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BIPOLAR-VALUED FUZZY SHEFFER STROKE BE-ALGEBRAS

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Abstract. This paper explores the theory of bipolar fuzzy sets within Sheffer stroke BE-algebras. We introduce the concept of bipolar fuzzy SBE-subalgebras and establish that their level cuts (negative and positive) are SBE-subalgebras, and vice versa. We also prove equivalence conditions for bipolar-valued fuzzy subalgebras in terms of ordinary fuzzy SBE-subalgebras. Further, we define and characterize bipolar fuzzy SBE-filters and implicative bipolar fuzzy SBE-filters, providing several equivalent formulations and showing that every implicative bipolar fuzzy SBE-filter is a bipolar fuzzy SBE-filter, although the converse is not generally true. Additionally, we investigate bipolar fuzzy SBE-ideals, proving that each bipolar fuzzy SBE-ideal is a bipolar fuzzy SBE-subalgebra, but not conversely. We present necessary and sufficient conditions under which the level sets of bipolar fuzzy SBE-ideals form ordinary SBE-ideals. Several illustrative examples accompany the theoretical results to demonstrate their applicability.

1 Introduction and Preliminaries

In 1913, Sheffer [22] introduced a new operation called a Sheffer, also known as the Sheffer stroke or NAND operator. It is well established that the Sheffer operation, when used alone and without the need for any other logical operators, is sufficient to construct a complete logical system. As a result, any axiom of a logical system can be restated solely using the Sheffer operation. In 1965, Zadeh [24] introduced the notion of fuzzy sets and it has been applied to many branches in mathematics. In 1998, Zhang [25] defined the concept of bipolar fuzzy sets, which combines polarity and fuzziness into a unified model and provides a theoretical basis for bipolar clustering and multi agent coordination. In 2000, Lee [13] introduced an extension of fuzzy sets named bipolar valued fuzzy sets. In 2004, Lee [12] investigated the relationship between interval valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets and it is shown that bipolar valued fuzzy sets can represent the satisfaction degree to counter property, but they can not express uncertainties in assigning membership degree. In 2009, He applied it to the BCK/BCI-algebras, and defined the concepts of bipolar fuzzy subalgebras and bipolar fuzzy ideals of a BCK/BCI-algebra (see, [11]). In 2007, Kim and Kim [8] introduced a wide class of abstract algebras: BE-algebras, as a generalization of (dual) BCK-algebras. In 2021, Prabhakar et al. [3] investigated the properties of ideals in transitive BE-algebras. In 2022, Kumar et al. [10] introduced the concept of radical filters in transitive BE-algebras. In 2011, Ahn et al. [1] fuzzified the concept of BE-algebras and related properties are investigated. They discussed characterizations of fuzzy BE-algebras in terms of level subalgebras of fuzzy BE-algebras. In 2014, Sambasiva Rao [18] studied the notions (fuzzification) of fantastic filters in BE-algebras and provided a necessary and sufficient condition for a filter of a BE-algebra to become a fantastic filter. In 2017, He [17] introduced the concept of fuzzy transitive filters with respect to fuzzy relations in BE-algebras. In 2022, Katican et al. [7] introduced the notion of a Sheffer stroke BE-algebra (briefly, SBE-algebra) and investigated a relationship between SBE-algebras and BE-algebras. In 2023, Oner et al. [15] considered the notion of Sheffer stroke BE-algebras and gave the new mathematical tools for dealing with uncertainties. While there are many papers about extensions of fuzzy BE-algebras (see, [4]-[6], [9], [2]-[21], [23]), we can not find papers about bipolar fuzzy Sheffer stroke BE-algebras. This serves as a motivation to extend the concept to the case of a BE-algebra.

In this paper, we define a bipolar fuzzy SBE-subalgebra and a level set of a bipolar fuzzy set on Sheffer stroke BE-algebras. It appears that these concepts are integral to understanding the behavior of bipolar fuzzy logic within the framework of Sheffer stroke BE-algebras. The study establishes a relationship between subalgebras and level sets in Sheffer stroke BE-algebras. Specifically, it proves that the level set of bipolar fuzzy SBE-subalgebras on this algebra is its subalgebra, and vice versa. This indicates a tight connection between these two concepts within the given algebraic structure. This suggests that there is a well-defined structure and order among these subalgebras, allowing for systematic analysis. The study describes a bipolar fuzzy SBE-ideal of a Sheffer stroke BE-algebra and provides some of its properties. Additionally, it is shown that every bipolar fuzzy SBE-ideal of a Sheffer stroke BE-algebra is also its bipolar fuzzy SBE-subalgebra, though the inverse is generally not true. This highlights the specific characteristics and behavior of bipolar fuzzy SBE-ideals within the given algebraic context. Of course, these results are not entirely novel, but these are extended by fuzzy theory to get more useful results.

In the following, we cite some elementary aspects that will be used in the sequel of this paper.

Definition 1.1. ([22]) Let $H = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:
(S1) $(\forall x, y \in H)\ (x|y = y|x)$,
(S2) $(\forall x, y \in H)\ ((x|x)|(x|y) = x)$,
(S3) $(\forall x, y, Z \in H)\ (x|((y|z)|(y|z)) = ((x|y)|(x|y))|z)$,
(S4) $(\forall x, y \in H)\ ((x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x)$.

Definition 1.2. ([7]) A Sheffer stroke BE-algebra (briefly, SBE-algebra) is a structure $(A, |, 1)$ of type (2, 0) such that 1 is the constant element in A and the Sheffer stroke $|$ that satisfies the following axioms:
(SBE-1) $(\forall x \in A)\ (x|(x|x) = 1)$,
(SBE-2) $(\forall x, y, z \in A)\ (x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z))))$.

Let $(A, |, 1)$ be a SBE-algebra. Define a relation $\leq$ on A by

$$x \leq y \Leftrightarrow x|(y|y) = 1,$$

for all $x, y \in A$. The ralation $\leq$ is not a partial order on A. It is only a reflexive relation on A (see ([7]).

Definition 1.3. ([7]) Let $(A, |, 1)$ be a SBE-algebra. A nonempty subset G of A is called an SBE-subalgebra of A if $x|(y|y) \in G$, for all $x, y \in G$.

Definition 1.4. ([7]) Let $(A, |, 1)$ be a SBE-algebra. A nonempty subset F of A is called a SBE-filter of A if it satisfies the following properties:

(i) $1 \in F$,

(ii) For all $x, y \in F$, $x|(y|y) \in F$ and $x \in F$ imply $y \in F$.

Definition 1.5. Let $(A, |, 1)$ be a SBE-algebra. A nonempty subset F of A is called an implicative SBE-filter of A if it satisfies the following properties:

(i) $1 \in F$,

(ii) For all $x, y, z \in F$, $(x|((y|(z|z))|(y|(z|z))) \in F$ and $(x|(y|y)) \in F$ imply $(x|(z|z)) \in F$.

Definition 1.6. ([7]) Let $(A, |, 1)$ be a SBE-algebra, $x, y \in A$ and define $U(x, y) = \{z \in A : x|((y|(z|z))|(y|(z|z))) = 1\}$. Then $U(x, y)$ is called an upper set of x and y. For $x, y \in A$, $U(x, y)$ is not a SBE-filter of A in general.

Definition 1.7. ([4]) Let $(A, |, 1)$ be a SBE-algebra. A nonempty subset I of A is called a SBE-ideal of A if it satisfies the following properties:
(SBEI-1) $(\forall x \in X)(\forall a \in I)$, $x|(a|a) \in I$,
(SBEI-2) $(\forall x \in X)(\forall a, b \in I)$, $(a|((b|(x|x))|(b|(x|x))))|(x|x) \in I$.

