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Spectral analysis of the D_α-matrix associated with the zero-divisor graph

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Abstract *Let R be a commutative ring with identity element $1 \neq 0$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph where each vertex represents a nonzero, non-unit element of R, and two distinct vertices g and h are connected by an edge if and only if their product is zero, i.e., $gh = 0$. In the context of a basic connected undirected graph G, let $D(G)$ denote its distance matrix, and let $Tr(G)$ represent the diagonal transmission matrix, where each diagonal entry gives the sum of distances from a vertex to all other vertices. Using these, the D_α-matrix is defined $D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G)$ for any $\alpha \in [0,1]$. We compute the D_α spectra of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ for a general class of n, represented as $n = {\xi_1}^F {\xi_2}^L$, where ξ_1, ξ_2 are different primes with $\xi_1 < \xi_2$ and $F, L \in \mathbb{N}$.*

1 Notations and Introduction

In this article, a commutative ring having identity $1 \neq 0$ shall be denoted by R. If there exist $0 \neq c \in R$ such that $bc = 0$, then a nonzero element $b \in R$ is the zero divisor of R. $Z(R)$ represents the set of zero-divisors of R, and $Z(R)' = Z(R)/\{0\}$. The set that includes every non-zero and non-unit elements in ring R is denoted by $Z(R)'$. For a positive integer n, $\mathbb{Z}_n$ represents the ring of integers modulo n.

The graph $G = (V, E)$ has been defined, where V denotes the set of vertices and E denotes the set of edges of G. When two distinct vertices of graph G, ζ_1 and ζ_2 are adjacent to each other in graph G, the notation $\zeta_1 \sim \zeta_2$ represents this. The neighborhood of a vertex ζ in graph G is the set of vertices that are adjacent to it; this neighborhood is denoted by the notation $N_G(\zeta)$. K_m denotes the complete graph with m vertices . $deg(\zeta)$, the degree of vertex ζ, represents the number of edges incident with $\zeta \in V$. If $deg(\zeta) = 0$, then ζ is referred to as an isolated vertex. For any vertex ζ, G is k-regular if $deg(\zeta) = k$. Let A be any square matrix and let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ be its different eigenvalues with multiplicities of $u_1, u_2, u_3, \ldots, u_k$ respectively. The $spectrum$ of A is then denoted by $\sigma(A)$, which is defined by

$$\sigma(A) = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_k \\ u_1 & u_2 & u_3 & \ldots & u_k \end{Bmatrix}.$$

The adjacency matrix $A(G)$ for a graph G is a n-dimensional square matrix that can be found by using

$$A(G) = (a_{ij}) = \begin{cases} 1, & \zeta_i \sim \zeta_j \\ 0, & otherwise. \end{cases}$$

First, in 1988, Beck [4] presented and studied the idea of zero-divisor graphs in relation to commutative rings. In Beck's definition, the graph's vertices are ring elements. Later, in 1999, Anderson and Livingston [2] modified Beck's definition, wherein the graph's vertices are defined as nonzero zero-divisors of the ring R. The vertices u and v are adjacent for all $u, v \in Z(R)'$ if and only if $uv = 0$ such that $u \neq v$. This vertex set is $Z(R)'$. The symbol for zero-divisor graph is $\Gamma(R)$. See [1, 2] for further details on the zero-divisor graph.

The generalized distance matrix $D_\alpha(G)$ was presented by Cui et al. [17] as the convex combination of $T(G)$ and $D(G)$ for $\alpha \in [0,1]$, $D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G)$ where $D(G)$ represents the distance matrix of G and $T(G)$ the transmission matrix of G. The D_α spectrum of the zero-divisor graph of $\mathbb{Z}_n$ is found in this paper for various values of n. Readers can refer to [3, 6, 8, 10, 11, 12, 13, 14, 15, 16] for additional study on different graphs. The definitions, lemmas, and theorems that are utilized to support the main results are analyzed in section 2. D_α eigenvalues of $\Gamma(\mathbb{Z}_n)$ are looked into in section 4, for $n = {\xi_1}^F {\xi_2}^L$, where ξ_1, ξ_2 are distinct primes with $\xi_1 < \xi_2$ and $F, L \in \mathbb{N}$ with $F \leq L$.

