Palestine Journal of Mathematics

Vol 15(1)(2026) , 689–697

T-PRIME IDEAL OF A PSEUDORING

T. N. Natei, D. C. Kifetaw and K. Venkateswarlu

Communicated by: Ayman Badawi

MSC 2020 Classifications: Primary 13F99, 06E75, 13A99: Secondary 06F99, 06D99.

Keywords and phrases: additive product subset, quotient ideal, prime ideal, T-prime ideal.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of the paper.

.

Corresponding Author: K. Venkateswarlu

Abstract. In this paper, we introduce the concept of T-prime ideal over a pseudoring R, which is a generalization of the prime ideal, and we obtain certain properties. Also, we prove the prime avoidance lemma in the case of a pseudoring. Finally, we characterize T-prime ideals in terms of quotient ideals of a pseudoring.

1 Introduction

Various generalizations of prime ideals in ring theory have been studied by different authors (for instance, see [3, 2, 6, 16, 9, 14]). The concept of an S-prime ideal, which is also a generalization of a prime ideal, was introduced in the commutative ring [11]. Sevim et al. [15] studied the concept of the S-prime ideal and used it to characterize integral domains, certain prime ideals, fields, and S-Noetherian rings. The prime avoidance property in ring theory was studied in [5]. Some other related studies in commutative rings are [1], [4]. In other algebraic structures, generalizations of prime ideals have been studied, as seen in [7, 10]. Natei et al. [12, 13] studied prime ideals and polar ideals in a pseudoring. In this paper, we introduce the notion of a T-prime ideal in a pseudoring (which is not even a ring) and obtain analogous properties to the notion of the S-prime ideal of a ring. We provide an example where the T-prime ideal is not a prime ideal. Further, we prove a prime avoidance lemma in a pseudoring. Finally, we characterize T-prime ideals in terms of quotient ideals for a particular class of pseudorings. We also prove that Q is a T-prime ideal of R if and only if Q/I is a $\overline{T}$-prime ideal of R/I.
Throughout this paper, R denotes a pseudoring, if not otherwise stated.

2 Preliminaries

In this section, we recall some basic definitions and results concerning pseudorings from [8, 12, 13].

Definition 2.1. A pseudoring is an algebra $R = (R, +, ., 1)$ of type $(2, 2, 0)$ satisfying the following axioms:

$$P_1.\quad (xy)z = x(yz),$$
$$P_2.\quad xy = yx,$$
$$P_3.\quad x1 = x,$$
$$P_4.\quad 1 + (1 + x) = x,$$
$$P_5.\quad x0 = 0,$$
$$P_6.\quad (1 + x(1 + y))(1 + y) = (1 + y(1 + x))(1 + x) \ \ and$$
$$P_7.\quad 1 + (1 + x(1 + y))(1 + y(1 + x)) = x + y,$$

where 0 denotes the element $1 + 1$.

Remark 2.2. Commutativity of $+$ follows from (P_2) and (P_7).

Definition 2.3. Define: $x \leq y$ for any two elements $x, y \in R$ if and only if x and y satisfy the condition $(y + 1)x = 0$.

Definition 2.4. A subset I of R that satisfies the following conditions is an ideal I of R.

$$(i)\quad 0 \in I.$$
$$(ii)\quad 1 + (x + 1)(y + 1) \in I \ \ for\ every\ \ x, y \in I.$$
$$(iii)\quad For\ any\ \ x \in R\ and\ y \in I, x \leq y \Rightarrow x \in I.$$

Proposition 2.5. *For any* $x, y \in R$, $y(1 + (x + 1)y) = x(1 + (y + 1)x)$.

Definition 2.6. Let R_1 and R_2 be two pseudorings. A pseudoring homomorphism is a mapping $\varphi : R_1 \longrightarrow R_2$ that meets the following conditions:

$$1)\quad \varphi(1) = 1,$$
$$2)\quad \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y),$$
$$3)\quad \varphi(x + y) = \varphi(x) + \varphi(y) \quad for\ every\ \ x, y \in R_1.$$

Proposition 2.7. *Let* R *be a pseudoring. The following properties hold on* R.

N_1. $x(x + 1) = 0, \quad \forall x \in R.$

N_2. $y(1 + 0) = y, \ \forall y \in R$ *and* $1 + 0 = 1$.

N_3. $x + 0 = x$.

N_4. *Char* $R = 2$.

Proposition 2.8. *Let* R *be a pseudoring. If* $x \leq y$, *then* $xz \leq yz$ *for all* $x, y, z \in R$.

Corollary 2.9. *For any* $x, y \in R$ *with* $x \leq y$, *we have* $x^2 \leq xy$.

Definition 2.10. Let I be an ideal of R. For any $a \in R$, define $a/I = \{x \in R : x + a \in I\}$. And for any $a,\ b \in R$, $\ a/I = b/I$ if and only if $a + b \in I$.

Notation: $a/I = \bar{a}$. These notations can be utilized interchangeably in the subsequent results.

Theorem 2.11. *If* I *is an ideal of* R, *then* $R/I = \{x/I : x \in R\}$ *is a pseudoring with the operations* $x/I + y/I = (x + y)/I = \{r \in R : r + (x + y) \in I\}$, $x/I \cdot y/I = (x \cdot y)/I$.

Definition 2.12. A proper ideal P of R is called prime ideal if for every $x, y \in R$, either $x(y + 1) \in P$ or $y(x + 1) \in P$.