Definition 1.8. ([15]) Let $(A, |, 1)$ be a SBE-algebra. A fuzzy set μ in A is called a fuzzy SBE-subalgebra of A if it satisfies:

$$(\forall x, y \in A)\left(\ \mu(x|(y|y)) \geq \min\{\mu(x), \mu(y)\}\ \right).$$

Definition 1.9. ([4]) Let $(A, |, 1)$ be a SBE-algebra. A fuzzy set μ in A is called a fuzzy SBE-ideal of A if it satisfies:

$$(\forall x, y, z \in L)\left(\begin{array}{l} \mu(x|(y|y)) \geq \mu(y) \\ \mu((x|((y|(z|z))|(y|(z|z))))|(z|z)) \geq \min\{\mu(x), \mu(y)\} \end{array}\right).$$

Definition 1.10. ([15]) Let $(A, |, 1)$ be a SBE-algebra. A fuzzy set μ in A is called a fuzzy SBE-filter of A if
(a) $(\forall x \in A)\ \mu(1) \geq \mu(x))$,
(b) $(\forall x, y \in A)\ \mu(y) \geq \min\{\mu(x|(y|y)), \mu(x)\}$.

Lemma 1.11. ([7]) Let $(A, |, 1)$ be a SBE-algebra. Then

(i) $x|(1|1) = 1$,

(ii) $1|(x|x) = x$,

(iii) $x|((y|(x|x))|(y|(x|x))) = 1$,

(iv) $x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 1$,

(v) $(x|1)|(x|1) = x$,

(vi) $((x|y)|(x|y))|(x|x) = 1$ and $((x|y)|(x|y))|(y|y) = 1$,

(vii) $x|((x|y)|(x|y)) = x|y = ((x|y)|(x|y))|y$.

Definition 1.12. ([7]) A SBE-algebra $(A, |, 1)$ is called

(i) commutative if $(x|(y|y))|(y|y) = (y|(x|x))|(x|x)$, for all $x, y \in A$,

(ii) self-distributive if $x|((y|(z|z))|(y|(z|z))) = (x|(y|y))|((x|(z|z))|(x|(z|z)))$ for all $x, y, z \in A$.

Definition 1.13. A SBE-algebra $(A, |, 1)$ is called transitive if

$$(\forall x, y, z \in A)((y|(z|z)) \leq ((x|(y|y))|((x|(z|z))|(x|(z|z)))))$$

.

Lemma 1.14. ([7]) Let $(A, |, 1)$ be a SBE-algebra. Then

(i) If $x \leq y$, then $y|y \leq x|x$,

(ii) $x \leq y|(x|x)$,

(iii) $y \leq (y|(x|x))|(x|x)$,

(iv) If S is self-distributive, then $x \leq y$ implies $y|z \leq x|z$,

(v) If S is self-distributive, then $y|(z|z) \leq (z|(x|x))|((y|(x|x))|(y|(x|x)))$.

Definition 1.15. ([25]) Let L be a nonempty set. A bipolar fuzzy set B in L is an object having the form $B = \{(x, f^-(x), f^+(x)) \mid x \in L\}$, where $f^+ : X \to [0, 1]$ and $f^- : X \to [-1, 0]$ are mappings. We use the positive membership degree $f^+(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $f^-(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $f^+(x) \neq 0$ and $f^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $f^+(x) = 0$ and $f^-(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x to be such that $f^+(x) = 0$ and $f^-(x) = 0$ when the membership function of the property overlaps that of its counter property over some portion of L. For the sake of simplicity, we shall use the symbol $f = (L, f^-, f^+)$ for the bipolar fuzzy set $B = \{(x, f^-(x), f^+(x)) \mid x \in L\}$.

Lemma 1.16. ([23]) Let $a, b, c \in \mathbb{R}$. Then the following statements hold:

(i) $a - \min\{b, c\} = \max\{a - b, a - c\}$,

(ii) $a - \max\{b, c\} = \min\{a - b, a - c\}$.

2 Bipolar fuzzy sets in Sheffer stroke BE-algebras

In this section, the paper present the notions of bipolar fuzzy SBE-subalgebras in the framework of Sheffer stroke BE-algebras. It is important to mention that, unless specified differently, the symbol L denotes a Sheffer stroke BE-algebra throughout the discussion.

Definition 2.1. A bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is called a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$ if

$$(\forall x, y \in L) \left(\begin{array}{l} f^-(x|(y|y)) \leq \max\{f^-(x), f^-(y)\} \\ f^+(x|(y|y)) \geq \min\{f^+(x), f^+(y)\} \end{array} \right). \tag{2.1}$$

Example 2.2. Let $L = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with the binary operation "|" given in the following table:

Table 1. Sheffer Stroke BE-algebra $(L, |)$

$\mid$	0	1	2	3	4	5	6	7
0	4	4	6	1	1	1	1	3
1	4	5	6	7	0	1	2	3
2	6	6	6	1	1	1	1	6
3	1	7	1	7	0	1	7	1
4	1	0	1	0	0	1	0	1
5	1	1	1	1	1	1	1	1
6	1	2	1	7	0	1	2	1
7	3	3	6	1	1	1	1	3

Then $(L, |)$ is a Sheffer stroke BE-algebra. Define a bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L by the table below:

Table 2. Bipolar-valued fuzzy set $f = (L, f^-, f^+)$

L	0	1	2	3	4	5	6	7
f^-	–0.7	–0.9	–0.7	–0.7	–0.7	–0.7	–0.8	–0.7
f^+	0.8	0.85	0.8	0.7	0.8	0.8	0.8	0.7

It is straightforward to verify that the bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$.

Definition 2.3. Let $f = (L, f^-, f^+)$ be a bipolar fuzzy set on an SBE-algebra L and $(s, t) \in [-1, 0] \times [0, 1]$. The sets $L(f^-, s) = \{x \in L : f^-(x) \leq s\}$ and $U(f^+, t) = \{x \in L : f^+(x) \geq t\}$ are called negative s-cut of $f = (L, f^-, f^+)$ and positive t-cut of $f = (L, f^-, f^+)$, respectively.

Example 2.4. Let $f = (L, f^-, f^+)$ be the bipolar fuzzy set given in Example 2.2. For $s = -0.8$,

$$L(f^-, -0.8) = \{x \in L \mid f^-(x) \leq -0.8\} = \{1, 6, \}.$$

For $t = 0.8$,

$$U(f^+, 0.8) = \{x \in L \mid f^+(x) \geq 0.8\} = \{0, 1, 2, 4, 5, 6\}.$$

Theorem 2.5. *A bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$ if and only if its negative s-cut and positive t-cut are SBE-subalgebras of $\mathcal{L} = (L, |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

Proof. Assume that $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$ and $L(f^-, s) \neq \emptyset \neq U(f^+, t)$ for all $(s, t) \in [-1, 0] \times [0, 1]$.
Let $x, y, a, b \in L$ be such that $(x, a) \in L(f^-, s) \times U(f^+, t)$ and $(y, b) \in L(f^-, s) \times U(f^+, t)$. Hence $f^-(x) \leq s$, $f^-(y) \leq s$, $f^+(a) \geq t$ and $f^+(b) \geq t$. Then we get

$$f^-(x|(y|y)) \leq \max\{f^-(x), f^-(y)\} \leq s$$

and

$$f^+(a|(b|b)) \geq \min\{f^+(a), f^+(b)\} \geq t.$$

Thus, $(x|(y|y), a|(b|b)) \in L(f^-, s) \times U(f^+, t)$. Therefore, $L(f^-, s)$ and $U(f^+, t)$ are SBE-subalgebras of $\mathcal{L} = (L, |)$.
Conversely, let $f = (L, f^-, f^+)$ be a bipolar-valued fuzzy set in L for which its negative s-cut and positive t-cut are SBE-subalgebras of $\mathcal{L} = (L, |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.
Suppose that $f^-(a|(b|b)) > \max\{f^-(a), f^-(b)\}$ or $f^+(x|(y|y)) < \min\{f^+(x), f^+(y)\}$ for some $a, b, x, y \in L$. Then $a, b \in L(f^- s)$ or $x, y \in U(f^+, t)$ where $s = \max\{f^-(a), f^-(b)\}$ and $t = \min\{f^+(x), f^+(y)\}$. But $a|(b|b) \notin L(f^-, s)$ or $(x|(y|y)) \notin U(f^+, t)$, a contradiction. Therefore, $f^-(a|(b|b)) \leq \max\{f^-(a), f^-(b)\}$ and $f^+(x|(y|y)) \geq \min\{f^+(x), f^+(y)\}$ for all $a, b, x, y \in L$. Consequently, $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. □