2 Preliminaries

Definition 2.1. Let $G(V,E)$ be a graph of order m having vertex set $\{v_1, v_2, \ldots, v_m\}$ and $H_k(V_k, E_k)$ be disjoint graphs of order m_k, $1 \leq k \leq m$. The graph $H_1, H_2, \ldots, H_m$ formed the generalized join graph $G[H_1, H_2, \ldots, H_m]$ and whenever v_k ans v_l are adjacent in G, joined each vertex of H_k to every vertex of H_l.

The number of positive integers smaller than or equal to s_1 that are relatively prime to s_1 is indicated by Eulur's phi function $\phi(s_1)$. If $s_1 = \xi_1^{h_1}\xi_2^{h_2}\cdots\xi_k^{h_k}$, where $h_1, h_2, \ldots, h_k$ are positive integers and $\xi_1, \xi_2, \ldots, \xi_k$ are distinct primes, then s_1 is in $prime\ decoposition$.

Lemma 2.2. *[9]* If $s_1 = \xi_1^{h_1}\xi_2^{h_2}\cdots\xi_k^{h_k}$ is a prime decomposition of s_1, then $\tau(s_1) = (h_1 + 1)(h_2+1)\cdots(h_k+1)$.

Let $c_1, c_2, \ldots, c_k$ be the proper divisors of n. For $1 \leq i \leq k$, consider the following sets

$$A_{c_i} = \{x \in \mathbb{Z}_n : gcd(x,n) = c_i\}.$$

Moreover, observe that for $i \neq j$, $A_{c_i} \cap A_{c_j} = \phi$. As a result, the vertex set of $\Gamma(\mathbb{Z}_n)$ has a partition formed by the sets $A_{c_1}, A_{c_2}, \ldots, A_{c_k}$. $V(\Gamma(\mathbb{Z}_n)) = A_{c_1} \cup A_{c_2} \cup \cdots \cup A_{c_k}$, as a result. The following lemma provides information about the cardinality of each A_{c_i}.

Lemma 2.3. *[18, Lemma 2.1] Let c_i be the divisor of n then $|A_{c_i}| = \phi(\frac{n}{c_i})$ for $1 \leq i \leq k$.*

Let $c_1, c_2, \ldots, c_k$ be the distinct proper divisor of n and let γ_n be the simple graph with vertex set $\{c_1, c_2, \ldots, c_k\}$. The graph γ_n has two different vertices c_i and c_j, that are adjacent if and only if n divides $c_i c_j$. If the prime decomposition of $n = \xi_1^{n_1}\xi_2^{n_2}\ldots\xi_r^{n_r}$ then the order of graph γ_n is as follows:

$$|V_{\gamma_n}| = \prod_{i=1}^{r}(n_i+1) - 2.$$

Corollary 2.4. *[5] Let c_r be the divisor of n. Then the following assertions are true:*
(1) The complete graph $K_{\phi(\frac{n}{c_r})}$ or its complement graph $\overline{K}_{\phi(\frac{n}{c_r})}$ are two possible outcomes for induced subgraph $\Gamma(A_{c_r})$ of $\Gamma(Z_n)$ on the vertex set A_{c_r} for $r \in 1, 2, \ldots, k$. In fact, $\Gamma(A_{c_r})$ is $K_{\phi(\frac{n}{c_r})}$ if and only if $n | {c_r}^2$.
(2) For $r \in \{1, 2, \ldots, k\}$ and $r \neq j$, a vertex of $\Gamma(A_{c_r})$ is adjacent to either all or none of the vertices of $\Gamma(A_{c_j})$.

Because of this, the partition $A_{c_1}, A_{c_2}, \ldots, A_{c_k}$ of $V(\Gamma(\mathbb{Z}_n))$ is equitable in that, for each $i, j \in \{1, 2, \ldots, k\}$, every vertex of the A_{c_i} has an equal number of neighbors in A_{c_j}.

