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Semi-nil-clean property in bi-amalgamated algebras along ideals

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Abstract. In this paper, we study the possible transfer of the notion of semi-nil-clean rings in various context of commutative ring extensions such as homomorphic image, direct product, amalgamations of rings along ideals and pullback with applications into trivial ring extensions to obtain new original classes of rings satisfying this property.

1 Introduction

Throughout this paper all rings are commutative with identity. We denote respectively by $Nilp(A)$, $Per(A)$ the ideal of all nilpotent elements of the ring A and the set of all periodic elements of A. An element $x \in A$ is called nilpotent if there exists some integer $k \geq 1$ such that $x^k = 0$, an element $y \in A$ is called periodic if there exist distinct positive integers $m, n > 0$ such that $y^m = y^n$, an element $z \in A$ is called potent if there exists some integer $k \geq 2$ such that $z^k = z$. Notice that every idempotent, potent and nilpotent element is periodic. In [7], *P. Danchev* et al. proved that a ring A is periodic if and only if for all $x \in A$ there exists some integer $k \geq 2$ such that $x - x^k \in Nilp(A)$ if and only if each element $x \in A$ can be expressed in the form $x = a + b$ where b is a nilpotent element and $a^k = a$ for some integer $k \geq 2$. Recall that a ring A is UU (i.e., every Unit is Unipotent) if every unit $u \in A$ is unipotent, that is, u can be expressed as $u = 1 + b$ for some nilpotent element b of A.

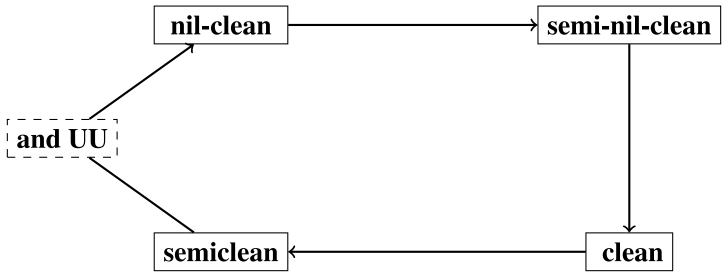
Nicholson introduced in [17] the notion of clean ring (that is a ring in which every element is a sum of a unit and an idempotent). *Y. Ye* introduced in [18] the concept of semiclean ring (that is a ring in which every element is a sum of a unit and a periodic element), as a generalization of the notion of clean ring. *A. J. Diesl* introduced in [9] a new class of rings and called it nil-clean ring (that is a ring in which every element can be expressed as a sum of an idempotent and a nilpotent), that is, in fact, a subclass of clean rings. Recall that every nil-clean element is clean and every clean element is semiclean. Notice, also, that nil-clean ring is periodic.

In [8], P. V. Danchev and W. W. McGovern introduced a generalization of the nil-clean rings, and they called it the weakly-nil-clean rings, recall that a ring is said to be weakly-nil-clean if each of its elements can be written as a sum or difference of a nilpotent and an idempotent. *M. Chhiti* and *S. Moindze* have investigated in [6], the transfer of nil-clean and weakly nil-clean properties to bi-amalgamations algebras.

In [16], *Han* and *Nicholson* showed that the group ring $\mathbb{Z}_{(7)}[C_3]$ is not clean, with $\mathbb{Z}_{(7)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \ and \ gcd(7, n) = 1\}$ and C_3 is a cyclic group of order 3, while *Y. Ye* proved in [18], that for all prime number p and for all cyclic group C_3 of order 3, the group ring $\mathbb{Z}_{(p)}[C_3]$ is semiclean, where $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \ and \ gcd(p, n) = 1\}$.

In [4], *N. Bisht* introduced the notion of semi-nil-clean ring (that is a ring in which every element can be written as a sum of a nilpotent element and a periodic element). Notice that every nil-clean ring is semi-nil-clean, every semi-nil-clean ring is clean and every clean ring is semiclean since every idempotent element is periodic and every periodic element is clean see [18, Lemma 5.1)]. Moreover, it is showed in [3] that, if a ring A is semiclean and UU then the ring A is nil-clean.