Theorem 2.6. *A bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$ if and only if the fuzzy sets f_c^- and f^+ are fuzzy SBE-subalgebras of $\mathcal{L} = (L, |)$, where $f_c^- : L \to [0, 1]$, $x \mapsto 1 - f^-(x)$.*

Proof. Assume that $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. It is clear that f^+ is a fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. For every $x, y \in L$,

$$\begin{aligned} f_c^-(x|(y|y)) &= 1 - f^-(x|(y|y)) \\ &\geq 1 - \max\{f^-(x), f^-(y)\} \\ &= \min\{1 - f^-(x), 1 - f^-(y)\} \\ &= \min\{f_c^-(x), f_c^-(y)\}. \end{aligned}$$

Hence f_c^- is a fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. Conversely, let $f = (L, f^-, f^+)$ be a bipolar-valued fuzzy set of $\mathcal{L} = (L, |)$ for which f_c^- and f^+ are fuzzy SBE-subalgebras of $\mathcal{L} = (L, |)$. Let $x, y \in L$. Then

$$\begin{aligned} 1 - f^-(x|(y|y)) &= f_c^-(x|(y|y)) \\ &\geq \min\{f_c^-(x), f_c^-(y)\} \\ &= \min\{1 - f^-(x), 1 - f^-(y)\} \\ &= 1 - \max\{f^-(x), f^-(y)\} \\ f^-(x|(y|y)) &\leq \max\{f^-(x), f^-(y)\}. \end{aligned}$$

Hence $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. □

Theorem 2.7. *Given a nonempty subset F of L, let $f_F = (L, f_F^-, f_F^+)$ be a bipolar-valued fuzzy set in L defined as follows:*

$$f_F^- : L \to [-1,0], a \mapsto \begin{cases} s^- & if\ a \in F, \\ t^- & otherwise, \end{cases}$$

and

$$f_F^+ : L \to [0,1], x \mapsto \begin{cases} s^+ & if\ x \in F, \\ t^+ & \text{otherwise.} \end{cases}$$

where $s^- < t^-$ in $[-1,0]$ and $s^+ > t^+$ in $[0,1]$. Then $f_F = (L, f_F^-, f_F^+)$ be a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$ if and only if F is a SBE-subalgebra of $\mathcal{L} = (L, |)$. Moreover, we have

$$F = L_{f_F} = \{x \in L : f_F^-(x) = f_F^-(1), f_F^+(x) = f_F^+(1)\}.$$

Proof. Assume that $f_F = (L, f_F^-, f_F^+)$ is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. Let $x, y \in L$ be such that $x, y \in F$. Then we have:

$$f^-(x|(y|y)) \leq \max\{f^-(x), f^-(y)\} = s^-,$$

$$f^+(x|(y|y)) \geq \min\{f^+(x), f^+(y)\} = s^+,$$

and so $f^-(x|(y|y)) = s^-$ and $f^+(x|(y|y)) = s^+$. This shows that $(x|(y|y)) \in F$. Therefore F is a SBE-subalgebra of $\mathcal{L} = (L, |)$. Conversely, let F be a SBE-subalgebra of $\mathcal{L} = (L, |)$ and $x, y \in L$.

If $x, y \in F$, then $(x|(y|y))|(x|(y|y)) \in F$, so $f^-(x|(y|y)) = s^- = \max\{f^-(x), f^-(y)\}$ and $f^+(x|(y|y)) = s^+ = \min\{f^-(x), f^-(y)\}$.

If $x \notin F$ of $y \notin F$, then $f^-(x|(y|y)) \leq t^- = \max\{f^-(x), f^-(y)\}$ and $f^+(x|(y|y)) \geq t^+ = \min\{f^-(x), f^-(y)\}$. Therefore, $f_F = (L, f_F^-, f_F^+)$ is a bipolar-valued fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. Since F is a SBE-subalgebra of $\mathcal{L} = (L, |)$, we get

$$\begin{aligned} L_{f_F} &= \{x \in L : f_F^-(x) = f_F^-(1), f_F^+(x) = f_F^+(1)\} \\ &= \{x \in L : f_F^-(x) = s^-, f_F^+(x) = s^+\} \\ &= \{x \in L : x \in F\} \\ &= F. \end{aligned}$$

□

Lemma 2.8. *Let $f = (L, f^-, f^+)$ be a bipolar fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. Then*

$$(\forall\, x \in L) \left(\begin{array}{c} f^-(1) \leq f^-(x) \\ f^+(1) \geq f^+(x) \end{array} \right). \tag{2.2}$$

Proof. Let $f = (L, f^-, f^+)$ be a bipolar fuzzy SBE-subalgebra of $\mathcal{L} = (L, |)$. Then by (SBE-1),

$$f^-(1) = f^-(x|(x|x)) \leq \max\{f^-(x), f^-(x)\} = f^-(x),$$

$$f^+(1) = f^+(x|(x|x)) \geq \min\{f^+(x), f^+(x)\} = f^+(x),$$

for all $x \in L$. □

Proposition 2.9. *A bipolar fuzzy SBE-subalgebra $f = (L, f^-, f^+)$ of a SBE-algebra L satisfies $f^+(x) \leq f^+(x|(y|y))$, $f^-(x|(y|y)) \leq f^-(x)$ for all $x, y \in L$ if and only if f^+ and f^- are constants.*

Proof. Let f be a bipolar fuzzy SBE-subalgebra of a SBE-algebra L satisfying $f^+(x) \leq f^+(x|(y|y))$, $f^-(x|(y|y)) \leq f^-(x)$ for all $x, y \in L$. Since $f^+(1) \leq f^+(1|(x|x)) = f^+(x)$ and $f^-(x) = f^-(1|(x|x)) \leq f^-(1)$ from Lemma 1.11 (2), it follows from Lemma 2.8 that $f^+(x) = f^+(1)$ and $f^-(x) = f^-(1)$ for all $x \in L$. Hence f^+ and f^- are constants. Conversely, it is obvious by the fact that f^+ and f^- are constants. □

3 On bipolar fuzzy SBE-filters in SBE-algebras

In this section, we introduce the notions of bipolar fuzzy SBE-filters in the framework of Sheffer stroke BE-algebras and investigate their related properties.

Definition 3.1. A bipolar fuzzy set f on an SBE-algebra L is called a bipolar fuzzy SBE-filter of L if

$$(\forall x, y \in L)\left(\begin{array}{l} f^-(1) \leq f^-(y) \leq \max\{f^-(x|(y|y)), f^-(x)\} \\ f^+(1) \geq f^+(y) \geq \min\{f^+(x|(y|y)), f^+(x)\} \end{array}\right).$$

Example 3.2. Let $(L, |)$ be a Sheffer stroke BE-algebra given in Example 2.2. Define a bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L by the table below:

Table 3. Bipolar-valued fuzzy set $f = (L, f^-, f^+)$

L	0	1	2	3	4	5	6	7
f^-	−0.72	−0.85	−0.72	−0.83	−0.83	−0.72	−0.83	−0.72
f^+	0.2	0.9	0.2	0.1	0.1	0.1	0.1	0.2

It is routine to verify that the bipolar fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar fuzzy SBE-filter of $\mathcal{L} = (L, |)$.

