0.$$

(ii) If $D \in (\ell_1(\mathcal{J}^{(k,1)}) : c_0)$, then

$$\|\mathcal{L}_D\|_\chi = \limsup_r \left(\sup_s |\theta_{rs}| \right)$$

and \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sup_s |\theta_{rs}| \right) = 0.$$

Proof. It can be seen as a parallel to Theorem 5.9. □

Lemma 5.12. [31] If Λ has the AK property or $\Lambda = \ell_\infty$, and $D \in (\Lambda : c)$, then

$$\frac{1}{2} \limsup_r \|D_r - d\|_\Lambda^\diamond \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \|D_r - d\|_\Lambda^\diamond$$

and \mathcal{L}_D is compact if and only if

$$\lim_r \|D_r - d\|_\Lambda^\diamond = 0,$$

where $d = (d_s)$ and $d_s = \lim_r d_{rs}$.

Theorem 5.13. If $D \in (\ell_p(\mathcal{J}^{(k,1)}) : c)$ for $1 < p < \infty$, then

$$\frac{1}{2} \limsup_r \left(\sum_s |\theta_{rs} - \theta_s|^q \right)^{\frac{1}{q}} \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \left(\sum_s |\theta_{rs} - \theta_s|^q \right)^{\frac{1}{q}}$$

and \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sum_s |\theta_{rs} - \theta_s|^q \right)^{\frac{1}{q}} = 0.$$

Proof. Let $D \in (\ell_p(\mathcal{J}^{(k,1)}) : c)$. By Lemma 5.8, we have $\theta \in (\ell_p : c)$. Using Lemma 5.12, we obtain the inequality

$$\frac{1}{2} \limsup_r \|\Theta_r - \theta\|_{\ell_p}^\diamond \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \|\Theta_r - \theta\|_{\ell_p}^\diamond.$$

Applying Lemma 5.6-(c), we conclude that

$$\frac{1}{2} \limsup_r \left(\sum_s |\theta_{rs} - \theta_s|^q \right)^{\frac{1}{q}} \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \left(\sum_s |\theta_{rs} - \theta_s|^q \right)^{\frac{1}{q}}.$$

Finally, by Lemma 5.12, we deduce that \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sum_s |\theta_{rs} - \theta_s|^q \right)^{\frac{1}{q}} = 0.$$

□

Theorem 5.14. *If $D \in (\ell_\infty(\mathcal{J}^{(k,1)}) : c)$, in that case*

$$\frac{1}{2} \limsup_r \left(\sum_s |\theta_{rs} - \theta_s| \right) \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \left(\sum_s |\theta_{rs} - \theta_s| \right)$$

and \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sum_s |\theta_{rs} - \theta_s| \right) = 0.$$

Proof. It can be seen as a parallel to Theorem 5.13. □

Theorem 5.15. *If $D \in (\ell_1(\mathcal{J}^{(k,1)}) : c)$, then*

$$\frac{1}{2} \limsup_r \left(\sup_s |\theta_{rs} - \theta_s| \right) \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \left(\sup_s |\theta_{rs} - \theta_s| \right)$$

and \mathcal{L}_D is compact if and only if

$$\lim_r \left(\sup_s |\theta_{rs} - \theta_s| \right) = 0.$$

Proof. It can be seen as a parallel to Theorem 5.13. □

6 Conclusion

In this paper, we explored the Jordan-type matrix operator $\mathcal{J}^{(k,1)}$ and its effects on the sequence spaces it induces. We examined the properties of these sequence spaces, their duals, and the matrix mappings and compact operators that act on them. The results highlight the intricate relationships between the sequence spaces and the operators defined on them, demonstrating the significant role of the Jordan-type matrix operator in shaping the structure of these spaces.

The findings of the study lay the groundwork for further research in the study of matrix operators and their applications in functional analysis and sequence space theory.

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