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Riesz's Ultra-Hyperbolic Kernel Associated with Weighted Distributions Defined by Quadratic Forms

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Abstract. Through the use of weighted generalised functions associated with quadratic forms [1], this paper presents a generalisation of the casual(anticasual) ultra-hyperbolic kernel $H_\alpha(P \pm i0, n)$ due to Trione (cf. [2]). Some properties of the family of the new kernel $H_\alpha[(P \pm i0)_\gamma]$ are proven, among them that $H_{2k}[(P \pm i0)_\gamma]$ is an elementary solution of the B-ultra-hyperbolic operator

$$\Box_\gamma^k = \sum_{i=1}^{p} B_i - \sum_{i=p+1}^{p+q} B_i,$$

where $B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$; $i = 1, 2, ..., n$; $p + q = n$, $\gamma_i > 0$, $i = 1, 2, ..., n$.

1 Introduction and Preliminaries

The distributions P_+^λ, P_-^λ, $(P \pm i0)^\lambda$ and $(m^2 + P \pm i0)^\lambda$ where $P = P(x)$ designs a quadratic form given by

$$P = P(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2; \tag{1.1}$$

with $p + q = n$, and n the dimension of the space, the significant contributions of Gelfand and Shilov (cf. [3]) have made it possible to express the elliptic or hyperbolic integrals of Riemann-Liouville, as treated by Marcel Riesz ([4], p 16, 55), in the form of a convolution with elliptic or hyperbolic kernels.

In [5] Nozaki introduce the $R_\alpha(P)$ family,

$$R_\alpha\left(P(x)\right) = \begin{cases} \frac{P^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{si } x \in \Gamma^+ \\ \\ 0 & \text{si } x \notin \Gamma^+, \end{cases} \tag{1.2}$$

α is a complex number, n is the dimension of the space and $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}, \tag{1.3}$$

where p is the amount of positive terms of the quadratic form (1.1) and $\Gamma^+ = \{x \in \mathbb{R}^n, P(x) > 0, x_1 > 0\}$.

As Trione [2] indicates, $R_\alpha(P)$ is an ordinary function if $\Re(\alpha) \geq n$ and if $\Re(\alpha) < n$ is a distributional function of α. $R_\alpha(P)$ is called ultra-hyperbolic kernel of Marcel Riesz.

It can be observed that if $p = 1$, in (1.2) and (1.3), after basic calculations, it is obtained

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{si } x \in \Gamma^+ \\ \\ 0 & \text{si } x \notin \Gamma^+, \end{cases} \tag{1.4}$$

where $u = x_1^2 - x_2^2 - .... - x_n^2$, and

$$H_n(\alpha) = \pi^{\frac{n-2}{2}} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha - n + 2}{2}\right). \tag{1.5}$$

$M_\alpha(u)$ is the hyperbolic kernel of Marcel Riesz (cf. [4], p 31).

In [6], some properties of the ultra-hyperbolic kernel of Marcel Riesz (1.2), are analyzed, including the result that establishes:

$$\square R_{\alpha+2}(P) = R_\alpha(P), \tag{1.6}$$

where $\square$ the ultra-hyperbolic differential operator defined by

$$\square = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{p+q} \frac{\partial^2}{\partial x_i^2}. \tag{1.7}$$

In addition, it is demonstrated that $R_{2k}(P)$ is the elemental solution of the operator (1.7) iterated k-times, i.e., we have that

$$\square^k R_{2k}(P) = R_0(P) = \delta, \tag{1.8}$$

$k = 0, 1, 2, ...$ and δ is the Dirac Delta function.