Lemma 3.3. *A bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-filter of $\mathcal{L} = (L, |)$ if and only if*

$$(\forall\, x, y \in L)\left(x \leq y \;\Rightarrow \left\{\begin{array}{l} f^-(y) \leq f^-(x) \\ f^+(x) \leq f^+(y) \end{array}\right.\right), \tag{3.1}$$

$$(\forall x, y \in L)\left(\begin{array}{l} f^-((x|y)|(x|y)) \leq \max\{f^-(x), f^-(y)\} \\ f^+((x|y)|(x|y)) \geq \min\{f^+(x), f^+(y)\} \end{array}\right). \tag{3.2}$$

Proof. Assume that $x \leq y$. Then $x|(y|y) = 1$. Thus,

$$f^-(y) \leq \max\{f^-(x|(y|y)), f^-(x)\} = \max\{f^-(1), f^-(x)\} \leq f^-(x)$$

and

$$f^+(x) = \min\{f^+(1), f^+(x)\} = \min\{f^+(x|(y|y)), f^+(x)\} \leq f^+(y).$$

Since $y \leq y|(((x|x)|(x|x))|((x|x)|(x|x))) = x|(x|y)$,

$$\begin{array}{rcl} f^-((x|y)|(x|y)) & \leq & \max\{f^-(x), f^-(x|(((x|y)|(x|y))|((x|y)|(x|y))))\} \\ & = & \max\{f^-(x), f^-(x|(x|y))\} \\ & \leq & \max\{f^-(x), f^-(y)\} \end{array}$$

and

$$\begin{array}{rcl} f^+((x|y)|(x|y)) & \geq & \min\{f^+(x), f^+(x|(((x|y)|(x|y))|((x|y)|(x|y))))\} \\ & = & \min\{f^+(x), f^+(x|(x|y))\} \\ & \geq & \min\{f^+(x), f^+(y)\} \end{array}$$

for all $x, y \in L$. Conversely, let f be a bipolar fuzzy set on L satisfying (1) and (2). Since $x \leq 1$, $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$, we get $f^-(1) \leq f^-(x)$. Also, $((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = (x|(y|y))|((x|(y|y))|(x|(y|y))) = 1$. It follows that $(x|(x|(y|y)))|(x|(x|(y|y))) \leq y$. Then

$$f^+(y) \geq f^+((x|(x|(y|y)))|(x|(x|(y|y)))) \geq \min\{f^+(x|(y|y)), f^+(x)\}$$

and

$$f^-(y) \leq f^-((x|(x|(y|y)))|(x|(x|(y|y)))) \leq \max\{f^-(x|(y|y)), f^-(x)\}$$

for all $x, y \in L$. Therefore, f is a bipolar fuzzy SBE-filter of L. □

Lemma 3.4. *Let f be a bipolar fuzzy SBE-filter of a SBE-algebra L. Then*

(i) $f^-(y|(x|x)) \leq f^-(x)$ *and* $f^+(y|(x|x)) \geq f^+(x)$,

(ii) $f^-(x|(y|y)) \leq \max\{f^-(x), f^-(y)\}$ *and* $f^+(x|(y|y)) \geq \min\{f^+(x), f^+(y)\}$,

(iii) $f^-((x|(y|y))|(y|y)) \leq f^-(x)$ *and* $f^+((x|(y|y))|(y|y)) \geq f^+(x)$,

(iv) $f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \leq \max\{f^-(x), f^-(y)\}$ *and* $f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) \geq \min\{f^+(x), f^+(y)\}$.

for all $x, y, z \in L$.

Proof. Let f be a bipolar fuzzy SBE-filter of a SBE-algebra L. Then
(i). It is proved from Lemma 1.14 (2) and (3.1).
(ii). It follows from (1) that $f^-(x|(y|y)) \leq f^-(y) \leq \max\{f^-(x), f^-(y)\}$ and $f^+(x|(y|y)) \geq f^+(y) \geq \min\{f^+(x), f^+(y)\}$ for all $x, y \in L$.
(iii). We get from Lemma 1.14 (3) and (3.1).
(iv). It is obtained from (3) and (SBE-2) that
$f^-((x|((y|(z|z))|(y|(z|z))))|(z|z))$

$$\begin{aligned} &\leq \max\{f^-(x), f^-(x|(((x|((y|(z|z))|(y|(z|z))))|(z|z))|((x|((y|(z|z))| \\ &\quad (y|(z|z))))|(z|z))))\} \\ &= \max\{f^-(x), f^-((y|((x|(z|z))|(x|(z|z))))|((x|(z|z))|(x|(z|z))))\} \\ &\leq \max\{f^-(x), f^-(y)\}, \end{aligned}$$

$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z))$

$$\begin{aligned} &\geq \min\{f^+(x), f^+(x|(((x|((y|(z|z))|(y|(z|z))))|(z|z))|((x|((y|(z|z))| \\ &\quad (y|(z|z))))|(z|z))))\} \\ &= \min\{f^+(x), f^+((y|((x|(z|z))|(x|(z|z))))|((x|(z|z))|(x|(z|z))))\} \\ &\geq \min\{f^+(x), f^+(y)\}, \end{aligned}$$

for all $x, y, z \in L$. □

Theorem 3.5. *Let f be a bipolar fuzzy set on a SBE-algebra L. Then f is a bipolar fuzzy SBE-filter of L if and only if*

$$(\forall\, x, y, z \in L)\left(z \in U(x,y) \Rightarrow \begin{cases} f^-(z) \leq \max\{f^-(x), f^-(y)\} \\ f^+(z) \geq \min\{f^+(x), f^+(y)\} \end{cases} \right) \quad (3.3)$$

Proof. Assume that f is a bipolar fuzzy SBE-filter of L and $z \in U(x,y)$. Since $x|((y|(z|z))|(y|(z|z))) = 1$, we get $x \leq y|(z|z)$. Then it follows from (3.1) that

$$f^-(z) \leq \max\{f^-(y|(z|z)), f^-(y)\} \leq \max\{f^-(x), f^-(y)\}$$

and

$$f^+(z) \geq \min\{f^+(y|(z|z)), f^+(y)\} \geq \min\{f^+(x), f^+(y)\}$$

for all $x, y, z \in L$. Conversely, let f be a bipolar fuzzy set on L satisfying the condition (3.3). Since $x|((x|(1|1))|(x|(1|1))) = 1$ from Lemma 1.11 (1), we have that $1 \in U(x,x)$ for all $x \in L$. Then $f^-(1) \leq \max\{f^-(x), f^-(x)\} = f^-(x)$ and $f^+(1) \geq \min\{f^+(x), f^+(x)\} = f^+(x)$ for all $x \in L$. Since $x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 1$, $y \in U(x, x|(y|y))$. Then by (3.3), $f^-(y) \leq \max\{f^-(x|(y|y)), f^-(x)\}$ and $f^+(y) \geq \min\{f^+(x|(y|y)), f^+(x)\}$ for all $x, y \in L$. Hence, f is a bipolar fuzzy SBE-filter of L. □

Definition 3.6. A bipolar fuzzy set f on a SBE-algebra L is called an implicative bipolar fuzzy SBE-filter of L if it satisfies

$$(\forall x, y \in L)\begin{pmatrix} f^{-}(1) \leq f^{-}(x) \\ f^{+}(1) \geq f^{+}(x) \end{pmatrix}. \tag{3.4}$$

$$(\forall x, y, z \in L)\begin{pmatrix} f^{-}(x|(z|z)) \leq \max\{f^{-}(x|((y|(z|z))|(y|(z|z)))), f^{-}(x|(y|y))\} \\ f^{+}(x|(z|z)) \geq \min\{f^{+}(x|((y|(z|z))|(y|(z|z)))), f^{+}(x|(y|y))\} \end{pmatrix}. \tag{3.5}$$

Example 3.7. Let $(L, |)$ be a Sheffer stroke BE-algebra given in Example 2.2. Define a bipolar-valued fuzzy set $f = (L, f^{-}, f^{+})$ in L by the table below:

Table 4. Bipolar-valued fuzzy set $f = (L, f^{-}, f^{+})$

L	0	1	2	3	4	5	6	7
f^{-}	−0.72	−0.85	−0.72	−0.83	−0.83	−0.72	−0.83	−0.72
f^{+}	0.8	0.85	0.8	0.6	0.6	0.6	0.6	0.8

It is routine to verify that the bipolar fuzzy set $f = (L, f^{-}, f^{+})$ in L is an implicative bipolar fuzzy SBE-filter of $\mathcal{L} = (L, |)$.

Lemma 3.8. *Every implicative bipolar fuzzy SBE-filter of a SBE-algebra L is a bipolar fuzzy SBE-filter of L.*

Proof. Assume that f is an implicative bipolar fuzzy SBE-filter of L. Then $f^{-}(1) \leq f^{-}(x)$ and $f^{-}(1) \geq f^{-}(x)$ for all $x \in L$. Since

$$f^{-}(y) = f^{-}(1|(y|y)) \leq \max\{f^{-}(1|((x|(y|y))|(x|(y|y)))), f^{-}(1|(x|x))\}$$

and

$$f^{+}(y) = f^{+}(1|(y|y)) \geq \min\{f^{+}(1|((x|(y|y))|(x|(y|y)))), f^{+}(1|(x|x))\}$$

for all $x, y, z \in L$, f is a bipolar fuzzy SBE-filter of L. □

The converse of Lemma 3.8 need not be true.

