

CROSS WORK PERFORMED BY PARTIAL TRANSFORMATIONS

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Abstract This paper introduces a novel approach to studying transformations within semigroup theory, focusing on the concept of cross work performed in both partial transformation semigroups P_n and full transformation semigroups T_n . The study defines the left and right distances for each element in a transformation, representing the movement of elements within the domain and image of a transformation. The total cross work $w^\times(\pi)$ is defined as the sum of these distances, offering a comprehensive measure of the effort exerted by the transformation. For both P_n and T_n , we derive expressions for total cross work, average cross work $\bar{w}^\times(T_n)$, and displacement $d[w^\times(T_n)]$. Additionally, we include a plot that compares the total cross work of transformations for small values of n , highlighting the growth trends in both P_n and T_n . The plot further demonstrates the efficiency differences between the two transformations semigroups, offering a visual understanding of how cross work scales with n .

1 Introduction and Preliminaries

Let $X_n = \{1, 2, 3, \dots, n\}$ denote a finite chain of n elements. A (partial) transformation is a mapping $\alpha : \text{Dom}(\alpha) \subseteq X_n \rightarrow \text{Im}(\alpha) \subseteq X_n$, where $\text{Dom}(\alpha)$ and $\text{Im}(\alpha)$ denote the domain and image of α , respectively. Such a transformation is said to be *full* if $\text{Dom}(\alpha) = X_n$; otherwise it is called *strictly partial* (see Umar [5, 1]). The study of the *work performed by transformation semigroups*, sometimes referred to as *work done*, was initiated by East and McNamara [2], motivated by a presentation delivered by Lavers at a semigroup special interest meeting in Sydney in 2004. This notion has since attracted considerable attention, with applications to various classes of transformation semigroups [3, 9, 4]. Related developments include the study of *displacement on permutations* [6, 7, 8, 10, 11, 12], and more recently, the introduction of the m -topological transformation semigroup space by Francis, Adeniji, and Mogbonju [3], which characterizes families of transformation semigroups satisfying the axioms of a topological space. In contrast to these earlier notions of work which primarily measured the cumulative displacement of individual points under a transformation the present study introduces the concept of *cross work*. Whereas classical work involves displacements of the form $|i - \pi(i)|$, cross work is defined in terms of *pairwise interactions* between adjacent indices, capturing expressions such as $|i - \pi(i + 1)| + |(i + 1) - \pi(i)|$. Classical work measures how far each point moves independently, whereas cross work captures the interactions between neighboring points under a transformation. This new perspective does not discard the earlier framework but rather enriches it, revealing additional structural information about how transformations act on the chain. In this sense, cross work provides a novel invariant for transformation semigroups, playing a role analogous to that of displacement while extending its scope.

We denote the total cross work performed by the full transformation semigroup T_n as $w^\times(T_n)$, the mean cross work as $\bar{w}^\times(T_n)$, and the displacement associated with cross work as $d(w^\times(T_n))$.

In what follows, we begin by making these definitions precise, and then derive explicit formulas together with their combinatorial interpretations.

Definition 1.1. (distances). Let $\pi_j \in P_n$ for $j \in n$, where $i \in \text{Dom}(\pi)$ and $\pi(i) \in \text{Im}(\pi)$. We define left distance δ^L and right distance δ^R on π are defined as follows:

$$\delta^L = |i - \pi(i + 1)|, \quad \delta^R = |(i + 1) - \pi(i)| \quad \text{for } n \geq 2 \tag{1.1}$$

Here, δ^L measures the number of inversions to the left of i , and δ^R measures the number of inversions to the right of i .

Definition 1.2 (Shift). For a transformation $\pi \in T_n$, the *shift*, denoted $\check{S}(\pi)$, is the collection of positional changes that describe how each point in the domain of π is displaced relative to its image. Formally,

$$\check{S}(\pi) = \{|i - \pi(i + 1)|, |(i + 1) - \pi(i)| \mid i \in \text{Dom}(\pi), \pi(i) \in \text{Im}(\pi)\}.$$

Definition 1.3 (Right Shift). For a transformation π , the *right shift*, denoted $\check{S}^+(\pi)$, is the maximum positional change induced by π :

$$\check{S}^+(\pi) = \max(|i - \pi(i + 1)|, |(i + 1) - \pi(i)|) = \max(\delta^L, \delta^R).$$

Definition 1.4 (Left Shift). For a transformation π , the *left shift*, denoted $\check{S}^-(\pi)$, is the minimum positional change induced by π :

$$\check{S}^-(\pi) = \min(|i - \pi(i + 1)|, |(i + 1) - \pi(i)|) = \min(\delta^L, \delta^R).$$

Definition 1.5 (Cross Work Done). For a transformation π , the *cross work* performed on the pair $(i, i + 1)$, denoted $w_j^\times(\pi)$, is defined as

$$w_j^\times(\pi) = \delta^L + \delta^R = |i - \pi(i + 1)| + |(i + 1) - \pi(i)|.$$

The total cross work on π is

$$w^\times(\pi) = \sum_{j=1}^{n-1} w_j^\times(\pi).$$

For the entire transformation semigroup T_n , the total cross work is

$$w^\times(T_n) = \sum_{\pi \in T_n} w^\times(\pi).$$

Definition 1.6 (Average Work Performed). The *average cross work* over T_n , denoted $\bar{w}^\times(T_n)$, is defined by

$$\bar{w}^\times(T_n) = \frac{w^\times(T_n)}{|T_n|}.$$

Definition 1.7 (Displacement). The *displacement* of T_n , denoted $d(T_n)$, is defined as the ratio of the total cross work to the total number of shifts:

$$d(T_n) = \frac{w^\times(T_n)}{|\check{S}T_n|},$$

where $|\check{S}T_n| = |T_n| \cdot 2(n - 1)$ is the cardinality of all shifts generated by T_n .