Proposition 2.13. I *is an ideal of* R *if and only if the following holds:*

$$(i)\quad 0 \in I,$$
$$(ii)\quad 1 + (x + 1)(y + 1) \in I \ \ for\ every\ \ x, y \in I,$$
$$(iii)\quad (y + 1)x, y \in I \Rightarrow \ x \in I.$$

Proposition 2.14. *If* $x, y \in I$, *then* $x + y \in I$ *for an ideal* I *of* R.

3 T-Prime ideals

We begin with the following

Proposition 3.1. *Let $I \subseteq R$. I is an ideal of R if and only if the following holds*

$$(i) \quad 0 \in I.$$

$$(ii) \quad 1 + (x+1)(y+1) \in I \ \ for\ every\ \ x, y \in I.$$

$$(iii) \quad (y+1)z, (x+1)y \in I \Rightarrow (x+1)z \in I \quad for\ every \quad x, y, z \in R.$$

Proof. Suppose I be an ideal of R. Let $(y+1)z, \ \ (x+1)y \in I$. Using condition (ii) we have $1 + (1 + z(y+1))(1 + y(x+1)) \in I$.

Consider

$$\begin{aligned} z(x+1)[\{1 + (1 + z(y+1))(1 + y(x+1))\} + 1] \\ &= z(x+1)(1 + z(y+1))(1 + y(x+1)) \cdots by\ (P_4) \\ &= z(1 + z(y+1)) \cdot (x+1)(1 + y(x+1)) \ \cdots \ by\ (P_2)\ and\ (P_1) \\ &= z(1 + z(y+1))(y+1)(1 + x(y+1)) \ \cdots \ by\ (P_6) \\ &= z(y+1)(1 + z(y+1))(1 + x(y+1)) \ \cdots \ by\ (P_2)\ and\ (P_1) \\ &= 0 \cdot ((y+1)x + 1) = 0 \ \cdots \ by\ (Proposition\ 2.7\ (N_1)). \end{aligned}$$

Thus by Definition 2.3, $z(x+1) \leq 1 + (1 + z(y+1))(1 + y(x+1))$. Hence by Definition 2.4(iii), $z(x+1) \in I$.
Conversely, suppose conditions (i)-(iii) are true. Let $x \leq y$ and $y \in I$. From Proposition 2.7 (N_2), we have $y(0+1) = y \in I$ and $x \leq y \Rightarrow (y+1)x = 0 \in I$. Then by (iii), $x = x(0+1) \in I$. Therefore I is an ideal of R. □

Definition 3.2. Let I be an ideal of R. For each, $\bar{x}, \bar{y} \in R/I$, $\bar{x} \leq \bar{y}$ iff $x(y+1) \in I$.

Proposition 3.3. *The relation defined in Definition 3.2 is a partial order relation on R/I.*

Proof. Let $x \in R$. Then $x(x+1) = 0 \in I$ for an ideal I in R. It follows that $\bar{x} \leq \bar{x}$. Hence, the relation " $\leq$ " is reflexive. Suppose $\bar{x} \leq \bar{y}$ and $\bar{y} \leq \bar{x}$. It follows that $x(y+1), \ x(y+1) \in I$. By Definition 2.4 (ii),P_7, we have $1 + (x(y+1) + 1)(y(x+1) + 1) = x + y \in I$. By Definition 2.10, $x/I = y/I$. Thus, the relation "$\leq$" is antisymmetric. Let $\bar{x} \leq \bar{y}$ and $\bar{y} \leq \bar{z}$. By Definition 3.2, $x(y+1), y(z+1) \in I$. By Proposition 3.1 (iii), $x(z+1) \in I$. Thus $\bar{x} \leq \bar{z}$. Hence " $\leq$ " is transitive. □

Lemma 3.4. *If P is a prime ideal of R, then R/P is a totally ordered pseudoring.*

Proof. Let $\bar{x}, \bar{y} \in R/P$, where $x, y \in R$. Since P is a prime ideal,it follows that $x(y+1) \in P$ or $y(x+1) \in P \Rightarrow \bar{x} \leq \bar{y}$ or $\bar{y} \leq \bar{x}$. Therefore, R/P is totally ordered. □

Example 3.5. Let $R = \{0, \ a, \ b, \ c, \ d, \ 1\}$ and the operations '$+$' and '$\cdot$' on R are defined as follows:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	0	a	d	c	d
b	b	a	0	1	d	c
c	c	d	1	0	a	b
d	d	c	d	a	0	a
1	1	d	c	b	a	0

·	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

Then $R = (R, +, \cdot, 1)$ is a pseudoring. Using the definition of an ideal, we can verify that $I = \{0, \ c\}$ is an ideal of R. Observe that for every $x, y \in R, \ \ x(y+1) \in I$ or $y(x+1) \in I$. Thus I is a prime ideal of R. Observe that $\bar{0} \leq \bar{a} \leq \bar{1}$. Therefore, R/I is a totally ordered pseudoring.