Example 3.9. Let $L = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ be a set with the binary operation "|" given in the following table:

$\vert$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	3	3	1	1	3	3	1	1	1	1	3	1	1	3
1	3	6	7	0	8	9	1	2	4	5	11	10	13	12
2	1	7	7	1	7	7	1	1	1	1	7	1	1	7
3	1	0	1	0	10	10	1	13	4	5	4	10	13	1
4	3	8	7	10	8	8	1	10	1	1	12	10	10	12
5	3	9	7	10	8	9	1	10	1	1	12	10	10	12
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	2	1	13	10	10	1	2	4	5	4	10	13	1
8	1	4	1	4	1	1	1	4	4	5	4	1	4	1
9	1	5	1	5	1	1	1	5	5	5	5	1	5	1
10	3	11	7	4	12	12	1	4	4	5	11	1	4	12
11	1	10	1	10	10	10	1	10	1	1	1	10	10	1
12	1	13	1	13	10	10	1	13	4	5	4	10	13	1
13	3	12	7	1	12	12	1	1	1	1	12	1	1	12

Then $(L, |)$ is a Sheffer stroke BE-algebra. Define a bipolar-valued fuzzy set $f = (L, f^{-}, f^{+})$ in L by the table below:

Table 5. Bipolar-valued fuzzy set $f = (L, f^-, f^+)$

L	0	1	2	3	4	5	6
f^-	−0.72	−0.85	−0.72	−0.72	−0.83	−0.83	−0.72
f^+	0.1	0.9	0.1	0.1	0.1	0.1	0.1

L	7	8	9	10	11	12	13
f^-	−0.72	0.72	−0.72	−0.72	−0.72	−0.72	−0.72
f^+	0.1	0.1	0.1	0.1	0.1	0.1	0.1

It is routine to verify that the bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-filter of $\mathcal{L} = (L, |)$. But it is not an implicative bipolar fuzzy SBE-filter of $\mathcal{L} = (L, |)$ since

$$f^-(10|(7|7)) = f^-(10|2) = f^-(7) = -0.72$$

and

$$\begin{aligned}\max\{f^-(10|((0|(7|7))|(0|(7|7)))), f^-(10|(0|0))\} &= \max\{f^-(10|((0|2)|(0|2))), f^-(10|3)\}\\ &= \max\{f^-(10|(1|1)), f^-(4)\}\\ &= \max\{f^-(10|6), f^-(4)\}\\ &= \max\{f^-(1), f^-(4)\}\\ &= \max\{-0.85, -0.83\}\\ &= -0.83.\end{aligned}$$

Therefore

$$f^-(10|(7|7)) \not\geq \max\{f^-(10|((0|(7|7))|(0|(7|7)))), f^-(10|(0|0))\}$$

Theorem 3.10. *Every bipolar fuzzy SBE-filter of a SBE-algebra L is a bipolar fuzzy SBE-subalgebra of L.*

Proof. Let f be a bipolar fuzzy SBE-filter of L. Since $((x|y)|(x|y))|(y|y) = 1$, $(x|y)|(x|y) \leq y$ for all $x, y \in L$. Then

$$f^-(x|(y|y)) \leq f^-(y) \leq f^-((x|y)|(x|y)) \leq \max\{f^-(x), f^-(y)\}$$

and

$$f^+(x|(y|y)) \geq f^+(y) \geq f^+((x|y)|(x|y)) \geq \min\{f^+(x), f^+(y)\}$$

for all $x, y \in L$. Thereby, f is a bipolar fuzzy SBE-subalgebra of L. □

Lemma 3.11. *Let f be a bipolar fuzzy SBE-subalgebra of a SBE-algebra L satisfying*

$$(\forall x, y, z \in L)\left(\begin{array}{l} f^-(x|(z|z)) \leq \max\{f^-(x|((y|(z|z))|(y|(z|z)))), f^-(x|(y|y))\} \\ f^+(x|(z|z)) \geq \min\{f^+(x|((y|(z|z))|(y|(z|z)))), f^+(x|(y|y))\} \end{array}\right). \quad (3.6)$$

Then f is a bipolar fuzzy SBE-filter of L.

Proof. Let f be a bipolar fuzzy SBE-subalgebra of L satisfying the condition (3.6). By Lemma 2.8, $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$ for all $x \in L$. Then it is obtained from Lemma 1.11 (2) that

$$\begin{aligned} f^-(y) &= f^-(1|(y|y))\\ &\leq \max\{f^-(1|((x|(y|y))|(x|(y|y)))), f^-(1|(x|x))\}\\ &= \max\{f^-(x|(y|y)), f^-(x)\}\end{aligned}$$

and

$$\begin{aligned} f^+(y) &= f^+(1|(y|y)) \\ &\geq \min\{f^+(1|((x|(y|y))|(x|(y|y)))), f^+(1|(x|x))\} \\ &= \min\{f^+(x|(y|y)), f^+(x)\} \end{aligned}$$

for all $x, y \in L$. Hence, f is a bipolar fuzzy SBE-filter of L. □

Theorem 3.12. *Let L be a self-distributive SBE-algebra. Then every bipolar fuzzy SBE-filter of L is an implicative bipolar fuzzy SBE-filter of L.*

Proof. Let f be a bipolar fuzzy SBE-filter of a self-distributive SBE-algebra L. Since f is a bipolar fuzzy SBE-filter of L, $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$ for all $x \in L$. Then $\min\{f^+(x|((y|(z|z))|(y|(z|z)))), f^+(x|(y|y))\}$

$$\begin{aligned} f^-(x|(z|z)) &\leq \max\{f^-((x|(y|y))|((x|(z|z))|(x|(z|z)))), f^-(x|(y|y))\} \\ &= \max\{f^-(x|((y|(z|z))|(y|(z|z)))), f^-(x|(y|y))\} \end{aligned}$$

$$\begin{aligned} f^+(x|(z|z)) &\geq \min\{f^+((x|(y|y))|((x|(z|z))|(x|(z|z)))), f^+(x|(y|y))\} \\ &= \min\{f^+(x|((y|(z|z))|(y|(z|z)))), f^+(x|(y|y))\} \end{aligned}$$

for all $x, y, z \in L$. Thus, f is an implicative bipolar fuzzy SBE-filter of L. □

Lemma 3.13. *Let f be a (implicative) bipolar fuzzy SBE-filter of a SBE-algebra L. Then the subsets $Lf^+ = \{x \in L : f^+(x) = f^+(1)\}, Lf^- = \{x \in L : f^-(x) = f^-(1)\}$ of L are (implicative) SBE-filters of L.*

Proof. Let f be a bipolar fuzzy SBE-filter of L. Then it is obvious that $1 \in Lf^+, Lf^-$. Assume that $x, x|(y|y) \in Lf^+, Lf^-$. Since $f^-(x) = f^-(1) = f^-(x|(y|y))$ and $f^+(x) = f^+(1) = f^+(x|(y|y))$, we get

$$f^-(y) \leq \max\{f^-(x|(y|y)), f^-(x)\} = \max\{f^-(1), f^-(1)\} = f^-(1)$$

and

$$f^+(y) \geq \min\{f^+(x|(y|y)), f^+(x)\} = \min\{f^+(1), f^+(1)\} = f^-(1)$$

which imply that $f^-(y) = f^-(1)$ and $f^+(y) = f^+(1)$. Then $y \in Lf^+, Lf^-$. Hence, Lf^+, Lf^- are SBE-filters of S. Let f be an implicative bipolar fuzzy SBE-filter of L. Suppose that $x|((y|(z|z))|(y|(z|z))), x|(y|y) \in Lf^+, Lf^-$. Since $f^-(x|((y|(z|z))|(y|(z|z)))) = f^-(1) = f^-(x|(y|y))$ and
$f^+(x|((y|(z|z))|(y|(z|z)))) = f^+(1) = f^+(x|(y|y))$, we have

$$\begin{aligned} f^-(x|(z|z)) &\leq \max\{f^-(x|((y|(z|z))|(y|(z|z)))), f^-(x|(y|y))\} \\ &= \max\{f^-(1), f^-(1)\} \\ &= f^-(1), \end{aligned}$$

$$\begin{aligned} f^+(x|(z|z)) &\geq \min\{f^+(x|((y|(z|z))|(y|(z|z)))), f^+(x|(y|y))\} \\ &= \min\{f^+(1), f^+(1)\} \\ &= f^+(1), \end{aligned}$$

which imply that $f^+(x|(z|z)) = f^+(1)$ and $f^-(x|(z|z)) = f^-(1)$. It follows that $x|(z|z) \in Lf^+, Lf^-$. Therefore, Lf^+, Lf^- are implicative SBE-filters of L. □

4 Ideals theory in SBE-algebras based on bipolar fuzzy sets

In this section, we define bipolar fuzzy SBE-ideals in Sheffer stroke BE-algebras and we prove that any bipolar fuzzy SBE-ideal in this algebra is also classified as a bipolar fuzzy SBE-subalgebra, though the converse does not generally hold.