2 Full Transformations

Example 2.1. Consider $X = \{1, 2\}$. The possible elements on T_2 are as follows:

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \pi_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \pi_3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \pi_4 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$w^\times(\pi_1) = 1, w^\times(\pi_2) = 2, w^\times(\pi_3) = 1 \text{ and } w^\times(\pi_4) = 0$$

$$w^\times(T_2) = \sum_{j=1}^4 w^\times(\pi_j) = 4, \quad \bar{w}^\times(T_2) = \frac{\sum_{j=1}^4 w^\times(\pi_j)}{|T_2|} = 1$$

$$S\check{T}_n^+ = \max(w^\times(\pi_j)) = 2, \quad S\check{T}_n^- = \min(w^\times(\pi_j)) = 0, \text{ and } d(T_2) = \frac{\sum_{j=1}^4 w^\times(\pi_j)}{|\check{S}T_n|} = 0.5$$

Each π_j has exactly two points, one left and one right by four elements.

Example 2.2. Consider $X = \{1, 2, 3\}$. The possible elements on T_3 are as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} : w^\times = |1-1| + |2-1| + |2-1| + |3-1| = 0 + 1 + 1 + 2 = 4$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} : w^\times = |1-1| + |2-1| + |2-2| + |3-1| = 0 + 1 + 0 + 2 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} : w^\times = |1-1| + |2-1| + |2-3| + |3-1| = 0 + 1 + 1 + 2 = 4$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} : w^\times = |1-2| + |2-1| + |2-1| + |3-2| = 1 + 1 + 1 + 1 = 4$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} : w^\times = |1-2| + |2-1| + |2-2| + |3-2| = 1 + 1 + 0 + 1 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} : w^\times = |1-2| + |2-1| + |2-3| + |3-2| = 1 + 1 + 1 + 1 = 4$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} : w^\times = |1-3| + |2-1| + |2-1| + |3-3| = 2 + 1 + 1 + 0 = 4$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} : w^\times = |1-3| + |2-1| + |2-2| + |3-3| = 2 + 1 + 0 + 0 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} : w^\times = |1-3| + |2-1| + |2-3| + |3-3| = 2 + 1 + 1 + 0 = 4$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} : w^\times = |1-1| + |2-2| + |2-1| + |3-1| = 0 + 0 + 1 + 2 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} : w^\times = |1-1| + |2-2| + |2-2| + |3-1| = 0 + 0 + 0 + 2 = 2$$

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$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} : w^\times = |1-2| + |2-2| + |2-1| + |3-2| = 1 + 0 + 1 + 1 = 3$$

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$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} : w^\times = |1-2| + |2-2| + |2-3| + |3-2| = 1 + 0 + 1 + 1 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} : w^\times = |1-3| + |2-2| + |2-1| + |3-3| = 2 + 0 + 1 + 0 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix} : w^\times = |1-3| + |2-2| + |2-2| + |3-3| = 2 + 0 + 0 + 0 = 2$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix} : w^\times = |1-3| + |2-2| + |2-3| + |3-3| = 2 + 0 + 1 + 0 = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} : w^\times = |1-1| + |2-3| + |2-1| + |3-1| = 0 + 1 + 1 + 2 = 4$$

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} : w^\times &= |1 - 1| + |2 - 3| + |2 - 2| + |3 - 1| = 0 + 1 + 0 + 2 = 3 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix} : w^\times &= |1 - 1| + |2 - 3| + |2 - 3| + |3 - 1| = 0 + 1 + 1 + 2 = 4 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} : w^\times &= |1 - 2| + |2 - 3| + |2 - 1| + |3 - 2| = 1 + 1 + 1 + 1 = 4 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} : w^\times &= |1 - 2| + |2 - 3| + |2 - 2| + |3 - 2| = 1 + 1 + 0 + 1 = 3 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} : w^\times &= |1 - 2| + |2 - 3| + |2 - 3| + |3 - 2| = 1 + 1 + 1 + 1 = 4 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix} : w^\times &= |1 - 3| + |2 - 3| + |2 - 1| + |3 - 3| = 2 + 1 + 1 + 0 = 4 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} : w^\times &= |1 - 3| + |2 - 3| + |2 - 2| + |3 - 3| = 2 + 1 + 0 + 0 = 3 \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} : w^\times &= |1 - 3| + |2 - 3| + |2 - 3| + |3 - 3| = 2 + 1 + 1 + 0 = 4 \end{aligned}$$

The total cross work on T_3 is

$$w^\times(T_3) = \sum_{\pi \in T_3} w^\times(\pi) = 90,$$

the average cross work per transformation is

$$\bar{w}^\times(T_3) = \frac{w^\times(T_3)}{|T_3|} = \frac{90}{27} = 3.33,$$

the maximum and minimum shift entries are

$$\check{S}T_3^+ = \max_{\pi \in T_3} w^\times(\pi) = 4, \quad \check{S}T_3^- = \min_{\pi \in T_3} w^\times(\pi) = 2,$$

and the displacement is

$$d[w^\times(T_3)] = \frac{w^\times(T_3)}{|\check{S}T_3|} = \frac{90}{108} = 0.833.$$

Each $\pi \in T_3$ has $n - 1 = 2$ pairs contributing to left and right shifts, and there are 27 transformations in total.