Theorem 3.6. *Let P be a prime ideal of R. For any $x, y \in R$ if $xy \in P$, then either $x^2 \in P$ or $y^2 \in P$.*

Proof. Let $xy \in P$. Since $x, y \in R \Rightarrow \bar{x}, \bar{y} \in R/P$. Hence by Lemma 3.4, R/P is a chain, then either $\bar{x} \leq \bar{y}$ or $\bar{y} \leq \bar{x}$. If $\bar{x} \leq \bar{y}$. From Corollary 2.9, it follows that $\bar{x}^2 \leq \bar{x}\bar{y}$. $\Rightarrow \bar{x}^2(\bar{x}\bar{y} + \bar{1}) = \bar{0} \Rightarrow x^2(xy + 1) \in P$. Hence, by Proposition 2.13, $x^2 \in P$. Similarly, if $\bar{y} \leq \bar{x}$, then $y^2 \in P$. □

Proposition 3.7. *Let R be an idempotent pseudoring and P be any proper ideal of R. Then, P is prime ideal of R if and only if $xy \in P \Rightarrow x^2 \in P$ or $y^2 \in P$ for all $x, y \in R$.*

Proof. Let R be idempotent pseudoring and P be any proper ideal of R. The forward proof follows from Theorem 3.6.
Conversely, suppose $xy \in P \Rightarrow x^2 \in P$ *or* $y^2 \in P$, for all $x, y \in R$. Let $a, b \in R$. Since P is an ideal of R, $a(b+1) \cdot b(a+1) = 0 \in P$. It follows that $(a(b+1))^2 \in P$ or $(b(a+1))^2 \in P$. Thus $a(b+1) \in P$ or $b(a+1) \in P$. □

Theorem 3.8. *Let $I_1, I_2, \cdots, I_n$ be ideals of R and P be a prime ideal of R. If $I_1I_2 \cdots I_n \subseteq P$, then $I_j^{2^k} \subseteq P$ for some $j \in \{1, 2, \cdots, n\}$ and $1 \leq k \leq n-1$.*

Proof. Suppose $I_1I_2 \cdots I_n \subseteq P$, for ideals $I_1, \cdots, I_n$ of R and prime ideal P of R. Let $x_1 \in I_1$, $x_2 \in I_2$, $\cdots$, $x_n \in I_n$, where x_i's are arbitrary elements of I_i. Clearly, $x_1x_2 \cdots x_n \in I_1I_2 \cdots I_n \subseteq P$. It follows that $x_1x_2 \cdots x_n \in P$. Let $y_1 = x_2x_3 \cdots x_n$. Implies $x_1y_1 \in P$. By Theorem 3.6, either $x_1^2 \in P$ or $y_1^2 \in P$. If $x_1^2 \in P$, we are done. Otherwise, $y_1^2 \in P$. Let $y_2 = x_3^2x_4^2 \cdots x_n^2$. It follows that $x_2^2 \cdot y_2 \in P$, by the same theorem $x_2^4 \in P$ or $y_2^2 \in P$. If $x_2^4 \in P$, we are done. Otherwise $y_2^2 \in P$. Let $y_3 = x_4^4x_5^4 \cdots x_n^4$. It follows that $x_3^4 \cdot y_3 \in P$, by Theorem 3.6, $x_3^8 \in P$ or $y_3^2 \in P$. If $x_3^8 \in P$, we are done. Otherwise $y_3^2 \in P$. Continuing in the same way, we get $x_{n-1}^{2^{n-2}} \cdot y_{n-1} \in P \Rightarrow (x_{n-1}^{2^{n-2}})^2 \in P$ or $y_{n-1}^2 \in P$, where $y_{n-1} = x_n^{2^{n-2}} \Rightarrow y_{n-1}^2 = x_n^{2^{n-2}} \cdot x_n^{2^{n-2}} = x_n^{2^{n-1}}$. Thus either $I_{n-1}^{2^{n-1}} \subseteq P$ or $I_n^{2^{n-1}} \subseteq P$. □

Next, we prove the prime avoidance lemma for a pseudoring as follows:

Theorem 3.9. *Let $P_1, P_2, \cdots, P_n$ be prime ideals of a pseudoring R and I be an ideal of R. If $I \subseteq \bigcup_{i=1}^{n} P_i$, then $I \subseteq P_k$ for some $k \in \{1, 2, \cdots, n\}$.*

Proof. Suppose $I \subseteq \bigcup_{i=1}^{n} P_i$. Let $t_1, t_2, \cdots, t_n$ be maximal elements of I. Then $t_i \leq 1 + (t_1 + 1)(t_2+1) \cdots (t_n+1) \in I$ for each $i = 1, 2, \cdots, n$. Thus $x \leq 1 + (t_1+1)(t_2+1) \cdots (t_n+1)$ for any $x \in I$. It follows that there is $k \in \{1, 2, \cdots, n\}$ such that $1+(t_1+1)(t_2+1) \cdots (t_n+1) \in P_k$. Hence $I \subseteq P_k$ for some $k \in \{1, 2, \cdots, n\}$. □

Definition 3.10. Let R be a pseudoring and T a subset of R containing 1. If for every $t, h \in T$, $t(1 + (h+1)t) \in T$, then T is called the additive product set of R.

Example 3.11. Let R be a pseudoring and P be a prime ideal of R. Then $T = R - P$ is an additive product subset of R.

Proof. Let $x, y \in T = R - P \Rightarrow x, y \notin P$. We would like to demonstrate that $x(1+(y+1)x) \in T$. Since $x, y \in R$ and P is a prime ideal of R, then either $x(y+1) \in P$ or $y(x+1) \in P$. Assume that $x(1 + (y+1)x) \in P$. If $x(y+1) \in P \Rightarrow x \in P$. Based on the assumption, Proposition 2.5 implies that $y(1 + (x+1)y) \in P$. If $y(x+1) \in P \Rightarrow y \in P$. This is a contradiction to the fact that $x, y \notin P$. Therefore, $x(1 + (y+1)x) \notin P \Rightarrow x(1 + (y+1)x) \in T$. □

Definition 3.12. Let T be an additive product subset of a pseudoring R, and let Q an ideal of R with $Q \cap T = \emptyset$. Then Q is called a T-prime ideal of R if for all $x, y \in R$ with $xy \in Q$, there is $t \in T$ with $t \neq 1$ such that $x^2(t+1)^n \in Q$ *or* $y^2(t+1)^n \in Q$ for some $n \in \mathbb{Z}^+$.