The following diagram always holds, where the implications cannot be reversed in general



Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. $Kabbaj$, $Louartiti$ and $Tamekkante$ defined and studied in [15] the following subring of $B \times C$: $A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j') \mid a \in A,\ j \in J,\ j' \in J'\}$, called the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). This ring construction is a natural generalization of the amalgamated algebras along an ideal introduced and studied by *M. D'Anna*, *C. A. Finocchiaro* and *M. Fontana* as the following:

Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this setting, we can consider the following subring of $A \times B$: $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$, called *the amalgamation of* A *and* B *along* J *with respect to* f, introduced and studied by *D'Anna*, *Finocchiaro* and *Fontana* in [10, 11].

Let A be a ring and M an A-module. The trivial ring extension of A by M (also called idealization of M over A) is the ring $R := A \ltimes M$ whose underlying group is $A \times M$ with multiplication given by $(a, m)(b, n) = (ab, an + bm)$. Trivial ring extensions have been studied extensively and considerable work, part of is summarized in *Huckaba's* book [14] and *Glaz's* book [13], has been concerned with these extensions. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. For instance, see [1, 14].

The main of this manuscript is to enlarge the class of nil-clean and to study the transfer of the notion of semi-nil-clean rings, to various context of commutative ring extensions such as homomorphic image, direct product, bi-amalgamated algebras and pullback, with applications to the transfer of this property to trivial ring extensions. Our results generate new classes of rings.

2 About semi-nil-clean property

The following result examines the transfer of semi-nil-clean property to homomorphic image and direct product.

Proposition 2.1.

(i) Let A be a commutative ring and I be an ideal of A. If A is a semi-nil-clean ring, then A/I so is. The converse holds when I is nil ideal of A.

(ii) Let $(A_i)_{i\in\Lambda}$ be a finite family of rings, Then. $\prod_{i\in\Lambda} A_i$ is semi-nil-clean if and only if A_i is semi-nil-clean for each $i \in \Lambda$. [4, Theorem 2.10].

Proof. (i) Suppose that a ring A is semi-nil-clean and let $x \in A$, let $f : A \to B$ be a ring homomorphism, such that B is a commutative ring. Thus $x = p + b$ where $p \in Per(A)$ and $b \in Nilp(A)$. Then, $f(x) = f(p)+f(b)$, it is easy to verify that $f(p) \in Per(B)$ and $f(b) \in Nilp(B)$. Hence, $f(x)$ is semi-nil-clean. Conversely, assume that I is nil ideal of A. Let $\overline{x} \in A/I$, then there exists $\overline{p} \in Per(A/I)$ such that $\overline{x} - \overline{p} \in Nilp(A/I)$. Then $x - p + I \in Nilp(A/I)$. Since I is nil ideal of A, then it follows that $x - p \in Nilp(A)$. Thus, $x = p + b$ for some nilpotent $b \in A$. Hence, A is a semi-nil-clean ring.

(ii) Assume that $\prod_{i\in\Lambda} A_i$ is semi-nil-clean. For each $i \in \Lambda$; A_i is a homomorphic image of $\prod_{i\in\Lambda} A_i$. Hence, by (1), A_i is semi-nil-clean for each $i \in \Lambda$. Conversely, assume that every A_i is semi-nil-clean and let $a = (a_i) \in \prod_{i\in\Lambda} A_i$. For each $i \in \Lambda$, let $a_i = p_i + b_i$, where $p_i \in Per(A_i)$ and $b_i \in Nilp(A_i)$. Then, in view of [2, Lemma 2.4]; $p = (p_i) \in Per(A)$, and it is clear that $b = (b_i) \in Nilp(A)$. hence, $a = (a_i)$ is semi-nil-clean as desired.

□

Recall that an element x is nilpotent if $x^k = 0$ for some integer $k \geq 1$. A ring is reduced if zero is the only nilpotent element. The next theorem gives a characterization of the semi-nil-clean commutative rings.