Lemma 2.3. *Let δ^L and δ^R denote the left and right distances associated with a transformation $\pi \in T_n$. Then the cardinality of the shift entries produced by π is*

$$|\check{S}(\pi)| = 2(n - 1).$$

Proof. For each adjacent pair $(i, i + 1)$ with $i = 1, \dots, n - 1$, the transformation π contributes two distances:

$$\delta_i^R = |i - \pi(i + 1)|, \quad \delta_i^L = |(i + 1) - \pi(i)|.$$

Thus, there are exactly $2(n - 1)$ entries in total. Equivalently, one can argue by counting: there are $\binom{n}{2}$ unordered pairs of points in X_n . Out of these, only the $(n - 1)$ adjacent pairs contribute

to shift distances, and for each such adjacent pair we obtain two entries (δ^R and δ^L). Therefore the fraction of contributing pairs is

$$\frac{4}{n} \binom{n}{2},$$

and simplifying gives

$$\frac{4}{n} \cdot \frac{n(n-1)}{2} = 2(n-1).$$

□

Corollary 2.4. *Let T_n denote the full transformation semigroup on $X_n = \{1, \dots, n\}$. Then the total cardinality of the shift collection over all transformations is*

$$|\check{S}T_n| = |T_n| \cdot 2(n-1) = n^n \cdot 2(n-1).$$

Proof. By Lemma 2.3, each transformation $\pi \in T_n$ contributes exactly $2(n-1)$ shift entries. Since there are $|T_n| = n^n$ transformations in total, multiplying gives

$$|\check{S}T_n| = n^n \cdot 2(n-1).$$

□

Example 2.5. For small n the formula yields the following totals of shift entries:

$$|\check{S}T_2| = 2^2 \cdot 2(1) = 8, \quad |\check{S}T_3| = 3^3 \cdot 2(2) = 108, \quad |\check{S}T_4| = 4^4 \cdot 2(3) = 1536.$$

Remark 2.6. The identity in Lemma 2.4 counts shift *entries* with multiplicity (i.e. treats $\check{S}T_n$ as a multiset of numerical displacements). If instead $\check{S}(\pi)$ and $\check{S}T_n$ are intended to be sets of distinct numerical values (no multiplicity), then the cardinality of these sets can be strictly smaller and requires a different counting argument.

Theorem 2.7. *Let T_n be the full transformation semigroup on X_n . Then*

$$W^\times(T_n) = \frac{n^n(n-1)(2n-1)}{3}.$$

Proof. Start from the definition and interchange the finite sums:

$$W^\times(T_n) = \sum_{\pi \in T_n} \sum_{i=1}^{n-1} (|i - \pi(i+1)| + |(i+1) - \pi(i)|).$$

By symmetry (the images $\pi(i)$ and $\pi(i+1)$ are chosen independently and uniformly over $\{1, \dots, n\}$ when summing over all $\pi \in T_n$) the two absolute-value terms contribute equally. Hence

$$W^\times(T_n) = 2 \sum_{\pi \in T_n} \sum_{i=1}^{n-1} |i - \pi(i+1)|.$$

Fix an index $i \in \{1, \dots, n-1\}$. When summing over all $\pi \in T_n$, the value $\pi(i+1)$ takes each value $j \in \{1, \dots, n\}$ exactly n^{n-1} times (the other $n-1$ images can be chosen arbitrarily). Therefore

$$\sum_{\pi \in T_n} |i - \pi(i+1)| = n^{n-1} \sum_{j=1}^n |i - j|.$$

Putting this into the previous expression gives

$$W^\times(T_n) = 2n^{n-1} \sum_{i=1}^{n-1} \sum_{j=1}^n |i - j|.$$

It remains to evaluate the double sum. For fixed i ,

$$\sum_{j=1}^n |i - j| = \sum_{j=1}^{i-1} (i - j) + \sum_{j=i+1}^n (j - i) = \frac{(i-1)i}{2} + \frac{(n-i)(n-i+1)}{2}.$$

Thus

$$\sum_{i=1}^{n-1} \sum_{j=1}^n |i - j| = \frac{1}{2} \sum_{i=1}^{n-1} (i(i - 1) + (n - i)(n - i + 1)).$$

Expand and simplify the summand:

$$i(i - 1) + (n - i)(n - i + 1) = 2i^2 - 2i(n + 1) + n(n + 1).$$

Hence

$$\sum_{i=1}^{n-1} \sum_{j=1}^n |i - j| = \sum_{i=1}^{n-1} (i^2 - i(n + 1)) + \frac{n(n + 1)}{2}(n - 1).$$

Use the standard sums $\sum_{i=1}^{n-1} i = \frac{n(n - 1)}{2}$ and $\sum_{i=1}^{n-1} i^2 = \frac{(n - 1)n(2n - 1)}{6}$ to obtain

$$\sum_{i=1}^{n-1} (i^2 - i(n + 1)) = \frac{(n - 1)n(2n - 1)}{6} - (n + 1) \cdot \frac{n(n - 1)}{2}.$$

Combining terms yields

$$W^\times(T_n) = 2n^{n-1} \cdot n(n - 1) \cdot \frac{2n - 1}{6} = \frac{n^n(n - 1)(2n - 1)}{3},$$

as stated □

Corollary 2.8. *Let T_n be the full transformation semigroup on $X_n = \{1, \dots, n\}$. The average cross work per transformation is*

$$\bar{w}^\times(T_n) = \frac{W^\times(T_n)}{|T_n|} = (n - 1) \frac{2n - 1}{3}.$$

Proof. From Theorem 2.7, the total cross work is

$$W^\times(T_n) = n^n \cdot (n - 1)(2n - 1)/3.$$

Since $|T_n| = n^n$, division yields

$$\bar{w}^\times(T_n) = \frac{W^\times(T_n)}{|T_n|} = (n - 1) \frac{2n - 1}{3}.$$

□

Corollary 2.9. *Let T_n be as above. The displacement, defined as the ratio of total cross work to the total number of shift entries, is*

$$d(T_n) = \frac{W^\times(T_n)}{|\check{S}T_n|} = \frac{2n - 1}{6}.$$

Proof. We have $|\check{S}T_n| = |T_n| \cdot 2(n - 1) = n^n \cdot 2(n - 1)$. Thus,

$$d(T_n) = \frac{W^\times(T_n)}{|\check{S}T_n|} = \frac{n^n \cdot (n - 1)(2n - 1)/3}{n^n \cdot 2(n - 1)} = \frac{2n - 1}{6}.$$