Definition 4.1. A bipolar fuzzy set f on an SBE-algebra L is called a bipolar fuzzy SBE-ideal of L if

$$(\forall x, y \in L)\left(\begin{array}{c} f^-(x|(y|y)) \leq f^-(y) \\ f^+(x|(y|y)) \geq f^+(y) \end{array}\right), \tag{4.1}$$

$$(\forall x, y, z \in L)\left(\begin{array}{c} f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \leq \max\{f^-(x), f^-(y)\} \\ f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) \geq \min\{f^+(x), f^+(y)\} \end{array}\right). \tag{4.2}$$

Example 4.2. Let $(L, |)$ be a Sheffer stroke BE-algebra given in Example 2.2. Define a bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L by the table below:

Table 6. Bipolar-valued fuzzy set $f = (L, f^-, f^+)$

L	0	1	2	3	4	5	6	7
f^-	–0.2	–0.9	–0.2	–0.1	–0.1	–0.1	–0.1	–0.2
f^+	0.8	0.85	0.8	0.6	0.6	0.6	0.6	0.8

It is routine to verify that the bipolar fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar fuzzy SBE-ideal of $\mathcal{L} = (L, |)$.

Theorem 4.3. *Every bipolar fuzzy SBE-ideal of an SBE-algebra L is a bipolar fuzzy SBE-subalgebra of L.*

Proof. It follows from (4.1). □

The converse of Theorem 4.3 is not true in general as seen in the following example.

Example 4.4. Let $(L, |)$ be a Sheffer stroke BE-algebra and $f = (L, f^-, f^+)$ a bipolar-valued fuzzy set in L given in Example 2.2. Then $f = (L, f^-, f^+)$ is bipolar fuzzy SBE-subalgebra of L. But it is not a bipolar fuzzy SBE-ideal of L since

$$f^-((1|((6|(3|3))|(6|(3|3))))|(3|3)) = f^-(3) = -0.7$$

and

$$\max\{f^-(1), f^-(6)\} = \max\{-0.9, -0.8\} = -0.8$$

but

$$f^-((1|((6|(3|3))|(6|(3|3))))|(3|3)) \nleq \max\{f^-(1), f^-(6)\}.$$

Theorem 4.5. *A bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L, |)$ if and only if its negative s-cut and positive t-cut are SBE-ideals of $\mathcal{L} = (L, |)$ whenever they are nonempty for all $(s, t) \times [-1, 0] \times [0, 1]$.*

Proof. Assume that $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L, |)$ and $L(f^-, s) \neq \emptyset \neq U(f^+, t)$ for all $(s, t) \times [-1, 0] \times [0, 1]$. Let $x, y, a, b \in L$ be such that $(y, b) \in L(f^-, s) \times U(f^+, t)$. Then $f^-(y) \leq s$ and $f^+(b) \geq t$. Then $f^-(x|(y|y)) \leq f^-(y) \leq s$ and $f^+(a|(b|b)) \geq f^+(b) \geq t$ and so $(x|(y|y), a|(b|b)) \in L(f^-, s) \times U(f^+, t)$. Let $x, y, z, a, b, c \in L$ be such that $(x, a) \in L(f^-, s) \times U(f^+, t)$ and $(y, b) \in L(f^-, s) \times U(f^+, t)$. Then $f^-(x) \leq s$, $f^-(y) \leq s$, $f^+(a) \geq t$ and $f^+(b) \geq t$. Then we get

$$f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \leq \max\{f^-(x), f^-(y)\} \leq s$$

and

$$f^+((a|((b|(c|c))|(b|(c|c))))|(c|c)) \geq \min\{f^+(a), f^+(b)\} \geq t.$$

Thus,

$$((x|((y|(z|z))|(y|(z|z))))|(z|z), (a|((b|(c|c))|(b|(c|c))))|(c|c)) \in L(f^-, s) \times U(f^+, t).$$

Therefore, $L(f^-, s)$ and $U(f^+, t)$ are SBE-ideals of $\mathcal{L} = (L, |)$. Conversely, let $f = (L, f^-, f^+)$ be a bipolar-valued fuzzy set in L for which its negative s-cut and positive t-cut are SBE-ideals of $\mathcal{L} = (L, |)$ whenever they are nonempty for all $(s,t) \times [-1,0] \times [0,1]$. Assume that $f^-(a|(b|b)) > f^-(b)$ for some $b \in L$. Then $b \in L(f^-, f^-(b))$ but $a|(b|b) \notin L(f^-, f^-(b))$, a contradiction. Hence $f^-(x|(y|y)) \leq f^-(y)$ for all $y \in L$. Suppose that $f^+(x|(y|y)) < f^+(y)$ for some $y \in L$. Then $y \in U(f^+, f^+(y))$ but $x|(y|y) \notin U(f^+, f^+(y))$, a contradiction. Hence $f^-(x|(y|y)) \geq f^-(y)$ for all $y \in L$. Suppose that $f^-((a|((b|(c|c))|(b|(c|c))))|(c|c)) >$ $\max\{f^-(a), f^-(b)\}$ or
$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) < \min\{f^+(x), f^+(y)\}$ for some $a, b, c, x, y, z \in L$. Then $a, b \in L(f^-, s)$ or $x, y \in U(f^+ t)$ where $s = \max\{f^-(a), f^-(b)\}$ and $t = \min\{f^+(x), f^+(y)\}$. But $(a|((b|(c|c))|(b|(c|c))))|(c|c) \notin L(f^-, s)$ or $(x|((y|(z|z))|(y|(z|z))))|(z|z) \notin U(f^+, t)$, a contradiction. Therefore,

$$f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \leq \max\{f^-(x), f^-(y)\}$$

and

$$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) \geq \min\{f^+(x), f^+(y)\}$$

for all $x, y \in L$. Consequently, $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L, |)$. □

Theorem 4.6. *A bipolar-valued fuzzy set $f = (L, f^-, f^+)$ in L is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L, |)$ if and only if the fuzzy sets f_c^- and f^+ are fuzzy SBE-ideals of $\mathcal{L} = (L, |)$, where $f_c^- : L \to [0,1]$, $x \mapsto 1 - f^-(x)$.*

Proof. Assume that $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L, |)$. It is clear that f^+ is a bipolar fuzzy SBE-ideal of $\mathcal{L} = (L, |)$. For every $x, y \in L$,

$$\begin{aligned} f_c^-(x|(y|y)) &= 1 - f^-(x|(y|y)) \\ &\geq 1 - f^-(y) \\ &= f_c^-(y), \end{aligned}$$

$$\begin{aligned} f_c^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) &= 1 - f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \\ &\geq 1 - \max\{f^-(x), f^-(y)\} \\ &= \min\{1 - f^-(x), 1 - f^-(y)\} \\ &= \min\{f_c^-(x), f_c^-(y)\}. \end{aligned}$$