Theorem 2.2. *Let A be a commutative ring and let n be a positive integer. The following are equivalent:*

- *(1) A is semi-nil-clean.*
- *(2) $A[X]/(X^n)$ is semi-nil-clean.*
- *(3) $A[[X]]/(X^n)$ is semi-nil-clean.*
- *(4) $A/Nilp(A)$ is periodic.*
- *(5) For all $a \in A$, there exists some integer $k \geq 2$ such that $a - a^k \in Nilp(A)$.*

Proof.
$(1) \Leftrightarrow (4)$ This follows easily by Proposition 2.1.
$(1) \Leftrightarrow (5)$ Assume that A is semi-nil-clean and let $a \in A$. Then $a = p+b$ where $p \in Per(A)$ and $b \in Nilp(A)$. Then $a-b \in Per(A)$ and in view of [7, Theorem 3.4], we get $(a-b)-(a-b)^k \in Nilp(A)$, we can deduce easily that $a - a^k \in Nilp(A)$ since $b \in Nilp(A)$. Conversely, suppose that for all $a \in A$, there exists an integer $k \geq 2$ such that $a - a^k \in Nilp(A)$. One can see easily that for all $b \in Nilp(A)$ we get $(a - b) - (a - b)^k \in Nilp(A)$ and in virtue of [7, Theorem 3.4], we get $a - b \in Per(A)$ thus $a - b = p \in Per(A)$. Hence $a = p + b$ is a semi-nil-clean decomposition of a as desired.
$(1) \Leftrightarrow (2)$: Consider the canonical surjection $f : R = A[X]/(X^n) \twoheadrightarrow A$,
such that $f(a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + (X^n)) = a_0$. Clearly, $Ker(f) = (X)$ where (X) is the ideal generated by X and $Ker(f)$ is the kernel of f. Since f is a surjective then $R/(X) \simeq A$. Since (X) is nil then by Proposition 2.1 the result follows.
$(1) \Leftrightarrow (3)$: The proof is similar to $(1) \Leftrightarrow (2)$.

□

It is clear that the polynomial ring over a field is not clean and by [18, Example 3.2], this ring is not semiclean the next proposition shows that the polynomial ring and the ring of formal power series are not semi-nil-clean.

Proposition 2.3. *For a commutative ring A the following hold:*

(i) The polynomial ring $A[X]$ is not semi-nil-clean.

(ii) The ring of formal power series $A[[X]]$ is not semi-nil-clean.

Proof.
(i) Assume that $A[X]$ is semi-nil-clean, then X so is. Thus, in view of Theorem 2.2, $X - X^k$ is nilpotent for some integer $k \geq 2$. Hence 1 is nilpotent which is impossible and so X is not semi-nil-clean. Therefore, $A[X]$ is not semi-nil-clean.
(ii) By the same argument $A[[X]]$ is not semi-nil-clean.

□

The following result follows immediately from Theorem 2.2.

Corollary 2.4. *Every subring of semi-nil clean ring is semi-nil-clean.*

Proof.
This follows immediately from Theorem 2.2 (5). □

3 Bi-amalgamation of semi-nil-clean property

Next, we will study the transfer of the semi-nil-clean property to bi-amalgamated algebras along ideals. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J') = I$. Recall that the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the following subring of $B \times C$:

$$A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j') \mid a \in A,\ j \in J,\ j' \in J'\}.$$

We start with the following result.

Theorem 3.1. *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). The following are equivalent.*

(i) $A \bowtie^{f,g} (J, J')$ is semi-nil-clean ring.

(ii) $f(A) + J$ and $g(A) + J'$ are semi-nil-clean.

Proof.
$(i) \Rightarrow (ii)$ In view of [15, Proposition 4.1 (2)], $f(A) + J$ and $g(A) + J'$ are homomorphic image of $A \bowtie^{f,g} (J, J')$ and the result follows by Proposition 2.1.
$(ii) \Rightarrow (i)$ Assume that $f(A) + J$ and $g(A) + J'$ are semi-nil-clean. Thus in virtue of Proposition 2.1, $(f(A)+J)) \times (g(A)+J')$ is semi-nil-clean, but $A \bowtie^{f,g} (J, J') \subset (f(A)+J)) \times (g(A)+J') \subset B \times C$. Hence by Corollary 2.4 we conclude that $A \bowtie^{f,g} (J, J')$ is semi-nil-clean. □

The next result investigates Theorem 3.1 in case J (resp., J') is nil ideal of B (resp., C).