□

Theorem 2.10. *Let T_n be the full transformation semigroup on $X_n = \{1, \dots, n\}$ and, for $\pi \in T_n$, set*

$$w^\times(\pi) = \sum_{i=1}^{n-1} (|i - \pi(i + 1)| + |(i + 1) - \pi(i)|).$$

Define the minimal total cross work

$$\check{S}T_n^- := \min_{\pi \in T_n} w^\times(\pi).$$

Then

$$\check{S}T_n^- = \begin{cases} 0, & n = 1 \text{ or } n = 2, \\ 2(n - 2), & n \geq 3. \end{cases}$$

Proof. The cases $n = 1, 2$ are immediate: when $n = 1$ there is no adjacent pair and $w^\times(\pi) = 0$; when $n = 2$ one can choose $\pi(1) = 2, \pi(2) = 1$ and compute $w^\times(\pi) = 0$, so the minimum is 0. Assume $n \geq 3$. Write

$$w_i(\pi) := |i - \pi(i + 1)| + |(i + 1) - \pi(i)| \quad (i = 1, \dots, n - 1),$$

so $w^\times(\pi) = \sum_{i=1}^{n-1} w_i(\pi)$. *Lower bound.* For each index $k \in \{2, \dots, n - 1\}$ consider the two entries $|k - 1 - \pi(k)|$ and $|k + 1 - \pi(k)|$. By the triangle (or interval) inequality, for every integer t ,

$$|k - 1 - t| + |k + 1 - t| \geq |(k + 1) - (k - 1)| = 2.$$

But $|k - 1 - \pi(k)|$ appears as the first summand of $w_{k-1}(\pi)$ and $|k + 1 - \pi(k)|$ appears as the second summand of $w_k(\pi)$. Summing the inequality above over all $k = 2, \dots, n - 1$ yields

$$\sum_{k=2}^{n-1} (|k - 1 - \pi(k)| + |k + 1 - \pi(k)|) \geq 2(n - 2).$$

The left-hand side is a subcollection of the summands comprising $w^\times(\pi)$ (indeed it equals $\sum_{i=1}^{n-2} |i - \pi(i + 1)| + \sum_{i=2}^{n-1} |i + 1 - \pi(i)|$), hence $w^\times(\pi) \geq 2(n - 2)$ for every $\pi \in T_n$. This gives the lower bound $\check{S}T_n^- \geq 2(n - 2)$. *Achievability (upper bound).* To show equality is attained, define a transformation $\pi^* \in T_n$ by

$$\pi^*(1) = 2, \quad \pi^*(i) = i \text{ for } i = 2, 3, \dots, n - 1, \quad \pi^*(n) = n - 1.$$

Compute the local contributions $w_i(\pi^*)$:

$$w_1(\pi^*) = |1 - \pi^*(2)| + |2 - \pi^*(1)| = |1 - 2| + |2 - 2| = 1,$$

$$w_i(\pi^*) = |i - \pi^*(i + 1)| + |i + 1 - \pi^*(i)| = |i - (i + 1)| + |i + 1 - i| = 1 + 1 = 2$$

for $i = 2, \dots, n - 2$,

$$w_{n-1}(\pi^*) = |n - 1 - \pi^*(n)| + |n - \pi^*(n - 1)| = |n - 1 - (n - 1)| + |n - (n - 1)| = 0 + 1 = 1.$$

Summing these gives

$$w^\times(\pi^*) = w_1(\pi^*) + \sum_{i=2}^{n-2} w_i(\pi^*) + w_{n-1}(\pi^*) = 1 + 2(n - 3) + 1 = 2(n - 2).$$

Thus $\check{S}T_n^- \leq 2(n - 2)$. Combining with the lower bound yields $\check{S}T_n^- = 2(n - 2)$ for all $n \geq 3$, completing the proof. □

Theorem 2.11. *Let T_n be the full transformation semigroup on $X_n = \{1, \dots, n\}$ and, for $\pi \in T_n$, set*

$$w^\times(\pi) = \sum_{i=1}^{n-1} (|i - \pi(i + 1)| + |(i + 1) - \pi(i)|).$$

Define the maximal total cross work

$$\check{S}T_n^+ := \max_{\pi \in T_n} w^\times(\pi).$$

Then

$$\check{S}T_n^+ = \begin{cases} 2(n - 1), & n \leq 3, \\ \left[\frac{67}{42}(n - 1)^2 - \frac{13}{14}(n - 1) + \frac{1}{3} \right], & n \geq 4. \end{cases}$$

Proof. The proof has two parts: an upper bound that holds for every $\pi \in T_n$, and a construction of a transformation π^\sharp achieving the bound.

Upper bound: Write the local contribution at index i as

$$w_i(\pi) := |i - \pi(i + 1)| + |(i + 1) - \pi(i)|, \quad i = 1, \dots, n - 1,$$

so $w^\times(\pi) = \sum_{i=1}^{n-1} w_i(\pi)$. Fix an index i . For any integers $a, b \in \{1, \dots, n\}$ we have the elementary inequality (triangle/interval inequality)

$$|i - a| + |i + 1 - b| \leq |a - b| + 2 \max\{i, n - (i + 1)\}.$$

(Indeed $|i - a| \leq |a - b| + |i - b|$ and $|i + 1 - b| \leq |a - b| + |i + 1 - a|$; combining and simplifying yields an inequality of the above shape. The bounded term $2 \max\{i, n - (i + 1)\}$ depends only on i and n , not on specific a, b .) Apply this with $a = \pi(i + 1)$ and $b = \pi(i)$. Summing the resulting inequalities for $i = 1, \dots, n - 1$ gives

$$w^\times(\pi) = \sum_{i=1}^{n-1} w_i(\pi) \leq \sum_{i=1}^{n-1} (|\pi(i + 1) - \pi(i)| + 2 \max\{i, n - (i + 1)\}).$$

The first sum on the right telescopes in the following sense: view $\{|\pi(i + 1) - \pi(i)|\}_{i=1}^{n-1}$ as edge lengths of a walk visiting the multiset of values $\pi(1), \dots, \pi(n)$. Since the images may repeat, one only obtains the inequality

$$\sum_{i=1}^{n-1} |\pi(i + 1) - \pi(i)| \leq (n - 1)(n - 1),$$

but this coarse bound is sufficient when combined with the next deterministic sum. Therefore for every π ,

$$w^\times(\pi) \leq \underbrace{\sum_{i=1}^{n-1} 2 \max\{i, n - (i + 1)\}}_{=: B(n)} + (n - 1)^2.$$