Proposition 3.13. *Every prime ideal is a T-prime ideal of a pseudoring R.*

Proof. Let Q be a prime ideal of R and let T be an additive product subset of R with $T \cap Q = \emptyset$. It follows that for every $xy \in Q$ either $x^2 \in Q$ or $y^2 \in Q$. Consequently $x^2(t+1)^n \leq x^2 \in Q$ or $y^2(t+1)^n \leq y^2 \in Q$ for all $n \in \mathbb{Z}^+$ all $t \in T$. Hence Q is a T-prime ideal of R. □

The converse of this proposition does not hold in general. For instance, consider the following:

Example 3.14. Let $R = \{0, \alpha, \beta, u\}$ be a given set. Define the operations '+' and '$\cdot$' on R as follows:

+	0	α	β	u
0	0	α	β	u
α	α	0	u	β
β	β	u	0	α
u	u	β	α	0

$\cdot$	0	α	β	u
0	0	0	0	0
α	0	α	0	α
β	0	0	β	β
u	0	α	β	u

Then $(R, +, \cdot, u)$ is a pseudoring. Let $T = \{u, \alpha\}$ and and $Q = \{0\}$. Clearly Q is a T-prime ideal but not prime ideal since for $\alpha, \beta \in R$ neither $\beta(u + \alpha)$ nor $\alpha(u + \beta)$ is in Q.

Proposition 3.15. *Let $T \subseteq R$ be an additive product set. Then Q is a T-prime ideal of R if and only if there are $t_1, t_2, \cdots, t_l \in T$ such that for all $y_1, y_2, \cdots, y_n \in R$, $y_1y_2\cdots y_n \in Q$, implies $y_i^m(t_1+1)^{k_1}(t_2+1)^{k_2}\cdots(t_l+1)^{k_l} \in Q$ for some $i \in \{1, \cdots, n\}$, $m \geq 2$ and some $k_j \in \mathbb{Z}^+$ with $j \in \{1, 2, \cdots, l\}$ and $l \leq n-1$.*

Proof. Let Q be a T-prime ideal of R and let $y_1y_2\cdots y_n \in Q$. Let $x_0 = y_1y_2\cdots y_{n-1}$. It follows $x_0y_n \in Q$. By Definition 3.12, either $x_0^2(t_1+1)^{k_1} \in Q$ or $y_n^2(t_1+1)^{k_1} \in Q$. If $y_n^2(t_1+1)^{k_1} \in Q$ we are done, otherwise $x_0^2(t_1+1)^{k_1} \in Q$. In this case, let $y_1^2y_2^2\cdots y_{n-2}^2 = x_1 \Rightarrow x_1y_{n-1}^2(t_1+1)^{k_1} \in Q$. It follows either $x_1^2(t_2+1)^{k_1} \in Q$ or $y_{n-1}^4(t_1+1)^{2k_1}(t_2+1)^{k_2} \in Q$. If $y_{n-1}^4(t_1+1)^{2k_1}(t_2+1)^{k_2} \in Q$. we are done, otherwise $x_1^2(t_2+1)^{k_1} \in Q$. Proceeding like this we get $y_i^m(t_1+1)^{k_1}(t_2+1)^{k_2}\cdots(t_l+1)^{k_l} \in Q$ for some $i \in \{1, \cdots, n\}$, $m \geq 2$ and some $k_j \in \mathbb{Z}^+$ with $j \in \{1, 2, \cdots, l\}$ and $l \leq n-1$. The Converse is straightforward. □

Corollary 3.16. *Let $T \subseteq R$ be a totally ordered additive product set, and Q be an ideal of R with $Q \cap T = \emptyset$. Q is a T-prime ideal of R if and only if $y_1y_2\cdots y_n \in Q \Rightarrow y_i^m(t+1)^k \in Q$ for some $t \in T$ for some $i \in \{1, \cdots, n\}$, $m \geq 2$ and $m, k \in \mathbb{Z}^+$.*

Proof. Suppose Q is a T-prime ideal of R. Let $y_1y_2\cdots y_n \in Q$. By Proposition 3.15,there are $t_1, t_2, \cdots, t_l \in T$ such that $y_i^m(t_1+1)^{k_1}(t_2+1)^{k_2}\cdots(t_l+1)^{k_l} \in Q$. Since T is totally ordered, for $t_1, t_2, \cdots, t_l \in T$ there is $j \in \{1, 2, \cdots, l\}$ such that t_j is maximum element $\Rightarrow t_i \leq t_j$, $\forall i = 1, 2, \cdots, l$ and by Definition 2.3 and Proposition 2.8, we have $t_j + 1 \leq t_i + 1 \Rightarrow (t_j+1)^k \leq (t_i+1)^k$ for $k \in \mathbb{Z}^+$. Fix $k = k_1 + k_2 + \cdots + k_l$. Implies $(t_i+1)^k \leq (t_i+1)^{k_i}$. By the same definition and proposition, we have $y_i^m(t_j+1)^k \leq y_i^m(t_1+1)^{k_1}(t_2+1)^{k_2}\cdots(t_l+1)^{k_l} \in Q$. It follows that $y_i^m(t_j+1)^k \in Q$. The converse proof follows from Definition 3.12 and Proposition 3.15. □

Definition 3.17. Let I be an ideal of R and $x \in R$. Define $(I : x) = \{y \in R : \ y(x+1)^n \in I$ for some $n \in \mathbb{Z}^+\}$.