Proposition 3.2. *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). The following items hold.*

(i) Suppose that J is a nil ideal of B, then:
$A \bowtie^{f,g} (J, J')$ is semi-nil-clean if and only if $g(A) + J'$ so is.

(ii) Suppose that J' is a nil ideal of C, then:
$A \bowtie^{f,g} (J, J')$ is semi-nil-clean if and only if $f(A) + J$ so is.

Proof.
(i) In virtue of [15, Proposition 4.1 (2)], $\dfrac{A \bowtie^{f,g} (J, J')}{J \times 0} \simeq g(A) + J'$. Since J is nil then $J \times 0$ so is and the proof follows by Proposition 2.1.
(ii) In virtue of [15, Proposition 4.1 (2)], $\dfrac{A \bowtie^{f,g} (J, J')}{0 \times J'} \simeq f(A) + J$. Since J' is nil then $0 \times J'$ so is and the proof follows by Proposition 2.1. □

Next, we examine the semi-nil-clean property in amalgamated algebras along an ideal introduced and studied by *D'Anna*, *Finocchiaro* and *Fontana* in [10] which is a special case of bi-amalgamated algebras along ideals introduced and studied in [15, Example 2.1].

Corollary 3.3. *Let (A, B) be a pair of commutative rings, $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. Then the following statements are equivalent.*

(i) $A \bowtie^{f} J$ is semi-nil-clean ring.

(ii) A and $f(A) + J$ are semi-nil-clean.

Proof.
$(i) \Rightarrow (ii)$ From [10, Proposition 5.1], A and $f(A)+J$ are homomorphic images of $A \bowtie^f J$, then the Proposition 2.1 complete the proof.
$(ii) \Rightarrow (i)$ Assume that A and $f(A)+J$ are semi-nil-clean, then $A\times(f(A)+J)$ is semi-nil-clean by Proposition 2.1. Thus $A \bowtie^f J$ is semi-nil clean by Corollary 2.4 since $A \bowtie^f J$ is a subring of $A\times(f(A)+J)$. □

When J is a nil ideal of B we obtain the following result.

Corollary 3.4. *Let A and B be two commutative rings, $f : A \longrightarrow B$ be a ring homomorphism and J be a nil ideal of B. Then the following hold. $A \bowtie^f J$ is semi-nil-clean if and only if so is A.*

Proof. From [10, Proposition 5.1 (3)], $\dfrac{A \bowtie^f J}{0\times J} \simeq A$. Since J is nil ideal, Proposition 2.1(1) complete the proof. □

Let A be a commutative ring and let I be a proper ideal of A. The amalgamated duplication of A along I introduced and studied in [12] is a special amalgamation given by:

$$A \bowtie I := A \bowtie^{id_A} I = \{(a, a+i) \mid a \in A, i \in I\}.$$

The next result follows immediately.

Corollary 3.5. *Let A be a commutative ring and I be an ideal of A. Then, $A \bowtie I$ is semi-nil-clean if and only if so is A.*

Proof. In this case $f = id_A$ and $f(A)+I = A$. □

Let A be a commutative ring and M be an A-module. Set $B = A \ltimes M$ be the trivial ring extension of A by M. Let $f : A \hookrightarrow B$ be the canonical embedding. After identifying M with $\{0\}\ltimes M$, M becomes an ideal of B, it is known that $A \ltimes M$ coincides with $A \bowtie^f M$, for more details one can see [10, Remark 2.8].

Corollary 3.6. *Let A be a ring and M be an A-module. Then $A \ltimes M$ is semi-nil-clean if and only if so is A.*

Proof.
Consider $B = A \ltimes M$ and $f : A \hookrightarrow B$ the canonical inclusion, the ideal $J := \{0\}\ltimes M$ is nil, since $M^2 = 0$. Then the result follows easily by Corollary 3.4. □

The following example shows the necessity of the supposition "J is a nil ideal" in Corollary 3.4.