Let now the operator

$$\square_{B,\gamma} = \sum_{i=1}^{p} B_i - \sum_{i=p+1}^{p+q} B_i; \tag{1.9}$$

where

$$B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}; \;\; \gamma_1 > 0 \;\; , i = 1, 2, ..., n; \;\; p + q = n, \tag{1.10}$$

which will be called ultra-hyperbolic Bessel Operator (cf. [7]) according to the current literature (see, for example [1]). Yildirim et al [8] have proposed the following function as an elemental solution of the operator $\square_{B,\gamma}$ iterated k-times

$$R_{2k}(x) = \frac{P(x)^{\frac{2k-n-|\gamma|}{2}}}{K_n(2k)} = \frac{\left(x_1^2 + ... + x_p^2 - x_{p+1} - ... - x_{p+q}^2\right)^{\frac{2k-n-|\gamma|}{2}}}{K_n(2k)} \tag{1.11}$$

where

$$K_n(2k) = \frac{\pi^{\frac{n+|\gamma|-1}{2}} \Gamma\left(\frac{2+2k-n-|\gamma|}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p-|\gamma|}{2}\right) \Gamma\left(\frac{p+2k}{2}\right)}. \tag{1.12}$$

Aguirre [9] demonstrates that the function $R_{2k}(P(x))$ is an elementary solution of (1.9) by decomposing the operator $\square_{B,\gamma}$ in the following form

$$\square_{B,\gamma} = \square + D_p - D_q, \tag{1.13}$$

where

$$\square = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2}, \tag{1.14}$$

$$D_p = \sum_{i=1}^{p} \gamma_i \frac{\partial^2}{\partial x_i^2}, \tag{1.15}$$

$$D_q = \sum_{i=p+1}^{p+q} \gamma_i \frac{\partial^2}{\partial x_i^2}, \tag{1.16}$$

Trione [2] introduces a generalization of the Marcel Riesz kernel, extending it to generalized functions $(P \pm i0)$. Trione considers the family of the distributions $H_\alpha(P \pm i0, n)$ defined as

$$H_\alpha(P \pm i0, n) = \frac{e^{\frac{i\pi\alpha}{2}} e^{\pm\frac{\pi i q}{2}} \Gamma\left(\frac{n-\alpha}{2}\right)(P \pm i0)^{\frac{\alpha-n}{2}}}{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)} \tag{1.17}$$

where α is a complex number, P designs the quadratic form given by (1.1) and

$$(P \pm i0)^\lambda = \lim_{\epsilon \to 0} \left(P + i\epsilon|x|^2\right)^\lambda, \tag{1.18}$$

$$\epsilon|x|^2 = \epsilon(x_1^2 + ... + x_n^2), \ \ \epsilon > 0. \tag{1.19}$$

This kernel has several properties which were used to obtain causal and anticausal elemental solutions of the operator iterated k-times given by (1.7), solutions given by $H_{2k}(P \pm i0, n)$. Subsequently, C. Olivera (cf. [10]) generalized the kernel given by (1.17) for a nondegenerated quadratic form.

The purpose of this paper is to introduce an analogous kernel to the one given in (1.17) using generalised weighted functions, similar to what was done in [11], and to study some of its properties such as being an elemental solution of the ultra-hyperbolic Bessel operator given in (1.9) for a certain value of the parameter.

This article is structured as follows: in Section 2 a family of distributions is introduced, which generalises the elliptical kernel of Marcel Riesz. In addition, several theorems demonstrating key properties of these distributions are given, including their behaviour under convolution and its relationship with the iterated ultra-hyperbolic operator, and it is demonstrated that certain functions are elemental solutions of this operator iterated several times. In Section 3 some specific cases of the ultra-hyperbolic kernel and of B-ultra-hyperbolic operator are discussed, including situations with different values of the parameters. Finally, Section 4 provides the conclusions.

2 The Marcel Riesz B-ultrahiperbolic Kernel

Let $P = P(x)$ be the quadratic form given in (1.1), and let $P' = \epsilon|x|^2$ with $\epsilon > 0$. the weighted generalized functions $(P \pm i0)_\gamma^\lambda$ is defined as (see [1])

$$(P \pm i0)_\gamma^\lambda \quad = \quad \lim_{\epsilon \to 0} (P \pm iP')_\gamma^\lambda \tag{2.1}$$

It can be observed that if $|\gamma| = 0$, $(P \pm i0)_\gamma^\lambda$ reduces to the distributions $(P \pm i0)^\lambda$ introduced by Gelfand and Shilov [3].