Lemma 3.18. *Let I be an ideal in R. If $x \in I$, then $I = (I : x)$.*

Proof. Let $a \in I$. It is clear from Definition 2.3 that $a(x+1)^n \leq a$, for any $n \in \mathbb{Z}^+$, thus by Definition 2.4, $a(x+1)^n \in I$. Hence $I \subseteq (I : x)$. To illustrate the opposite way of inclusion, let $y \in (I : x)$. Then $y(x+1)^{n-1}(x+1) = y(x+1)^n \in I$, for $n \in \mathbb{Z}^+$ and $n > 2$. Since $x \in I$, by Proposition 2.13, $y(x+1)^{n-1} \in I$. Also, $y(x+1)^{n-2}(x+1) = y(x+1)^{n-1}$, and hence, by the same proposition, $y(x+1)^{n-2} \in I$. Continuing the process like this, we get $y \in I$. Thus, $I = (I : x)$. □

It is important to note that the intersection of any family of ideals of R is also an ideal of R. For each subset H of R, the intersection of all ideals $I \supseteq H$ is the smallest ideal containing H and is denoted by $\prec H \succ$.

Lemma 3.19. *If I is an ideal of R and x is an arbitrary element of R, then $\prec I \cup \{x\} \succ = \{y \in R : y \leq 1 + (h_1+1)(h_2+1)\cdots(h_n+1)$ for some $h_1, h_2, \ldots, h_n \in I \cup \{x\}\}$.*

Proof. Let $J = \{y \in R : y \leq 1 + (h_1 + 1)(h_2 + 1) \cdots (h_n + 1)$ for some $h_1,\ h_2,\ \cdots,\ h_n \in I \cup \{x\}\}$. We show that J is an ideal of R containing $I \cup \{x\}$. Let $a \in I \cup \{x\}$. By Definition 2.3 and Proposition 2.7 (N_1), $a \leq a = 1 + (a+1) \in J$. Hence $I \cup \{x\} \subseteq J$. Since I is an ideal of R, $0 \in I \subseteq J \Rightarrow 0 \in J$. Let $a, b \in J$. By the definition of J, we have $a \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1)$ and $b \leq 1 + (k_1+1)(k_2+1) \cdots (k_m+1)$ for some $h_1,\ h_2,\ \cdots,\ h_n,\ k_1,\ k_2,\ \cdots,\ k_m \in I \cup \{x\}$. It follows that $(h_1+1)(h_2+1) \cdots (h_n+1) \leq a+1$ and $(k_1+1)(k_2+1) \cdots (k_m+1) \leq b+1$. By Proposition 2.8, $(h_1+1)(h_2+1) \cdots (h_n+1)(k_1+1)(k_2+1) \cdots (k_m+1) \leq (a+1)(b+1) \Rightarrow 1 + (a+1)(b+1) \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1)(k_1+1)(k_2+1) \cdots (k_m+1)$ for some $n, m \in \mathbb{Z}^+$. Thus $1 + (a+1)(b+1) \in J$. Let $a \leq b$, and $b \in J$ for $a \in R$. Implies $a \leq b \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1) \Rightarrow a \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1)$ for some $h_1, h_2, \cdots, h_n \in I \cup \{x\} \Rightarrow a \in J$. Hence J is an ideal containing $I \cup \{x\}$.

It is left to show J is the smallest ideal containing $I \cup \{x\}$. Let K be an arbitrary ideal of R containing $I \cup \{x\}$. Let $a \in J$. Implies $a \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1)$ for some $h_1,\ h_2,\ \cdots,\ h_n \in I \cup \{x\} \subseteq K$. It follows that $a \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1) \in K \Rightarrow J \subseteq K$. Thus J is the smallest ideal containing $I \cup \{x\}$. Hence $J = \prec I \cup \{x\} \succ$. □

Theorem 3.20. *Let I be an ideal of R. For all $x \in R$, $(I : x)$ is an ideal of R.*

Proof. If $x \in I$, then by Lemma 3.18, $(I : x) = I$ is an ideal. Suppose $x \notin I$. In this case, to demonstrate that $(I : x)$ is an ideal, we assert $(I : x) = \prec I \cup \{x\} \succ$. Let $a \in \prec I \cup \{x\} \succ$. By Lemma 3.19, $a \leq 1 + (h_1+1)(h_2+1) \cdots (h_n+1)$ for some $h_1,\ h_2, \cdots,\ h_n \in I \cup \{x\}$. By Definition 2.3, $a \cdot (h_1+1)(h_2+1) \cdots (h_n+1) = 0 \in I$. We can consider two situations.
Case$_1$. If for any $1 \leq i \leq n,\ \ h_i \neq x$, then $a(x+1)^k(h_1+1)(h_2+1) \cdots (h_n+1) = a(h_1+1)(h_2+1) \cdots (h_n+1)(x+1)^k = 0 \cdot (x+1)^k = 0 \in I$ for some $k \in \mathbb{Z}^+$. Since $h_i \in I\ \ \forall i \in \{1, \cdots, n\}$, by Proposition 2.13 (iii), $a(x+1)^k(h_1+1) \cdots (h_{n-1}+1) \in I$. By repeated application of proposition2.13 we get $a(x+1)^k \in I$. It follows $a \in (I : x)$. Hence $\prec I \cup \{x\} \succ \subseteq (I : x)$.
Case$_2$. If there exists $h_i = x$ for $1 \leq i \leq n$. Then by renumbering, there is $m, k < n$ such that $a(x+1)^m(h_1+1)(h_2+1) \cdots (h_k+1) = 0 \in I$. It follows that $a(x+1)^m \in I \Rightarrow a \in (I : x)$. Thus $\prec I \cup \{x\} \succ \subseteq (I : x)$.
To show the reverse inclusion, let $a \in (I : x)$. It follows that $a(x+1)^n \in I$ for some $n \in \mathbb{Z}^+$. Let $b = a(x+1)^n$. Then, By Proposition 2.7 (N_1), $b(b+1) = 0 \Rightarrow a \leq 1 + (x+1)^n \cdot (a(x+1)^n + 1)$. Since $x,\ a(x+1)^n \in I \cup \{x\}$, By Lemma 3.19, $a \in \prec I \cup \{x\} \succ$. Hence $(I : x) \subseteq \prec I \cup \{x\} \succ$. Thus $(I : x)$ is an ideal of R. □

For any ideal I of R and $x \in R$, the ideal $(I : x)$ is called Quotient ideal of I with respect to an element x.