Example 3.7. Let A be any semi-nil-clean ring (for instance, take $A := \mathbb{Z}_2$ the ring of integers modulo 2), $B := A[X]$ be the polynomial ring with coefficient in A, $J := XA[X]$ be the ideal of B generated by X and $f : A \hookrightarrow B$ be the natural injection. Then:

(i) A is semi-nil-clean.

(ii) $A \bowtie^f J$ is not semi-nil-clean.

Proof.
(i) This is Straightforward.
(ii) We claim that $A \bowtie^f J$ is not semi-nil-clean. Indeed, $f(A)+J = A+XA[X] = A[X]$ is not semi-nil-clean by Proposition 2.3. Then in virtue of Corollary 3.3, $A \bowtie^f J$ is not semi-nil-clean. □

It is known by [3, Theorem 2.6] that the ring $A \bowtie^f J$ is periodic if and only if A is periodic and $J \subset Per(B)$, therefore, the next theorem shows that the characterization for $A \bowtie^f J$ to be semi-nil-clean can be reconducted to the case where $A \bowtie^f J$ is periodic which is equivalent to say that A is periodic and $J \subset Per(B)$.

Theorem 3.8. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Let $\overline{A} = A/Nilp(A)$, $\overline{B} = B/Nilp(B)$, $\pi : B \to \overline{B}$ the canonical projection and $\overline{J} = \pi(J)$, consider the ring homomorphism $\overline{f} : \overline{A} \to \overline{B}$ such that $\overline{x} \mapsto \overline{f}(\overline{x}) = \overline{f(x)}$. Then, $A \bowtie^f J$ is semi-nil-clean if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is periodic.*

Proof. It is easy to see that $\overline{f}$ is well defined and it is a ring homomorphism. Consider the following map:

$$\begin{aligned} \psi : \quad A \bowtie^f J/Nilp(A \bowtie^f J) \quad &\longrightarrow \overline{A} \bowtie^{\overline{f}} \overline{J} \\ \overline{(a, f(a) + j)} \quad &\longmapsto (\overline{a}, \overline{f}(\overline{a}) + \overline{j}) \end{aligned}$$

It is proved in [5] that ψ is a ring isomorphism.
$\Rightarrow$) Suppose that $A \bowtie^f J$ is semi-nil-clean. Then by Proposition 2.1, $A \bowtie^f J/Nilp(A \bowtie^f J)$ is periodic. Hence $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is pereiodic.
$\Leftarrow$) Suppose that $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is periodic, then $A \bowtie^f J/Nilp(A \bowtie^f J)$ periodic. Then in view of Proposition 2.1, $A \bowtie^f J$ is semi-nil-clean. □

4 Semi-nil-clean property to pullback

Finally, we will examine the transfer of semi-nil-clean property to pullback.

Recall that pullback can be defined as follows: Let T be a ring, M is a nonzero ideal of T, p is the natural surjection $p : T \to T/M$ and D is a subring of T/M. Then $R := p^{-1}(D)$ is a subring of T and M is a common ideal of R and T and $D = R/M$. R is called a pullback ring associated to the following pullback diagram:

$$\begin{array}{ccc} R = p^{-1}(D) & \xrightarrow{\quad p/R \quad} & D = R/M \\ \Big\downarrow i & & \Big\downarrow j \\ T & \xrightarrow{\quad p \quad} & T/M \end{array}$$

where i and j are the natural injections.

We assume that $R \subset T$ and we refer to this as a diagram of type $\triangle$. Our next result investigates the transfer of semi-nil clean property to pullback of type $\triangle$.

Theorem 4.1. *For a diagram of type $\triangle$ the following hold:*

(i) If R is semi-nil-clean, then so is D.

(ii) If M is nil ideal then T is semi-nil-clean if and only if so is R.

Proof.
(i) This follows from Proposition 2.1, as D is a homomorphic image of R.
(ii) Suppose M is nil ideal, then in view of Proposition 2.1. □

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