Hence f_c^- is a bipolar fuzzy SBE-ideal of $\mathcal{L} = (L, |)$.
Conversely, let $f = (L, f^-, f^+)$ be a bipolar-valued fuzzy set of $\mathcal{L} = (L, |)$ for which f_c^- and f^+ are fuzzy SBE-ideals of $\mathcal{L} = (L, |)$. Let $x, y \in L$. Then

$$\begin{aligned} 1 - f^-(x|(y|y)) &= f_c^-(x|(y|y)) \\ &\geq f_c^-(y) \\ &= 1 - f^-(y) \\ f^-(x|(y|y)) &\leq f^-(y), \end{aligned}$$

$$\begin{aligned} 1 - f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) &= f_c^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \\ &\geq \min\{f_c^-(x), f_c^-(y)\} \\ &= \min\{1 - f^-(x), 1 - f^-(y)\} \\ &= 1 - \max\{f^-(x), f^-(y)\} \\ f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) &\leq \max\{f^-(x), f^-(y)\}. \end{aligned}$$

Hence $f = (L, f^-, f^+)$ is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L, |)$. □

Theorem 4.7. *Given a nonempty subset F of L, let $f_F = (L, f_F^-, f_F^+)$ be a bipolar-valued fuzzy set in L defined as follows:*

$$f_F^- : L \to [-1, 0], a \mapsto \begin{cases} s^- & if\ a \in F, \\ t^- & otherwise, \end{cases}$$

and

$$f_F^+ : L \to [0,1], x \mapsto \begin{cases} s^+ & if\ x \in F, \\ t^+ & \text{otherwise.} \end{cases}$$

where $s^- < t^-$ in $[-1,0]$ and $s^+ > t^+$ in $[0,1]$. Then $f_F = (L, f_F^-, f_F^+)$ be a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L,|)$ if and only if F is a SBE-ideal of $\mathcal{L} = (L,|)$.

Proof. Assume that $f_F = (L, f_F^-, f_F^+)$ is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L,|)$. Let $x, y \in L$ be such that $x, y \in F$. Using (3.4), we have $f^-(x|(y|y)) \le f^-(y) = s^-$, $f^+(x|(y|y)) \ge f^+(y) = s^+$. Hence $f^-(x|(y|y)) = s^-$ and $f^+(x|(y|y)) = s^+$. This shows that $x|(y|y) \in F$. Also,

$$f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \le \max\{f^-(x), f^-(y)\} = s^-,$$

$$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) \ge \min\{f^+(x), f^+(y)\} = s^+,$$

and so $f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) = s^-$ and
$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) = s^+$.
This shows that $(x|((y|(z|z))|(y|(z|z))))|(z|z) \in F$. Therefore, F is a SBE-ideal of $\mathcal{L} = (L,|)$.
Conversely, let F be a SBE-ideal of $\mathcal{L} = (L,|)$.
For every $x, y \in L$, if $y \in F$, then $x|(y|y) \in F$ which implies that
$f^-(x|(y|y)) = s^- = f^-(y)$ and $f^+(x|(y|y)) = s^+ = f^-(y)$.
If $x|(y|y) \notin F$, then $f^-(x|(y|y)) = t^- > f^-(y)$ and $f^+(x|(y|y)) = t^+ < f^-(y)$. For every $x, y, z \in L$, if $x, y \in F$, then $(x|((y|(z|z))|(y|(z|z))))|(z|z) \in F$ which implies that
$f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) = s^- = \max\{f^-(x), f^-(y)\}$ and
$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) = s^+ = \min\{f^-(x), f^-(y)\}$.
If $x \notin F$ or $y \notin F$, then
$f^-((x|((y|(z|z))|(y|(z|z))))|(z|z)) \le t^- = \max\{f^-(x), f^-(y)\}$ and
$f^+((x|((y|(z|z))|(y|(z|z))))|(z|z)) \ge t^+ = \min\{f^-(x), f^-(y)\}$.
Therefore, $f_F = (L, f_F^-, f_F^+)$ is a bipolar-valued fuzzy SBE-ideal of $\mathcal{L} = (L,|)$. □

Proposition 4.8. *If $f_i = \{(f_i^+, f_i^-) : i \in \Delta\}$ is a family of bipolar fuzzy SBE-ideals of a SBE-algebra L, then $\bigwedge_{i\in\Delta} f_i$ is a bipolar fuzzy SBE-ideal of L.*

Proof. Let $f_i = \{(f_i^+, f_i^-) : i \in \Delta\}$ be a family of bipolar fuzzy SBE-ideals of a SBE-algebra L. Let $x, y \in L$, we have

$$(\bigwedge_{i\in\Delta} f_i^+)(x|(y|y)) = \inf_{i\in\Delta}\{f_i^+(x|(y|y))\} \ge \inf_{i\in\Delta}\{f_i^+(y)\} = (\bigwedge_{i\in\Delta} f_i^+)(y),$$

$$(\bigwedge_{i\in\Delta} f_i^-)(x|(y|y)) = \sup_{i\in\Delta}\{f_i^-(x|(y|y))\} \le \sup_{i\in\Delta}\{f_i^-(y)\} = (\bigwedge_{i\in\Delta} f_i^-)(y).$$

Let $x, y, z \in L$, we have
$(\bigwedge_{i\in\Delta} f_i^+)((x|((y|(z|z))|(y|(z|z))))|(z|z))$

$$\begin{aligned} &= \inf_{i\in\Delta}\{f_i^+((x|((y|(z|z))|(y|(z|z))))|(z|z))\} \\ &\ge \inf_{i\in\Delta}\{\min\{f_i^+(x), f_i^+(y)\}\} \\ &= \min\{\inf_{i\in\Delta} f_i^+(x), \inf_{i\in\Delta} f_i^+(y)\} \\ &= \min\{(\bigwedge_{i\in\Delta} f_i^+)(x), (\bigwedge_{i\in\Delta} f_i^+)(y)\}, \end{aligned}$$

$(\bigwedge_{i\in\Delta} f_i^-)((x|((y|(z|z))|(y|(z|z))))|(z|z))$

$$\begin{aligned} &= \sup_{i\in\Delta}\{f_i^-((x|((y|(z|z))|(y|(z|z))))|(z|z))\} \\ &\le \sup_{i\in\Delta}\{\max\{f_i^-(x), f_i^-(y)\}\} \\ &= \max\{\sup_{i\in\Delta} f_i^-(x), \sup_{i\in\Delta} f_i^-(y)\} \\ &= \max\{(\bigwedge_{i\in\Delta} f_i^-)(x), (\bigwedge_{i\in\Delta} f_i^-)(y)\}. \end{aligned}$$

Hence $\bigwedge_{i\in\Delta} f_i$ is a bipolar fuzzy SBE-ideal of a SBE-algebra L. □

Proposition 4.9. *If f is a bipolar fuzzy SBE-ideal of an SBE-algebra L and $a, b \in L$, then $f^-((a|(b|b))|(b|b)) \leq f^-(a)$ and $f^+((a|(b|b))|(b|b)) \geq f^+(a)$.*

Proof. Assume that f is a bipolar fuzzy SBE-ideal of L and $a, b \in L$. Using Lemma 1.11 (2) and (4.2), we get

$$\begin{aligned} f^-((a|(b|b))|(b|b)) &= f^-((a|((1|(b|b))|(1|(b|b))))|(b|b)) \\ &\leq \max\{f^-(a), f^-(1)\} \\ &= \max\{f^-(a), f^-(a|(a|a))\} \\ &\leq f^-(a), \end{aligned}$$

$$\begin{aligned} f^+((a|(b|b))|(b|b)) &= f^+((a|((1|(b|b))|(1|(b|b))))|(b|b)) \\ &\geq \min\{f^+(a), f^+(1)\} \\ &= \min\{f^+(a), f^+(a|(a|a))\} \\ &\geq f^+(a). \end{aligned}$$

□

Proposition 4.10. *Every bipolar fuzzy SBE-ideal $f = (L, f^-, f^+)$ of an SBE-algebra L satisfies the following*

$$(\forall\, a, b \in L)\left(\quad a \leq b \;\Rightarrow \begin{cases} f^-(b) \leq f^-(a) \\ f^+(b) \geq f^+(a) \end{cases} \right) \tag{4.3}$$

Proof. Let $a, b \in L$ be such that $a \leq b$. Then $a|(b|b) = 1$. From Lemma 1.11 (2) and Proposition 4.9, we have $f^-(b) = f^-(1|(b|b)) = f^-((a|(b|b))|(b|b)) \leq f^-(a)$ and $f^+(b) = f^+(1|(b|b)) = f^+((a|(b|b))|(b|b)) \geq f^+(a)$. □