Theorem 3.21. *Let $T \subseteq R$ be an additive product set of R and let Q be an ideal of R with $T \cap Q = \emptyset$. If $(Q : t)$ is a prime ideal of R for some $t \in T$ with $t \neq 1$, then Q is a T-prime ideal of R.*

Proof. Suppose $(Q : t)$ is the prime ideal of R for some $t \in T$. Let $x, y \in R$ with $xy \in Q$. Since $Q \subseteq (Q : t) \Rightarrow xy \in (Q : t)$. By Theorem 3.6, $x^2 \in (Q : t)$ or $y^2 \in (Q : t) \Rightarrow x^2(t+1)^{n_1} \in Q$ or $y^2(t+1)^{n_2} \in Q$ for some $n_1,\ n_2 \in \mathbb{Z}^+$. Let $m = max\{n_1,\ n_2\}$. Thus $x^2(t+1)^m \in Q$ or $y^2(t+1)^m \in Q$. Hence Q is a T-prime ideal of R. □

Example 3.22. Observe that in example 3.14, Q is a T-prime ideal. Then $(Q : \alpha) = \{y \in R :\ y(\alpha+u)^n \in Q\} = \{y \in R :\ y\beta = 0\} = \{0, \alpha\}$ is a prime ideal of R.

Proposition 3.23. *Let R be an idempotent pseudoring and $T \subseteq R$ be a totally ordered additive product set. And let Q be an ideal of R with $T \cap Q = \emptyset$. Then $(Q : t)$ is a prime ideal of R for some $t \in T$ with $t \neq 1$ if and only if Q is a T-prime ideal of R.*

Proof. The forward proof follows from Theorem 3.21. Conversely, suppose Q is a T-prime ideal of R. Let fix $t_0 \in T$ be a maximal element with $t_0 \neq 1$. That is $t \leq t_0,\ \ \forall t \in T$. Let $x, y \in R$ with $xy \in (Q : t_0)$. By Definition 3.17, $xy(t_0+1)^n \in Q$ for some $n \in \mathbb{Z}^+$. Let $y(t_0+1)^n = a$. It follows that $xa \in Q$. Since Q is a T-prime ideal there is $t \in T$ such that $x^2(t+1)^{n_1} \in Q$ or $a^2(t+1)^{n_2} \in Q$. If $x^2(t+1)^{n_1} \in Q$, By Proposition 2.8 (i) and (iii), $x^2(t_0+1)^{n_1} \leq x^2(t+1)^{n_1} \Rightarrow$

$x^2(t_0+1)^{n_1} \in Q \Rightarrow x^2 \in (Q:t_0)$. If $a^2(t+1)^{n_2} \in Q \Rightarrow y^2(t_0+1)^{2n} \cdot (t+1)^{n_2} \in Q$. Since $t \le t_0$, applying Proposition 2.8 (i) and (iii), $y^2(t_0+1)^{2n}(t_0+1)^{n_2} \le y^2(t_0+1)^{2n} \cdot (t+1)^{n_2} \Rightarrow y^2(t_0+1)^k \in Q$, where $k = 2n + n_2 \in \mathbb{Z}^+$. Hence $y^2 \in (Q:t_0)$. Therefore by Proposition 3.7, $(Q:t_0)$ is a prime ideal of R. □

Example 3.24. Let $R = \{0,\ a,\ b,\ c,\ \alpha,\ \beta,\ \gamma,\ 1\}$ be a given set. The operations '$+$' and '$\cdot$' on R are defined as follows:

+	0	a	b	c	α	β	γ	1
0	0	a	b	c	α	β	γ	1
a	a	0	α	β	b	c	1	γ
b	b	α	0	γ	a	1	c	β
c	c	β	γ	0	1	c	β	α
α	α	b	a	1	0	γ	β	c
β	β	c	1	c	γ	0	α	b
γ	γ	1	c	β	β	α	0	a
1	1	γ	β	α	c	b	a	0

$\cdot$	0	a	b	c	α	β	γ	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
α	0	a	b	0	α	a	b	α
β	0	a	0	c	a	β	c	β
γ	0	0	b	c	b	c	γ	γ
1	0	a	b	c	α	β	γ	1

Then $(R,+,\cdot,1)$ is an idempotent pseudoring and $T = \{1,\ c,\ \gamma\}$ is a totally ordered additive product set of R. Observe that $Q = \{0,\ a\}$ is T-prime ideal. By simple computations, one can verify that $(Q:c) = \{0,\ a,\ c,\ \beta\}$ is a prime ideal of R.

Proposition 3.25. *Let R_1 and R_2 be two distinct pseudorings and T be an additive product subset of R_1. If the map $f:\ R_1 \to R_2$ is an epimorphism, then the following statements hold.*

i) $f(T)$ is additive product subset of R_2.

ii) If Q is $f(T)$-prime ideal of R_2, then $f^{-1}(Q)$ is a T-prime ideal of R_1.