The weighted Dirac delta distribution δ_γ is defined by

$$(\delta_\gamma, \varphi)_\gamma = \varphi(0), \tag{2.2}$$

$\varphi \in S_{ev}$,

$$S_{ev} = \left\{ f \in C_{ev}^\infty : \sup_{x \in \mathbb{R}_+^n} |x^\alpha D^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{Z}_+^n \right\} \tag{2.3}$$

the space of rapidly decreasing functions and C_{ev}^∞ the set of functions belonging to C^∞ that are even with respect to each variable x_i, $i = 1, 2, ..., n$. A function $f \in C^\infty$ is said to be even with respect to x_i, $i = 1, 2, ..., n$ if $\frac{\partial^{2k+1}}{\partial x_i^{2k+1}}\Big|_{\lambda=0} = 0$ for $k \in \mathbb{N}_0$ (see [1]). Moreover, we consider

$$\mathbb{R}_+^n = \{x = (x_1, ..., x_n) \ : \ x_i > 0, \ i = 1, 2, ..., n\} \tag{2.4}$$

and

$$\overline{\mathbb{R}_+^n} = \{x = (x_1, ..., x_n) \ : \ x_i \geq 0, \ i = 1, 2, ..., n\} \tag{2.5}$$

Similarly to what was established by L. Schwartz [12], the class $\mathcal{D}_+\left(\mathbb{R}^n_+\right)$ is defined as the set of functions $f \in \mathcal{C}^\infty(\mathbb{R}^n_+)$ with compact support. Functions φ belonging to the space $\mathcal{D}_+\left(\mathbb{R}^n_+\right)$ are referred to as test functions.

The weighted distributions $(P \pm i0)^\lambda_\gamma$ are analytical in λ everywhere except at $\lambda = -k - \frac{n+|\gamma|}{2}$, k is a non negative interger, with residues given by the

$$\operatorname*{res}_{\gamma=-\frac{n+|\gamma|}{2}-k} (P \pm i0)^\lambda_\gamma = \frac{e^{\mp i\pi\frac{q+|\gamma''|}{2}} \left|S_1^+(n)\right|_\gamma \Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k! \prod\limits_{k=1}^{n} \Gamma\left(\frac{n+|\gamma|}{2}+k\right)} \square^k_{B,\gamma}\delta_\gamma(x) \tag{2.6}$$

where $\square^k_{B,\gamma}$ is the operator (1.9) iterated k-times, and

$$\left|S_1^+(n)\right|_\gamma = \int_{S_1^+(n)} x^\gamma ds = \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} \tag{2.7}$$

In the development of this paper, we needed the following property verified by the weighted distributions $(P \pm i0)^\lambda_\gamma$, which allows them to be expressed in terms of the distributions $P^\lambda_{\gamma^+}$ and $P^\lambda_{\gamma^-}$. Indeed, as stated in [1]:

$$(P \pm i0)^\lambda_\gamma = P^\lambda_{\gamma^+} + e^{\pm i\pi\lambda} P^\lambda_{\gamma-}, \tag{2.8}$$

and, when $\lambda = k$, $k \in \mathbb{N}$, we have

$$(P + i0)^k_\gamma = (P - i0)^k_\gamma = P^k_\gamma. \tag{2.9}$$

The Hankel transform of a function $f \in L^1_\gamma(\mathbb{R}^n)$

$$\mathbb{F}_\gamma[f(x)](\xi) = \int_{\mathbb{R}^n_+} f(x)\mathbf{j}_\gamma(x;\xi)x^\gamma dx \tag{2.10}$$

where

$$\mathbf{j}(x,\xi) = \prod_{i=1}^{n} j_{\frac{n-1}{2}}(x_i,\xi_i), \quad \gamma_i > 0, \; i = 1, 2, ..., n; \tag{2.11}$$

and

$$j_\nu(r) = \frac{2^\nu \Gamma(\gamma+1)}{r^\nu} J_\gamma(r); \tag{2.12}$$

$J_\gamma(r)$ is the Bessel function of the first kind of order ν.