Lemma 2.5. $\Gamma(\mathbb{Z}_n) = \gamma_n[\Gamma(A_{c_1}), \Gamma(A_{c_1}), \ldots, \Gamma(A_{c_k})]$, *where $c_1, c_2, \ldots, c_k$ are all the proper divisor of n.*

3 Methodology

Research in graph theory continues to flourish because it provides a link between discrete structures and pure as well as applied mathematics. Using sophisticated mathematical tools, the study's method builds upon well-established ideas in algebra and graph theory to produce new results. Our efforts rely on using the content of existing research to expand on established findings and investigate fresh aspects of zero-divisor graphs.

The analysis in this paper heavily relies on the use of matrix theory and linear algebra. In particular, spectral graph theory provides a strong framework for studying the interaction between algebraic and graph-theoretical characteristics. A crucial tool for capturing the structural features of the zero-divisor graph of the ring $\mathbb{Z}_n$.

Analyzing the D_α *spectra of the zero-divisor graph* $\Gamma(\mathbb{Z}_n)$ *for a general class of* n, *represented as* $n = {\xi_1}^F {\xi_2}^L$, *where* ξ_1, ξ_2 *are different primes with* $\xi_1 < \xi_2$ *and* $F, L \in \mathbb{N}$ *with* $F \leq L$, *is the main goal of this study. We accomplish this by computing eigenvalues and deriving the characteristic polynomials associated with these networks using sophisticated spectral graph theory techniques and computational tools.*

4 Main results

In this section, we will highlight the primary results of this paper. The adjacency spectrum of complete graph K_l *and its complement graph* $\overline{K_l}$ *on* l *vertices is given by*

$$\sigma_A(K_l) = \left\{ \begin{array}{cc} l-1 & -1 \\ 1 & l-1 \end{array} \right\} \text{ and } \sigma_A(\overline{K_l}) = \left\{ \begin{array}{c} 0 \\ l \end{array} \right\} \text{ respectively.}$$

The following theorem provides the generalized join graph's D_α *spectrum in terms of the spectrum of adjacency matrix of regular graphs.*

Theorem 4.1. *[7] Let* H *be a connected graph of order* s. *Let* $\alpha \in [0,1]$. *If, for* $\varepsilon = 1, 2, \ldots, s$, G_ε *is a* r_ε *regular graph of order* m_ε, *then the* D_α *spectrum of the* H*-join graphs* $G_1, G_2, \ldots, G_s$ *is*

$$\sigma\big(D_\alpha(G)\big) = \bigcup_{\varepsilon=1}^{s} \Big(\sigma\big(M_\varepsilon(\alpha)\big) - \{\mu_\varepsilon(\alpha)\} \Big) \bigcup \sigma(Y),$$

where

$$\mathcal{K} = \begin{bmatrix} \mu_1(\alpha) & \delta d_{1,2}\sqrt{m_1 m_2} & \ldots & \delta d_{1,s}\sqrt{m_1 m_s} \\ \delta d_{2,1}\sqrt{m_2 m_1} & \mu_2(\alpha) & \ldots & \delta d_{2,s}\sqrt{m_2 m_s} \\ \vdots & \vdots & \ddots & \vdots \\ \delta d_{s,1}\sqrt{m_s m_1} & \delta d_{s,2}\sqrt{m_s m_2} & \ldots & \mu_s(\alpha) \end{bmatrix}. \tag{4.1}$$

Where $\delta = (1-\alpha)$, $\sigma(M_\varepsilon(\alpha)) = \alpha Tr(v) + (1-\alpha)(2(m_\varepsilon - 1) - r_\varepsilon), \alpha Tr(v) - (1-\alpha)(2 + \lambda_{m_\varepsilon}(A(G_\varepsilon))), \ldots, \alpha Tr(v) - (1-\alpha)(2 + \lambda_2(A(G_\varepsilon)))$, $\mu_\varepsilon(\alpha) = \alpha Tr(v) + (1-\alpha)(2(m_\varepsilon - 1) - r_\varepsilon), Tr(v) = 2(m_\varepsilon - 1) - r_\varepsilon + \varkappa'$, $\varkappa' = \sum_{j=1 j \neq \varepsilon}^{s} m_j d_{\varepsilon,j}$ *and* $d_{\varepsilon,j}$ *are distance from vertex* ε *to* j *for* $1 \leq \varepsilon, j \leq s$.