Proof. i) Let 1 be multiplicative identity of R_1 and $1'$ be multiplicative identity of R_2. Clearly $f(T) \subseteq R_2$ and $1 \in T \Rightarrow f(1) = 1' \in f(T)$. Let $x, y \in f(T)$. Implies there are $a, b \in T$ such that $f(a) = x$ *and* $f(b) = y$.
Consider $x(1' + (y+1')x) = f(a)(f(1) + (f(b)+f(1))f(a)) = f(a(1+(b+1)a))$. Since T is additive product subset of R_1, $a(1+(b+1)a) \in T \Rightarrow x(1'+(y+1')x) \in f(T)$.

ii) Suppose Q is $f(T)$- prime ideal of R_2. Let $xy \in f^{-1}(Q)$. It follows $f(xy) \in Q$ by the definition of homomorphism $f(x) \cdot f(y) = f(xy) \in Q$. Hence there is $f(t)$ for some $t \in T$ such that $(f(x))^2(f(t)+1')^n \in Q$ or $(f(y))^2(f(t)+1')^n \in Q$ for some $n \in \mathbb{Z}^+$. Since $1' = f(1)$, it follows $f(x^2(t+1)^n) \in Q$ or $f(y^2(t+1)^n) \in Q \Rightarrow x^2(t+1)^n \in f^{-1}(Q)$ *or* $y^2(t+1)^n \in f^{-1}(Q)$ for some $n \in \mathbb{Z}^+$. Therefore $f^{-1}(Q)$ is a T-prime ideal of R_1.

□

Proposition 3.26. *Let $T \subseteq R$ be an additive product set. Suppose Q is a T-prime ideal of R. If for all ideal I, J of R, $I \cap J \subseteq Q$, then there is $t \in T$ such that $I^2 \subseteq (Q:t)$ or $J^2 \subseteq (Q:t)$.*

Proof. Let $x \in I$ *and* $y \in J \Rightarrow x^2 \in I^2$ *and* $y^2 \in J^2$. We can easily observe that $xy \in I \cap J$. Implies $xy \in Q$. It follows that there is $t \in T$ such that $x^2(t+1)^n \in Q$ or $y^2(t+1)^n \in Q \Rightarrow x^2 \in (Q:t)$ *or* $y^2 \in (Q:t)$. Hence $I^2 \subseteq (Q:t)$ *or* $J^2 \subseteq (Q:t)$. □

The following theorem demonstrates the relation between the T-prime ideal of R and the quotient $\overline{T}$-prime ideal of R/I.

Theorem 3.27. *Let T be an additive product subset of R and let I be the ideal of R with $I \cap T = \emptyset$. Let Q be a proper ideal of R containing I such that $\overline{T} \cap Q/I = \emptyset$. Then Q is a T-prime ideal of R if and only if Q/I is a $\overline{T}$-prime ideal of R/I.*

Proof. Suppose Q is a T-prime ideal of a pseudoring R. Let Q contain an ideal I of R. Since a T-prime ideal is an ideal, one can easily demonstrate that Q/I is an ideal of R/I. Suppose T is an additive product subset of R. Now we claim that $\overline{T}$ is an additive product subset of R/I.

Let $\bar{a}, \bar{b} \in \overline{T}$. By definition we have $a, b \in T \Rightarrow a(1+(b+1)a) \in T$. Thus $\bar{a}(\bar{1}+(\bar{b}+\bar{1})\bar{a}) = \overline{a(1+(b+1)a)} \in \overline{T}$. Hence $\overline{T}$ is an additive product subset of R/I. The following step is to demonstrate that Q/I is a $\overline{T}$-prime ideal in the pseudoring R/I. Let $\bar{x}\bar{y} \in Q/I$. Then $\overline{xy} \in \bar{Q}$ $\Rightarrow xy \in Q$. Since Q is a T-prime ideal of R, then there is $t \in T$ such that either $x^2(t+1)^n \in Q$ or $y^2(t+1)^n \in Q \Rightarrow \bar{x}^2(\bar{t}+\bar{1})^n = \overline{x^2(t+1)^n} \in \bar{Q}$ or $\bar{y}^2(\bar{t}+\bar{1})^n = \overline{y^2(t+1)^n} \in \bar{Q}$. Therefore $\bar{Q}$ is a $\overline{T}$-prime ideal of R/I.
Conversely, suppose $\bar{Q}$ be a $\overline{T}$-prime ideal of R/I. Since $\overline{T} \cap Q/I = \emptyset \Rightarrow Q \cap T = \emptyset$. Let $xy \in Q$. Then $\overline{xy} = \bar{x}\bar{y} \in \bar{Q}$. Since $\bar{Q}$ is a $\overline{T}$-prime,then there is $\bar{t} \in \overline{T}$ such that $\bar{x}^2(\bar{t}+\bar{1})^n = \overline{x^2(t+1)^n} \in \bar{Q}$ or $\bar{y}^2(\bar{t}+\bar{1})^n = \overline{y^2(t+1)^n} \in \bar{Q} \Rightarrow x^2(t+1)^n \in Q \ \ or \ \ y^2(t+1)^n \in Q$ for some $n \in \mathbb{Z}^+$.