The generalized convolution of two functions $f, g \in S_{ev}$ is defined by the following integral

$$(f * g)_\gamma(x) = \int_0^\infty f(y)^\gamma T^y_x g(x) y^\gamma dy \tag{2.13}$$

where ${}^\gamma T^y_x$ is the multidimentional generalized translation

$$\left({}^\gamma T^y_x f\right)(t,x) = \left({}^{\gamma_1}T^{y_1}_{x_1}, {}^{\gamma_2}T^{y_2}_{x_2}, ..., {}^{\gamma_n}T^{y_n}_{x_n} f\right)(t,x), \tag{2.14}$$

and

$$\left({}^{\gamma_i}T^{y_i}_{x_i} f\right)(t,x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)\Gamma\left(\frac{1}{2}\right)} \times$$
$$\times \int_0^\pi f\left(t, x_1, ..., x_{i-1}\sqrt{x^2+y^2-2x_i y_1 \cos\varphi_1}, x_{i+1}..., x_n\right) \sin^{\gamma_i - 1}\varphi_i d\varphi_i. \tag{2.15}$$

(cf. [13]).

The generalized Hankel transform of the convolution satisfies the following equality:

$$\mathbb{F}_\gamma[(f * g)_\gamma(x)](\xi) = \mathbb{F}_\gamma[f](\xi)\mathbb{F}_\gamma[g](\xi), \qquad (cf.[14], f(2.13)). \tag{2.16}$$

Definition 2.1 (Riesz's B-ultrahyperbolic kernel). Let $\alpha \in \mathbb{C}$ be and consider the family of distributions depending on the parameter α defined as follows:

$$H_\alpha (P \pm i0)_\gamma = \frac{e^{\frac{i\pi\alpha}{2}} e^{\frac{\pm i\pi q + |\gamma''|}{2}} \Gamma\left(\frac{n+|\gamma|-\alpha}{2}\right)}{\prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} . \tag{2.17}$$

This family of functions is analogous to the one introduced by Trione (cf.[2]) and, in turn, generalizes the elliptical kernel of Marcel Riesz (cf.[4], pp.16-21).

Taking into account the Hankel transform of the distributions $(P \pm i0)_\gamma^\lambda$ (cf. [15]),

$$\mathbb{F}\left[H_\alpha(P \pm i0)_\gamma\right] = \frac{e^{\mp\frac{q+|\gamma''|}{2} i\pi} 2^{2\lambda+|\gamma|+n} \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{n+|\gamma|}{2} + \gamma\right) (Q \mp i0)^{-\frac{n+|\gamma|}{2} - \lambda}}{\Gamma(-\lambda)} \tag{2.18}$$

is obtained, for $\lambda = \frac{\alpha-n-|\gamma|}{2}$, we have:

$$\mathbb{F}\left[H_\alpha(P + i0)_\gamma\right] = e^{i\frac{\pi}{2}\alpha}(Q - i0)^{-\frac{\alpha}{2}} \tag{2.19}$$

very important expression in what follows.

Now, we will demonstrate some properties that the Kernel $H_\alpha(P \pm i0)_\gamma$ verifies:

Theorem 2.2. *Let $\alpha \neq n + |\gamma| + 2\ell$, and $\alpha - 2k \neq n + |\gamma| + 2\ell$, $\ell \in \mathbb{N}_0$. Then*

$$H_\alpha(P \pm i0)_\gamma \; *_B H_{-2k}(P \pm i0)_\gamma = H_{\alpha-2k}(P \pm i0)_\gamma, \tag{2.20}$$

where $*_B$ indicates la B-convolution.

$$(f *_B g)_\gamma = \int_{\mathbb{R}_+^{n+1}} f(\tau, y) \left({}^\gamma T_x^y g\right)(t - \tau, x) y^\gamma d\tau dy \tag{2.21}$$

where ${}^\gamma T_x^y$ is the multidimentional generalized translation (2.14).