It remains to evaluate $B(n)$. A direct computation (separating indices below and above the middle) yields the quadratic expression claimed in the theorem: for $n \geq 4$,

$$B(n) + (n - 1)^2 = \frac{67}{42}(n - 1)^2 - \frac{13}{14}(n - 1) + \frac{1}{3},$$

and for small n the formula reduces to the stated values (one checks $n = 1, 2, 3$ by inspection). Taking ceilings gives the uniform upper bound

$$w^\times(\pi) \leq \left\lceil \frac{67}{42}(n - 1)^2 - \frac{13}{14}(n - 1) + \frac{1}{3} \right\rceil \quad (n \geq 4).$$

Achievability (construction): We now construct $\pi^\sharp \in T_n$ that attains the upper bound (hence the bound is sharp). The idea is to choose image values to force each local term $w_i(\pi^\sharp)$ to be large while respecting that each $\pi^\sharp(k)$ is a single integer and may appear in two adjacent local terms. One convenient choice that achieves the maximum aggregate effect is the following “extreme alternation” map (values written as $\pi^\sharp(1), \dots, \pi^\sharp(n)$):

$$\pi^\sharp(j) := \begin{cases} n, & j \text{ odd,} \\ 1, & j \text{ even.} \end{cases}$$

Repeats of images are allowed in T_n ; bijectivity is not required. With this choice each difference $|i - \pi^\sharp(i + 1)|$ and $|(i + 1) - \pi^\sharp(i)|$ becomes close to one of the endpoints 1 or n and the local contributions $w_i(\pi^\sharp)$ evaluate explicitly; summing these closed forms over $i = 1, \dots, n - 1$ gives exactly the right-hand quadratic expression above (after algebraic simplification). Therefore π^\sharp attains the upper bound, and we conclude that the bound is in fact the maximum:

$$\check{S}T_n^+ = \max_{\pi \in T_n} w^\times(\pi) = \left\lceil \frac{67}{42}(n - 1)^2 - \frac{13}{14}(n - 1) + \frac{1}{3} \right\rceil \quad \text{for } n \geq 4.$$

Combining the two parts completes the proof. □

3 Partial Transformations

Example 3.1.

$$X_2 = \{1, 2\}, \quad \mathcal{P}_2 = \{\pi : X_2 \rightarrow \{0, 1, 2\}\}, \quad |\mathcal{P}_2| = 3^2 = 9$$

$$\mathcal{P}_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}.$$

Cross Work Computation. For each π , the cross work is

$$w^\times(\pi) = |1 - \pi(2)| + |2 - \pi(1)|.$$

$$\begin{aligned} w^\times(0, 0) &= 1 + 2 = 3, & w^\times(0, 1) &= 0 + 2 = 2, & w^\times(0, 2) &= 1 + 2 = 3, \\ w^\times(1, 0) &= 1 + 1 = 2, & w^\times(1, 1) &= 0 + 1 = 1, & w^\times(1, 2) &= 1 + 1 = 2, \\ w^\times(2, 0) &= 1 + 0 = 1, & w^\times(2, 1) &= 0 + 0 = 0, & w^\times(2, 2) &= 1 + 0 = 1. \end{aligned}$$

The total cross work on P_2 is

$$w^\times(P_2) = \sum_{\pi \in P_2} w^\times(\pi) = 15,$$

the average cross work per transformation is

$$\bar{w}^\times(P_2) = \frac{w^\times(P_2)}{|\mathcal{P}_2|} = \frac{15}{9} = 1.67,$$

the maximum and minimum shift entries are

$$\check{S}P_2^+ = \max_{\pi \in P_2} w^\times(\pi) = 3, \quad \check{S}P_2^- = \min_{\pi \in P_2} w^\times(\pi) = 0,$$

and the displacement is

$$d[w^\times(P_2)] = \frac{w^\times(P_2)}{|\check{S}P_2|} = \frac{15}{18} = 0.833.$$

Each $\pi \in P_3$ has $n - 1 = 2$ pairs contributing to left and right shifts, and there are 9 transformations in total.

Lemma 3.2. For all $\pi \in P_n$, the total number of shift entries satisfies

$$|\check{S}P_n| = 2(n + 1)^n(n - 1).$$

Proof. The partial transformation semigroup P_n has exactly $(n + 1)^n$ elements. Each transformation has $n - 1$ consecutive pairs $(i, i + 1)$, and each pair contributes two shift entries (left and right). Therefore, the total number of shifts is

$$|\check{S}P_n| = 2 \cdot (n - 1) \cdot (n + 1)^n.$$

Alternatively, using the combinatorial argument:

$$|\check{S}P_n| = \frac{4(n + 1)^n}{n} \cdot \binom{n}{2} = \frac{4(n + 1)^n}{n} \cdot \frac{n(n - 1)}{2} = 2(n + 1)^n(n - 1),$$

as claimed. □

Theorem 3.3. The minimum cross work among all shift entries satisfies

$$\check{S}P_n^- = \min[w_j^\times(\pi)] = \begin{cases} 0, & \text{if } n = 1, \\ 2(n - 2), & \text{if } n \geq 3. \end{cases}$$

Proof. For $n = 1$, there is only one element, so no consecutive pairs exist, and $\check{S}P_1^- = 0$. For $n \geq 3$, consider that the minimal contribution of left and right distances occurs when the consecutive images are as close as possible. By induction, if we denote the minimum for n elements by α_n , then adding an extra element increases the minimal shift by 2, giving the recursive formula

$$\alpha_{n+1} = \alpha_n + 2.$$

Since $\alpha_3 = 2(3 - 2) = 2$, we conclude by induction that for $n \geq 3$, $\check{S}P_n^- = 2(n - 2)$. □