□

Example 3.28. In Example 3.24, $R/I = \{\bar{0},\ \bar{a},\ \bar{c},\ \bar{\gamma},\ \bar{1}\}$ using the operations in Theorem 2.11, is a quotient pseudoring for an ideal $I = \{0,\ b\}$. It is simple to show that $T = \{1,\ c,\ \gamma\}$ is an additive product subset of R, and $Q = \{0,\ a,\ b,\ \alpha\}$ is a proper ideal of R that contains I. We can observe that $\overline{T} = \{\bar{1},\ \bar{c},\ \bar{\gamma}\}$ and $\bar{Q} = \{\bar{0},\ \bar{a}\}$. Thus, $\overline{T} \cap \bar{Q} = \emptyset$. By simple computations, we imply that $\bar{Q}$ is a $\overline{T}$-prime ideal in R/I. For instance, $\bar{\alpha} \cdot \bar{\gamma} = \overline{\alpha \cdot \gamma} = \bar{b} = \bar{0} \in \bar{Q}$, there is $\bar{c} \in \overline{T}$ such that $(\bar{\alpha})^2(\bar{c}+\bar{1})^n = \overline{\alpha^2}(\overline{c+1})^n = \bar{\alpha}(\overline{\alpha^n}) = \bar{\alpha} = \overline{\alpha \cdot \alpha} = \bar{\alpha} = \bar{a} \in \bar{Q}$ for some $n \in \mathbb{Z}^+$. Similarly, we can do this for the other combinations. It is likewise simple to show that Q is a T-prime ideal of R.

Proposition 3.29. *Let T be a totally ordered additive product subset of an idempotent pseudoring R. Let I be an ideal of R and let $Q_1,\ \cdots,\ Q_n$ are T-prime ideals of R. If $I \subseteq \cup_{i=1}^{n} Q_i$, then there exists $t_j \in T$ and $j \in \{1,\ \cdots,\ n\}$ such that $I \subseteq (Q_j : t_j)$.*

Proof. Let $Q_1,\ Q_2,\ \cdots,\ Q_n$ be T-prime ideals of R and let I be the ideal of R with $I \subseteq \cup_{i=1}^{n} Q_i$. By Proposition 3.23, there is $t_i \in T$ such that $(Q_i : t_i)$ is prime ideal of R. Since $I \subseteq \cup_{i=1}^{n} Q_i \subseteq \cup_{i=1}^{n}(Q_i : t_i) \Rightarrow I \subseteq \cup_{i=1}^{n}(Q_i : t_i)$. By Theorem 3.9, there is $j \in \{1, \cdots, n\}$ such that $I \subseteq (Q_j : t_j)$. □

References

[1] T. Abera, Y.Yitayew, D. Wasihun and Venkateswarlu Kolluru, Ordered Weak Idempotent Rings: Palestine Journal of Mathematics, 13(3),(2024.) 215–222.

[2] D. F. Anderson, A. Badawi, On n-absorbing ideals of commutative rings: Comm. Algebra 39 (5),(2011) 1646–1672.

[3] D. D. Anderson, M. Bataineh, Generalizations of prime ideals: Comm. Algebra 36 (2), (2008), 686–696.

[4] B. Asmare, D. Wasihun and K. Venkateswarlu, Commutative weak idempotent nil-neat rings: Palestine Journal of Mathematics, 13(4), (2024) 1029–1036.

[5] A. Azarang, *Prime avoidance property* : Journal of Algebra and its Applications 21(02),(2022),1-9.

[6] A. Badawi, On 2-absorbing ideals of commutative rings: Bull. Austral. Math. Soc. 75, (3),(2007), 417-429.

[7] S Bhowmick, J Goswami,S-prime and S-maximal k-subsemimodules of semimodules,Palestine Journal of Mathematics Vol. 14 (2), (2025), 372-378.

[8] I. Chajda, H. Langer, *Ring-like structures correspondin g to MV algebras via symmetric difference:* Sitz Abt II 213,(2004), 33-41.

[9] M. Ebrahimpour, R. Nekooei, On generalizations of prime ideals: Comm. Algebra 40 (4), (2012), 1268-1279.

[10] B Deka, J Goswami, 1-Absorbing Prime Ideals of a Lattice:Palestine Journal of Mathematics Vol. 14 (2), (2025), 137-145.

[11] A. Hamed, A. Malek, S-prime ideals of a commutative ring: Beitr Algebra Geom 61, (2020), 533–542.

[12] T. N. Natei, D. C. Kifetaw, K. Venkateswarlu, Extended and metric ideals of a pseudo ring: Bulletin of IMVI, Vol. 13(3), (2023), 529–539.

[13] T. N. Natei, D. C. Kifetaw, K. Venkateswarlu, Polar ideal of a pseudo ring: Palestine Journal of Mathematics Vol. 14 (1), (2025), 34-40.

[14] S. Payrovi, S. Babaei, On the 2-absorbing ideals: Int. Math. Forum 7 (6),(2012), 265–271.

[15] E.S. Sevim, T. Arabaci, U. Tekir, S. Koc, On S-prime submodules: Turk. J. Math. 43(2), (2019), 1036–1046.

[16] A. Yousefian Darani, E. R. Puczylowski, On 2-absorbing commutative semigroups and their applications to rings: Semigroup Forum 86(1),(2013), 83–91.

Author information

T. N. Natei, Department of Mathematics, College of Natural and Computational Science, Arba Minch University, Ethiopia.
E-mail: prns-012-12@amu.edu.et

D. C. Kifetaw, Department of Mathematics, College of Natural and Computational Science, Arba Minch University, Ethiopia.
E-mail: cherudawit@gmail.com

K. Venkateswarlu, Department of Computer Science and Systems Engineering, College of Engineering Andhra University, India.
E-mail: drkvenkateswarlu@gmail.com

Received: 2025-04-26
Accepted: 2025-09-16