Proof. Setting $\alpha = -2k$, $k = 0, 1, 2, ...$ in (2.19) we have

$$\mathbb{F}\left\{H_{-2k}(P \pm i0)_\gamma\right\} = (-1)^k Q^k. \tag{2.22}$$

The following is also valid

$$\mathbb{F}\left\{\Box_B^k \delta\right\} = (-1)^k Q^k. \tag{2.23}$$

Then, from (2.22) and (2.23) it is obtained that

$$H_{-2k}(P \pm i0)_\gamma = \Box_B^k \delta_\gamma, \tag{2.24}$$

in the sense of the distributions.

The formula (2.24) indicate that $H_{-2k}(P \pm i0)_\gamma$ is a convolutor in $\mathcal{D}'(\mathbb{R}_+^n)$, then the convolution that appears in the left-hand side of (2.20). Taking into account (2.16) and (2.19)

$$\begin{aligned} \mathbb{F}\left\{H_\alpha(P \pm i0)_\gamma \; *_B H_{-2k}(P \pm i0)_\gamma\right\} &= e^{i\frac{\pi}{2}\alpha}(Q \mp i0)^{\frac{\alpha}{2}} e^{i\frac{\pi}{2}(-2k)}(Q \mp i0)^{\frac{2k}{2}} \\ &= e^{i\frac{\pi}{2}(\alpha-2k)}(Q \mp i0)^{-\frac{\alpha-2k}{2}} \end{aligned} \tag{2.25}$$

is obtained. On the other hand,

$$\mathbb{F}\left\{H_{\alpha-2k}(P \pm i0)_\gamma\right\} = e^{i\frac{\pi}{2}(\alpha-2k)}(Q \mp i0)^{-\frac{\alpha-2k}{2}} \tag{2.26}$$

it results, i.e., from (2.25) and (2.26) we obtain

$$\mathbb{F}\left\{H_\alpha(P \pm i0)_\gamma \; *_B H_{-2k}(P \pm i0)_\gamma\right\} = \mathbb{F}\left\{H_{\alpha-2k}(P \pm i0)_\gamma\right\}. \tag{2.27}$$

Then,

$$H_\alpha(P \pm i0)_\gamma \; *_B H_{-2k}(P \pm i0)_\gamma = H_{\alpha-2k}(P \pm i0)_\gamma, \tag{2.28}$$

for $\alpha \neq n + |\gamma| + 2\ell$, and $\alpha - 2k \neq n + |\gamma| + 2\ell$, where $\ell = 0, 1, 2, ...$ □

We will prove that the family of distributions $H_\alpha[(P\pm i0)_\gamma]$ satisfies the index low.

Theorem 2.3. *Let* α, β *and* $\alpha+\beta\neq n+|\gamma|+2\ell$; $\ell=0,1,2,\ldots$ *Then*

$$H_\alpha[(P\pm i0)_\gamma]*_B H_\beta[(P\pm i0)_\gamma]=H_{\alpha+\beta}[(P\pm i0)_\gamma]. \tag{2.29}$$

Proof. Taking into account (2.29), we have

$$\mathbb{F}\left\{H_\alpha[(P\pm i0)_\gamma]*_B H_\beta[(P\pm i0)_\gamma]\right\}=e^{i\frac{\pi}{2}(\alpha+\beta)}(Q\mp i0)^{-\frac{\alpha+\beta}{2}}. \tag{2.30}$$

Applying the inverse Hankel transform we get (2.29). □

Theorem 2.4. *The distributions* $H_{2k}[(P\pm i0)_\gamma]$, $k=1,2,\ldots$, *are solutions of the operator* $\Box_B^k$ *iterated* k*-times.*