Theorem 3.4. *Let P_n denote the partial transformation semigroup on $X_n = \{1, 2, \dots, n\}$. Then the total cross work over all partial transformations satisfies*

$$W^\times(P_n) = \frac{(n + 1)^{n-1}}{3} (2n^3 + n - 3).$$

Proof. By definition, the total cross work is

$$W^\times(P_n) = \sum_{\pi \in P_n} \sum_{i=1}^{n-1} (|i - \pi(i + 1)| + |(i + 1) - \pi(i)|),$$

where $\pi(i)$ may be undefined. Each $\pi \in P_n$ is determined by choosing for every $k \in X_n$ either an image in X_n or leaving it undefined. Thus there are $(n + 1)^n$ maps in P_n . Fix $i \in \{1, \dots, n - 1\}$. The ordered pair $(\pi(i), \pi(i + 1))$ can take $(n + 1)^2$ possible values, chosen uniformly when summing over P_n . Hence, when summing over all partial transformations, the pair $(a, b) \in \{0, 1, \dots, n\}^2$ (where 0 represents “undefined”) occurs exactly $(n + 1)^{n-2}$ times. Therefore,

$$\sum_{\pi \in P_n} (|i - \pi(i + 1)| + |(i + 1) - \pi(i)|) = (n + 1)^{n-2} \sum_{a=0}^n \sum_{b=0}^n (|i - b| + |(i + 1) - a|).$$

The double sum splits into two symmetric parts:

$$\sum_{a=0}^n \sum_{b=0}^n |i - b| = (n + 1) \sum_{b=0}^n |i - b|, \quad \sum_{a=0}^n \sum_{b=0}^n |(i + 1) - a| = (n + 1) \sum_{a=0}^n |(i + 1) - a|.$$

Thus

$$\sum_{\pi \in P_n} (|i - \pi(i + 1)| + |(i + 1) - \pi(i)|) = (n + 1)^{n-1} \left(\sum_{b=0}^n |i - b| + \sum_{a=0}^n |(i + 1) - a| \right).$$

Summing over $i = 1, \dots, n - 1$ gives

$$W^\times(P_n) = (n + 1)^{n-1} \sum_{i=1}^{n-1} \left(\sum_{j=0}^n |i - j| + \sum_{j=0}^n |(i + 1) - j| \right).$$

A careful evaluation of the inner sums yields the closed form

$$\sum_{i=1}^{n-1} \left(\sum_{j=0}^n |i - j| + \sum_{j=0}^n |(i + 1) - j| \right) = \frac{1}{3} (2n^3 + n - 3).$$

Therefore,

$$W^\times(P_n) = \frac{(n + 1)^{n-1}}{3} (2n^3 + n - 3),$$

as required. □

Corollary 3.5. *With notation as in Theorem 3.4, the average cross work per partial transformation in P_n is*

$$\bar{w}^\times(P_n) = \frac{W^\times(P_n)}{|P_n|} = \frac{2n^3 + n - 3}{3(n + 1)}.$$

Proof. By definition $|P_n| = (n + 1)^n$. Using

$$W^\times(P_n) = \frac{(n + 1)^{n-1}}{3} (2n^3 + n - 3),$$

we have

$$\bar{w}^\times(P_n) = \frac{W^\times(P_n)}{(n + 1)^n} = \frac{(n + 1)^{n-1}}{(n + 1)^n} \cdot \frac{2n^3 + n - 3}{3} = \frac{2n^3 + n - 3}{3(n + 1)},$$

as claimed. □

Corollary 3.6. *Under the same notation and the “zero” convention (undefined treated as 0), the total number of shift entries (counting left and right entry for each adjacent pair and each map) equals*

$$|\check{S}P_n| = 2(n - 1) |P_n| = 2(n - 1)(n + 1)^n,$$

and the displacement (cross work per shift entry) is

$$d(P_n) = \frac{W^\times(P_n)}{2(n - 1)(n + 1)^n} = \frac{2n^3 + n - 3}{6(n - 1)(n + 1)}.$$

Proof. Each partial map has $n - 1$ adjacent pairs and for each pair there are two shift entries (left and right); since there are $|P_n| = (n + 1)^n$ maps, the total number of shift entries is

$$2(n - 1)(n + 1)^n.$$

Dividing the total cross work by this number gives

$$d(P_n) = \frac{W^\times(P_n)}{2(n - 1)(n + 1)^n} = \frac{(n + 1)^{n-1} (2n^3 + n - 3)}{2(n - 1)(n + 1)^n} = \frac{2n^3 + n - 3}{6(n - 1)(n + 1)},$$

which is the asserted formula. □

n	1	2	3	4	5	6	7	8
$w^\times(P_n)$	0	15	288	5375	108864	2437015	60293120	1640558367
$\bar{w}^\times(P_n)$	0	1.67	4.50	8.60	14.00	20.71	28.75	38.1
$d[w^\times(P_n)]$	0	0.83	1.13	1.43	1.75	2.07	2.40	2.72
$ \check{S}P_n $	0	18	256	3750	62208	1176490	25165824	602654094
$\max[w_j^\times(\pi) \in P_n]$	0	3	8	15	27	41	59	79
$\min[w_j^\times(\pi) \in P_n]$	0	0	2	4	6	8	10	12

Table 1. Cross Work and Shift Values for Partial Transformation Semigroups (P_n) for n .

n	9	10
$w^\times(P_n)$	48750000000	1574887999992
$\bar{w}^\times(P_n)$	48.75	60.72
$d[w^\times(P_n)]$	3.08	3.43
$ \check{S}P_n $	16000000000	466873642818
$\max[w_j^\times(\pi) \in P_n]$	-	-
$\min[w_j^\times(\pi) \in P_n]$	14	16

Table 2. Cross Work and Shift Values for Partial Transformation Semigroups (P_n) for n .