The D_α *spectrum can be found using the example below by applying the preceding Theorem 4.1.*

Example 4.2. The D_α spectrum of the zero divisor graph $\Gamma(\mathbb{Z}_{36})$.

For $n = 36$, the proper divisors are $2, 3, 4, 6, 9, 12$ and 18. Consequently, the graph $2 \sim 18 \sim 4 \sim 9 \sim 12 \sim 6 \sim 18 \sim 12 \sim 3$ is γ_n. By applying Lemma 2.5 we have,

$$\Gamma(\mathbb{Z}_{36}) = \gamma_{36}[\Gamma(A_2), \Gamma(A_3), \Gamma(A_4), \Gamma(A_6), \Gamma(A_9), \Gamma(A_{12}), \Gamma(A_{18})].$$

Therefore, by Lemma 2.3 and Corollary 2.4, we have

$$\Gamma(\mathbb{Z}_{36}) = \gamma_{36}[\overline{K}_6, \overline{K}_4, \overline{K}_6, K_2, \overline{K}_2, K_2, K_1].$$

Using Theorem 4.1, the values of $\varkappa_i'$'s are

$$(\varkappa_1', \varkappa_2', \varkappa_3', \varkappa_4', \varkappa_5', \varkappa_6', \varkappa_7') = (39, 48, 35, 39, 40, 33, 28)$$

and the D_α eigenvalues denoted by, λ_i's are

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (51\alpha - 2, 56\alpha - 2, 47\alpha - 2, 41\alpha - 1, 44\alpha - 2, 35\alpha - 1, 29\alpha - 1).$$

The D_α spectrum of $\Gamma(\mathbb{Z}_{36})$ according to Theorem 4.1 is given by,

$$\begin{Bmatrix} 51\alpha - 2 & 56\alpha - 2 & 47\alpha - 2 & 41\alpha - 1 & 44\alpha - 2 & 35\alpha - 1 \\ 5 & 3 & 5 & 1 & 1 & 1 \end{Bmatrix}.$$

And the matrix's characteristic polynomial given in 4.2 , can be used to determine the remaining eigenvalues.

$$\mathcal{K} = \begin{bmatrix} 39\alpha + 10 & 6\delta\sqrt{6} & 12\delta & 4\delta\sqrt{3} & 6\delta\sqrt{3} & 4\delta\sqrt{3} & \delta\sqrt{6} \\ 6\delta\sqrt{6} & 48\alpha + 6 & 6\delta\sqrt{6} & 4\delta\sqrt{2} & 4\delta\sqrt{2} & 2\delta\sqrt{2} & 4\delta \\ 12\delta & 6\delta\sqrt{6} & 35\alpha + 10 & 4\delta\sqrt{3} & 2\delta\sqrt{3} & 4\delta\sqrt{3} & \delta\sqrt{6} \\ 4\delta\sqrt{3} & 4\delta\sqrt{2} & 4\delta\sqrt{3} & 39\alpha + 1 & 4\delta & 2\delta & \delta\sqrt{2} \\ 6\delta\sqrt{3} & 4\delta\sqrt{2} & 2\delta\sqrt{3} & 4\delta & 40\alpha + 2 & 2\delta & 2\delta\sqrt{2} \\ 4\delta\sqrt{3} & 2\delta\sqrt{2} & 4\delta\sqrt{3} & 2\delta & 2\delta & 33\alpha + 1 & \delta\sqrt{2} \\ \delta\sqrt{6} & 4\delta & \delta\sqrt{6} & \delta\sqrt{2} & 2\delta\sqrt{2} & \delta\sqrt{2} & 28\alpha \end{bmatrix}. \tag{4.2}$$

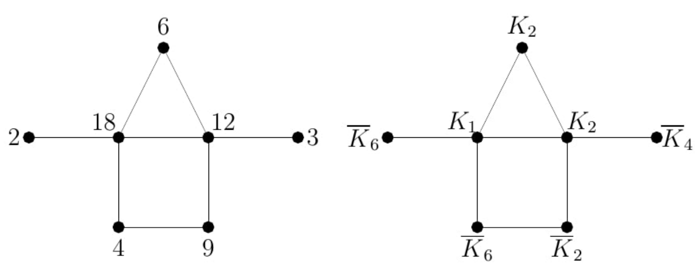

Figure 1. Proper divisor graph and zero divsior graph of $\mathbb{Z}_{36}$

Finding the D_α eigenvalues for $\Gamma(\mathbb{Z}_n)$, where $n = \xi_1^F \xi_2^L$, is the goal of this article. In this case, ξ_1 is less than ξ_2, ξ_1 and ξ_2 are prime numbers. When F and L are positive integers and $F \leq L$, we will demonstrate this.