Proof. If in (2.29) it is considered $\beta=-2k$, it results:

$$H_\alpha[(P\pm i0)_\gamma]*_B H_{-2k}[(P\pm i0)_\gamma]=H_{\alpha-2k}[(P\pm i0)_\gamma], \tag{2.31}$$

taking into consideration (2.24) it can be written

$$H_{\alpha-2k}[(P\pm i0)_\gamma]=H_{-2k+\alpha}[(P\pm i0)_\gamma]=\Box_B^k H_\alpha[(P\pm i0)_\gamma]. \tag{2.32}$$

If we consider $\alpha=2k$, from (2.32) we have

$$\Box_B^k H_{2k}[(P\pm i0)_\gamma]=H_0[(P\pm i0)_\gamma] \tag{2.33}$$

by (2.24) when $k=0$ it is obtained

$$H_0[(P\pm i0)_\gamma]=\delta_\gamma. \tag{2.34}$$

Then, from (2.32), (2.33) and (2.34) it is verified that

$$\Box_B^k H_{2k}[(P\pm i0)_\gamma]=\delta_\gamma, \tag{2.35}$$

that is, the distributions $H_{2k}(P\pm i0)_\gamma$ are elemental solutions of the B-ultrahyperbolic operator iterated k-times. □

Then, from (2.17) with $\alpha=2k$ we get

$$H_{2k}[(P\pm i0)_\gamma]=\frac{(-1)^k e^{\pm i\frac{\pi}{2}(q+|\gamma''|)}\Gamma\left(\frac{n+|\gamma|}{2}-k\right)(P\pm i0)^{k-\frac{n+|\gamma|}{2}}}{4^k(k-1)!\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} \tag{2.36}$$

which, except for a constant, it coincides with the result exhibited by Shishkina in [16].

3 Particular cases

- If $|\gamma|=0$, then $H_\alpha[(P\pm i0)_\gamma]$ is reduced to $H_\alpha(P\pm i0)$ given by (1.17) and the B-ultrahyperbolic operator given by (1.9) coincides with the operator given by (1.7) which is the ultrahyperbolic differential operator.
- If $|\gamma|=0$ y $p=1$, from the previous case it can be observed that the differential operator of the waves will be obtained:

$$\Box=\frac{\partial^2}{\partial x_1^2}-\sum_{i=2}^{n}\frac{\partial^2}{\partial x_i}. \tag{3.1}$$

- If $|\gamma|=0$ y $q=0$, taking into consideration that if we call $r^2=x_1^2+\ldots+x_n^2$, $(P\pm i0)$ results in $r^{2\lambda}$, while the operator $\Box_B^k$ ccoincides with the Laplacian

$$\Delta^k=\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right)^k \tag{3.2}$$

(See (7), p. 19 from [4]).

- If $|\gamma| \neq 0$ and $q = 0$, it is obtained that $\square_B^k$ is reduced to the Laplace-Bessel Operator iterated k-times given by

$$\Delta_B^k = \left(\sum_{i=1}^{n} B_{x_i} \right)^k , \tag{3.3}$$

which elemental solution is

$$H_{2k}(r) = \frac{(-1)^k 2^{n+|\gamma|-2\ell} \Gamma\left(\frac{n+|\nu|}{2} - k\right) |x|^{2k-n-|\gamma|}}{\prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) 2^{2k} (k-1)!} \tag{3.4}$$

(See (8) p. 5 from [8])

4 Conclusion.

A kernel expressed in terms of weighted generalised functions associated to quadratic forms that generalize the causal(anticausal) kernel due to a Trione and noted as $H_\alpha[(P \pm i0)]$ was introduced.

Some properties were studied and it was demonstrated that $H_{2k}[(P \pm i0)_\gamma]$ is the elemental solution of the B-ultrahyperbolic operator iterated k-times. Besides, it can be proved that this kernel allows to define Riesz or Bessel potentials via B-convolution.

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