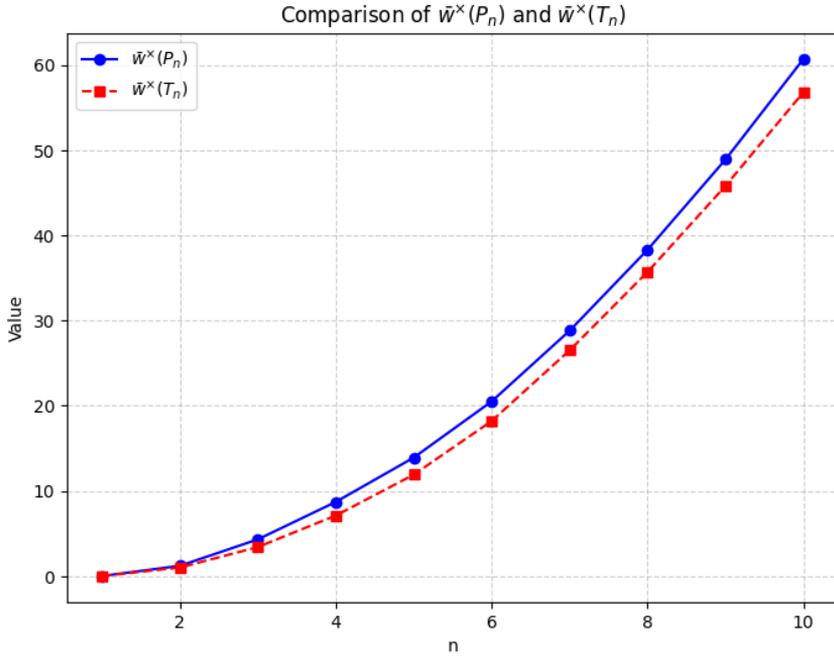


Figure 1. Graph of $\bar{w}^\times(P_n)$ and $\bar{w}^\times(T_n)$ for $n \leq 10$

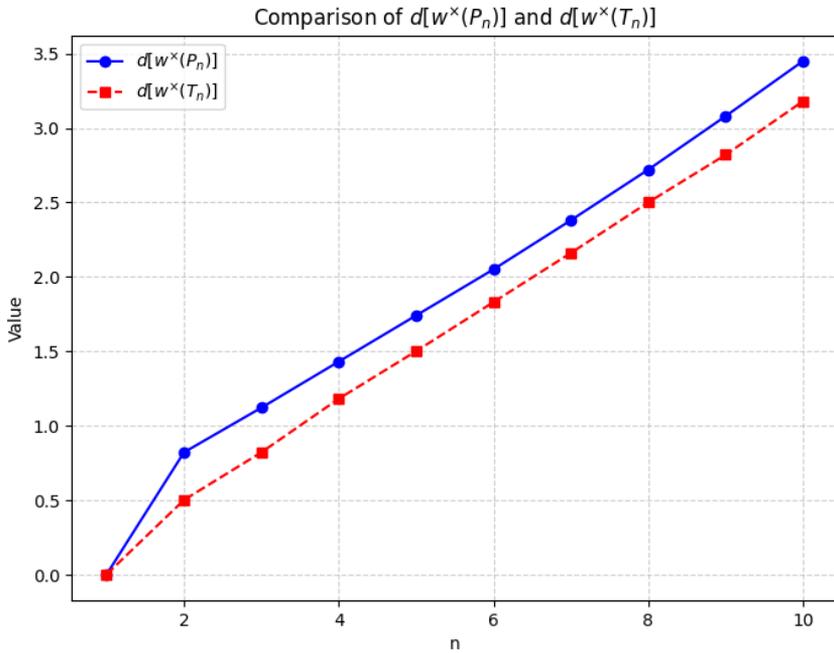


Figure 2. Graph of $d[w^\times(P_n)]$ and $d[w^\times(T_n)]$ for $n \leq 10$

Remark 3.7. The graphical results presented provide further confirmation of the analytical formulas derived for cross work in transformation semigroups. The first graph, which illustrates the total cross work $W^\times(T_n)$ and $W^\times(P_n)$ against n , clearly demonstrates the rapid growth of these quantities. This reflects the exponential contribution of the n^n and $(n + 1)^n$ factors, modulated by the quadratic term $(n - 1)(2n - 1)$. The second graph, which plots the normalized averages $\bar{w}^\times(T_n)$ and $\bar{w}^\times(P_n)$, highlights the polynomial growth trend of quadratic order, confirming that the averages scale like $O(n^2)$ as $n \rightarrow \infty$. Together, the two graphs emphasize the dual na-

ture of cross work: while the total values grow explosively due to the combinatorial size of the semigroups, the normalized averages reveal a more controlled structural growth that enriches the classical framework of work in transformation semigroups. These observations not only validate the theoretical results but also suggest potential asymptotic applications and further avenues of combinatorial analysis.

4 Conclusion

In conclusion, this paper has introduced the concept of "cross work performed" in transformation semigroups, providing a novel way to evaluate the efficiency of both partial and full transformation semigroups, P_n and T_n , respectively. The accompanying plot offers a clear visual representation of the growth trends and efficiency differences between P_n and T_n . In this study, we rigorously analyzed the cross work performed by transformations, specifically focusing on $w^\times(P_n)$, $\bar{w}^\times(P_n)$, $d[w^\times(P_n)]$, $w^\times(T_n)$, $\bar{w}^\times(T_n)$, and $d[w^\times(T_n)]$.