Theorem 4.3. *The D_α spectrum of the $\Gamma(\mathbb{Z}_{\xi_1{}^F \xi_2{}^L})$ where $F = 2\eta \leq 2\beta = L$ consists of the eigenvalues,*

$$\begin{aligned} \lambda_\varepsilon &= 2\alpha\varrho' + \alpha\phi(\xi_1^F)\xi_2^{L-1} - \alpha(\xi_1^x - 1) - 2, \ \textit{for } \varepsilon = x = 1, 2, \dots, \eta, \dots, F - 1, \\ \lambda_S &= 2\alpha\varrho' + \alpha\phi(\xi_1^F)(\xi_2^{L-1} - 1) - \alpha(\xi_1^L - 1) - 2, \\ \lambda_\varepsilon &= 2\alpha\varrho' + \alpha\xi_1^{F-1}\phi(\xi_2^L) - \alpha(\xi_2^x - 1) - 2, \ \textit{for } x = 1, 2, \dots, L - 1, \ \textit{and} \\ &\quad \varepsilon = F + 1, \dots, F + L - 1, \\ \lambda_{F+L} &= 2\alpha\varrho' + \alpha\phi(\xi_2^L)(\xi_1^{F-1} - 1) - \alpha(\xi_2^L - 1) - 2, \\ \lambda_\varepsilon &= 2\alpha\varrho' - \alpha(\xi_1\xi_2^x - 1) - 2, \ \textit{for } x = 1, 2, \dots, L, \ \textit{and } \varepsilon = F + L + 1, \dots, F + 2L, \\ &\ \vdots \end{aligned}$$

$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^\eta\xi_2^x - 1) - 2,\ for\ x = 1,2,\ldots,\beta-1,\ and\ \varepsilon = F+\beta L+1,\ldots,F+\beta L+\beta-1,$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^\eta\xi_2^x - 1) - 1,\ for\ x = \beta,\ldots,L,\ and\ \varepsilon = F+\beta L+\beta,\ldots,F+(\beta+1)L,$$
$$\vdots$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^F\xi_2^x - 1) - 2,\ for\ x = 1,2,\ldots,\beta-1,\ and\ \varepsilon = F+FL+1,\ldots,\beta-1,$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^F\xi_2^x - 1) - 1,\ for\ x = \beta,\ldots,L,\ and\ \varepsilon = F+FL+\beta,\ldots,F+FL+L-1,$$

with multiplicities $\phi(\xi_1^{F-1}\xi_2^L)-1, \phi(\xi_1^F\xi_2^{L-x})-1, \phi(\xi_1^{F-\varepsilon}\xi_2^{L-x})-1,\ldots,\phi(\xi_1^\beta\xi_2^{L-x})-1,\ldots,\phi(\xi_2^{L-x})-1$, *respectively, where* $\varepsilon = 1,\ldots,F$ *and* $x = 1,\ldots,L$. D_α*-eigenvalues that remain for the graph* $\Gamma(\mathbb{Z}_{\xi_1^F\xi_2^L})$ *are eigenvalues of the matrix* (4.1).