Proposition 4.11. *If f is a bipolar fuzzy SBE-ideal of an SBE-algebra L, then*

$$(\forall a \in L)\left(\begin{array}{l} f^-(1) \leq f^-(a) \\ f^+(1) \geq f^+(a) \end{array} \right). \tag{4.4}$$

Proof. Let $a \in L$. By using (SBE-1) and (4.1), we have $f^-(1) = f^-(a|(a|a)) \leq f^-(a)$ and $f^+(1) = f^+(a|(a|a)) \geq f^+(a)$. □

Proposition 4.12. *Let f be a bipolar fuzzy set in an SBE-algebra L, which satisfies*

$$(\forall a, b, c \in L)\left(\begin{array}{l} f^-(1) \leq f^-(a) \\ f^+(1) \geq f^+(a) \\ f^-(a|(c|c)) \leq \max\{f^-(a|((b|(c|c))|(b|(c|c)))), f^-(b)\} \\ f^+(a|(c|c)) \geq \min\{f^+(a|((b|(c|c))|(b|(c|c)))), f^+(b)\} \end{array} \right). \tag{4.5}$$

Then

$$(\forall\, a, b \in L)\left(\quad a \leq b \;\Rightarrow \begin{cases} f^-(b) \leq f^-(a) \\ f^+(b) \geq f^+(a) \end{cases} \right) \tag{4.6}$$

Proof. Let $a, b \in L$ be such that $a \leq b$. Then

$$\begin{aligned} f^-(b) &= f^-(1|(b|b)) \\ &\leq \max\{f^-(1|((a|(b|b))|(a|(b|b)))), f^-(a)\} \\ &= \max\{f^-(1|(1|1)), f^-(a)\} \\ &= \max\{f^-(1), f^-(a)\} \\ &= f^-(a), \end{aligned}$$

$$\begin{aligned} f^+(b) &= f^+(1|(b|b)) \\ &\geq \min\{f^+(1|((a|(b|b))|(a|(b|b)))), f^+(a)\} \\ &= \min\{f^+(1|(1|1)), f^+(a)\} \\ &= \min\{f^+(1), f^+(a)\} \\ &= f^+(a). \end{aligned}$$

□

Theorem 4.13. *A bipolar fuzzy set f in a transitive SBE-algebra L is a bipolar fuzzy SBE-ideal of L if and only if it satisfies (4.5).*

Proof. Assume that f is a bipolar fuzzy SBE-ideal of L. By Proposition 4.11, we have $f^-(1) \leq f^-(a)$ and $f^+(1) \geq f^+(a)$ for all $a \in L$. Since L is transitive, we get $(b|(c|c))|(c|c) \leq (a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c)))$, for all $a, b, c \in L$. Thus,
$((b|(c|c))|(c|c))|((a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c)))|(a|((b|(c|c))|(b|(c|c)))|$
$((a|(c|c))|(a|(c|c)))) = 1$, for all $a, b, c \in L$.
We consider

$$\begin{aligned} f^-(a|(c|c)) &= f^-(1|((a|(c|c))|(a|(c|c)))) \\ &= f^-(((b|(c|c))|(c|c)|((a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c)))| \\ &\quad (a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c)))))|((a|(c|c))|(a|(c|c))))) \\ &\leq \max\{f^-((b|(c|c))|(c|c)), f^-(a|((b|(c|c))|(b|(c|c))))\} \\ &\leq \max\{f^-(a|((b|(c|c))|(b|(c|c)))), f^-(b)\}, \end{aligned}$$

and

$$\begin{aligned} f^+(a|(c|c)) &= f^+(1|((a|(c|c))|(a|(c|c)))) \\ &= f^+(((b|(c|c))|(c|c)|((a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c)))| \\ &\quad (a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c)))))|((a|(c|c))|(a|(c|c))))) \\ &\geq \min\{f^+((b|(c|c))|(c|c)), f^+(a|((b|(c|c))|(b|(c|c))))\} \\ &\geq \min\{f^+(a|((b|(c|c))|(b|(c|c)))), f^+(b)\} \end{aligned}$$

for all $a, b, c \in L$. Hence the conditions (4.5) are true. Conversely, using (4.5), (SBE-1), and Lemma 1.11 (2), we have

$$\begin{aligned} f^-(a|(b|b)) &\leq \max\{f^-(a|((b|(b|b))|(b|(b|b))), f^-(b)\} \\ &= \max\{f^-(a|(a|1)), f^-(b)\} \\ &= \max\{f^-(1), f^-(b)\} \\ &= f^-(b), \end{aligned}$$

and

$$\begin{aligned} f^+(a|(b|b)) &\geq \min\{f^+(a|((b|(b|b))|(b|(b|b))), f^+(b)\} \\ &= \min\{f^+(a|(a|1)), f^+(b)\} \\ &= \min\{f^+(1), f^+(b)\} \\ &= f^+(b) \end{aligned}$$

$$\begin{aligned} f^-((a|(b|b))|(b|b)) &\leq \max\{f^-((a|(b|b))|((a|(b|b))|(a|(b|b))), f^-(a)\} \\ &= \max\{f^-(1), f^-(a)\} \\ &= f^-(a), \end{aligned}$$

$$\begin{aligned} f^+((a|(b|b))|(b|b)) &\geq \min\{f^+((a|(b|b))|((a|(b|b))|(a|(b|b))), f^+(a)\} \\ &= \min\{f^+(1), f^+(a)\} \\ &= f^+(a) \end{aligned}$$

for all $a, b \in L$. Since f satisfies (4.5) and by Proposition 4.12, we have f is order-preserving. Since L is transitive, we see that

$f^-((b|(c|c))|(c|c)) \geq f^-(a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c))))$ and
$f^+((b|(c|c))|(c|c)) \leq f^+(a|((b|(c|c))|(b|(c|c)))|((a|(c|c))|(a|(c|c))))$,
for all $a, b, c \in L$. Hence we have
$f^-((a|((b|(c|c))|(b|(c|c))))|(c|c))$

$$\begin{aligned} &\leq \max\{f^+((a|((b|(c|c))|(b|(c|c))))|((a|(c|c))|(a|(c|c)))), f^-(a)\} \\ &\leq \max\{f^-((b|(c|c))|(c|c)), f^-(a)\} \\ &\leq \max\{f^-(a), f^-(b)\} \end{aligned}$$

and
$f^+((a|((b|(c|c))|(b|(c|c))))|(c|c))$

$$\begin{aligned} &\geq \min\{f^+((a|((b|(c|c))|(b|(c|c))))|((a|(c|c))|(a|(c|c)))), f^+(a)\} \\ &\geq \min\{f^+((b|(c|c))|(c|c)), f^+(a)\} \\ &\geq \min\{f^+(a), f^+(b)\}. \end{aligned}$$

Therefore, f is a bipolar fuzzy SBE-ideal of L. □

Corollary 4.14. *Let f be a bipolar fuzzy set in a self-distributive SBE-algebra L. Then f is a bipolar fuzzy SBE-ideal of L if and only if it satisfies conditions* (4.5)*.*

Proof. Straightforward. □

5 Conclusion

In this paper, we introduced the concept of a bipolar fuzzy SBE-subalgebra and explored the level set of a bipolar fuzzy set within the context of Sheffer stroke BE-algebras. These concepts were crucial for understanding the dynamics of bipolar fuzzy logic in this algebraic framework. We established a connection between subalgebras and level sets in Sheffer stroke BE-algebras, demonstrating that the level set corresponding to bipolar fuzzy SBE-subalgebras was indeed a subalgebra, and the reverse held true as well. This revealed a strong interrelation between these two ideas within the algebraic structure. This structure indicated a clear hierarchy and organization among these subalgebras, facilitating systematic investigation. The paper also defined a bipolar fuzzy SBE-ideal in a Sheffer stroke BE-algebra and outlined some of its key properties. Notably, we proved that any bipolar fuzzy SBE-ideal in this algebra was also classified as a bipolar fuzzy SBE-subalgebra, though the converse did not generally hold. This distinction emphasized the unique attributes and behaviors of bipolar fuzzy SBE-ideals within the algebraic context. While the results presented were not entirely new, they built upon existing knowledge in bipolar fuzzy theory to yield more applicable insights.

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