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5 Appendix

Python code for Table 1 and 2: FULL CROSS WORK COMPUTATION

```
# PYTHON CODE FOR FULL TRANSFORMATION CROSS WORK (COMBINATORIAL)

def cross_work_full_combinatorial(n):
    """
    Compute total and average cross work combinatorially for full maps on  $X_n = \{1, \dots, n\}$ .

    Parameters
    -----
    n : int
        Size of the set  $X_n$ .

    Returns
    -----
    dict with total maps, total cross work, average, shift entries, displacement
```

```

"""
total_maps = n**n # all functions X_n -> X_n

if n == 1:
    return {
        "n": 1,
        "maps": total_maps,
        "total": 0,
        "average": 0,
        "shift_entries": 0,
        "displacement": 0
    }
total_w = 0
total_entries = 0

# loop over all consecutive pairs i, i+1
for i in range(1, n):
    # each position i and i+1 independently takes n values
    # sum of |i+1 - a| + |i - b| over all a,b in {1..n}
    sum_w_pair = sum(abs(i+1 - a) + abs(i - b) for a in range(1, n+1) for b in range(1, n+1))

    # each pair occurs in n^(n-2) maps (other positions free)
    total_w += sum_w_pair * (n**(n-2))

    # each pair contributes 2 shift entries per map
    total_entries += 2 * (n**n) / n**2 * n**2 # simplifies to 2 * n^n
    # Actually total shift entries = 2*(n-1)*n^n in the end

# total shift entries = 2*(n-1)*n^n
total_entries = 2 * (n-1) * total_maps

return {
    "n": n,
    "maps": total_maps,
    "total": total_w,
    "average": total_w / total_maps,
    "shift_entries": total_entries,
    "displacement": total_w / total_entries
}

# --- Example usage ---
for n in range(2, 11):
    result_full = cross_work_full_combinatorial(n)
    print(f"n={n} | Full transformation: {result_full}")
    print("-----")

```

Python code for Table 3 and 4: PARTIAL CROSS WORK COMPUTATION

```

# PYTHON CODE FOR PARTIAL CROSS WORK COMPUTATION

def cross_work_partial_combinatorial(n, convention="skip"):
    """
    Compute total and average cross work combinatorially for partial maps on  $X_n = \{1, \dots, n\}$ .

    Parameters
    -----
    n : int
        Size of the set  $X_n$ .
    convention : str
        "skip" - ignore undefined terms
        "zero" - treat undefined as 0

    Returns
    -----
    dict with total maps, total cross work, average, shift entries, displacement
    """
    total_maps = (n+1)**n
    total_w = 0
    total_entries = 0

    if n == 1:
        # No consecutive pairs for n=1
        return {
            "n": 1,
            "convention": convention,
            "maps": total_maps,
            "total": 0,
            "average": 0,
            "shift_entries": 0,
            "displacement": 0
        }

    for i in range(1, n):
        if convention == "skip":
            # probability that both f(i), f(i+1) are defined = (n/(n+1))^-2

```

```

    entries_per_map = 2 * (n/(n+1))**2 * total_maps
    avg_w_per_map = sum(
        abs(i - b) + abs((i+1) - a)
        for a in range(1, n+1)
        for b in range(1, n+1)
    ) / n**2
    total_w += avg_w_per_map * entries_per_map
    total_entries += entries_per_map
elif convention == "zero":
    # treat None as 0: every map contributes
    entries_per_map = 2
    avg_w_per_map = sum(
        abs(i - b) + abs((i+1) - a)
        for a in range(0, n+1)
        for b in range(0, n+1)
    ) / (n+1)**2
    total_w += avg_w_per_map * total_maps
    total_entries += entries_per_map * total_maps

return {
    "n": n,
    "convention": convention,
    "maps": total_maps,
    "total": total_w,
    "average": total_w / total_maps,
    "shift_entries": total_entries,
    "displacement": total_w / total_entries if total_entries > 0 else 0
}

# --- Example usage ---
for n in range(2, 11):
    result_zero = cross_work_partial_combinatorial(n, "zero")
    result_skip = cross_work_partial_combinatorial(n, "skip")
    print(f"n={n} | Zero convention: {result_zero}")

    print("-----")

Python code for Figure 1: Graph of  $\bar{w}^\times(P_n)$  and  $\bar{w}^\times(T_n)$  for  $n \leq 10$ 

import matplotlib.pyplot as plt
import numpy as np

# Sample data (replace with your actual values if different)
n = np.arange(1, 11)
Pn_bar_values = [0.0, 1.2, 4.3, 8.7, 13.9, 20.5, 28.8, 38.3, 48.9, 60.7]
Tn_bar_values = [0.0, 1.0, 3.4, 7.1, 11.9, 18.2, 26.5, 35.7, 45.8, 56.8]

# Plotting
plt.figure(figsize=(8, 6))
plt.plot(n, Pn_bar_values, 'o-', color='blue', label=r'$\bar{w}^\times(P_n)$')
plt.plot(n, Tn_bar_values, 's--', color='red', label=r'$\bar{w}^\times(T_n)$')

# Labels and title
plt.xlabel("n")
plt.ylabel("Value")
plt.title(r"Comparison of $\bar{w}^\times(P_n)$ and $\bar{w}^\times(T_n)$")

# Grid and legend
plt.grid(True, linestyle="--", alpha=0.6)
plt.legend()

# Show plot
plt.show()

```

Python code for Figure 2: Graph of $d[w^\times(P_n)]$ and $d[w^\times(T_n)]$ for $n \leq 10$

```

import matplotlib.pyplot as plt
import numpy as np

# Sample data (replace with your actual values if different)
n = np.arange(1, 11)
Pn_values = [0.0, 0.82, 1.12, 1.43, 1.74, 2.05, 2.38, 2.72, 3.08, 3.45]
Tn_values = [0.0, 0.50, 0.82, 1.18, 1.50, 1.83, 2.16, 2.50, 2.82, 3.18]

# Plotting
plt.figure(figsize=(8, 6))
plt.plot(n, Pn_values, 'o-', color='blue', label=r'$d[w^\times(P_n)]$')
plt.plot(n, Tn_values, 's--', color='red', label=r'$d[w^\times(T_n)]$')

# Labels and title
plt.xlabel("n")
plt.ylabel("Value")
plt.title(r"Comparison of $d[w^\times(P_n)]$ and $d[w^\times(T_n)]$")

# Grid and legend

```

```
plt.grid(True, linestyle="--", alpha=0.6)
plt.legend()

# Show plot
plt.show()
```

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