Proof. If $n = {\xi_1}^F{\xi_2}^L, \xi_1$ and ξ_2 are distinct primes and $F = 2\eta \leq 2\beta = L$. Then the proper divisors of n are

$$\Big\{\xi_1,\xi_1^2,\ldots,\xi_1^\eta,\ldots,\xi_1^F,\xi_2,\xi_2^2,\ldots,\xi_2^\beta,\ldots,\xi_2^L,\xi_1\xi_2,\xi_1\xi_2^2,\ldots,\xi_1\xi_2^\beta,\ldots,\xi_1\xi_2^L,\ldots,\xi_1^\eta\xi_2,\xi_1^\eta\xi_2^2,\ldots,$$
$$\xi_1^\eta\xi_2^\beta-1,\xi_1^\eta\xi_2^\beta,\ldots,\xi_1^\eta\xi_2^L,\ldots,\xi_1^F\xi_2,\xi_1^F\xi_2^2,\ldots,\xi_1^F\xi_2^\beta-1,\xi_1^F\xi_2^\beta,\ldots,\xi_1^F\xi_2^L-1\Big\}.$$

By Lemma 2.2, order of $\gamma_{\xi_1^F\xi_2^L}$ is $(F+1)(L+1)-2 = FL+F+L-1$. From the definition of $\gamma_{\xi_1^F\xi_2^L}$, we have,

$$\xi_1^\varepsilon \sim \xi_1^x\xi_2^L, \varepsilon+x \geq S,\ for\ \varepsilon = 1,2,\ldots,S$$
$$\xi_2^\varepsilon \sim \xi_1^s\xi_2^x, \varepsilon+x \geq T,\ for\ \varepsilon = 1,2,\ldots,T$$
$$\xi_1\xi_2^\varepsilon \sim \xi_1^k\xi_2^x, \varepsilon+x \geq T,\ for\ \varepsilon = 1,2,\ldots,T\ and\ k \geq 2\eta-1$$
$$\vdots$$
$$\xi_1^\eta\xi_2^\varepsilon \sim \xi_1^k\xi_2^x, \varepsilon+x \geq T,\ for\ \varepsilon = 1,2,\ldots,T\ and\ k \geq \eta$$
$$\vdots$$
$$\xi_1^\eta\xi_2^\varepsilon \sim \xi_1^k\xi_2^x, \varepsilon+x \geq T,\ for\ \varepsilon = 1,2,\ldots,T-1\ and\ k \geq 0.$$

In view of Lemma 2.5, the cardinalities of A_{c_ε} for $\varepsilon = 1,2,\ldots,F, x = 1,2,\ldots,L$ *and* $k = 1,2,\ldots,L-1$, are given by

$|A_{\xi_1^\varepsilon}| = \phi(\xi_1^F - \varepsilon\xi_2^L), |A_{\xi_2^x}| = \phi(\xi_1^F\xi_2^{L-x}),\ldots,|A_{\xi_1^\eta\xi_2^x}| = \phi(\xi_1^\eta\xi_2^{L-x}),\ldots,|A_{\xi_1^{F-1}\xi_2^x}| = \phi(\xi_1\xi_2^{L-x})$, $|A_{\xi_1^F\xi_2^k}| = \phi(\xi_2^{L-k})$.

From Lemma 2.4, the induced subgraphs $\Gamma(A_{c_{\xi_1^\varepsilon}})$ are,

$$K_\varepsilon = \begin{cases} \Gamma(A_{c_{\xi_1^\varepsilon}}) = \overline{K}_{\phi(\xi_1^{F-\varepsilon}\xi_2^L)}, & 1 \leq \varepsilon \leq F, \\ \Gamma(A_{c_{\xi_2^x}}) = \overline{K}_{\phi(\xi_1^F\xi_2^{L-x})}, & 1 \leq x \leq L, \\ \Gamma(A_{c_{\xi_1^\varepsilon\xi_2^x}}) = \overline{K}_{\phi(\xi_1^{F-\varepsilon}\xi_2^{L-x})}, & 1 \leq \varepsilon \leq \eta-1 \text{ and } 1 \leq x \leq L \\ \text{ or } \eta \leq \varepsilon \leq F \text{ and } 1 \leq x \leq \beta-1, \\ \Gamma(A_{c_{\xi_1^\varepsilon q^x}}) = K_{\phi(\xi_1^{F-\varepsilon}q^{L-x})}, & \eta \leq \varepsilon \leq F \text{ and } \beta \leq x \leq L. \end{cases} \tag{4.3}$$

The joined union of the zero-divisor graph $\Gamma(\mathbb{Z}_{{\xi_1}^F{\xi_2}^L})$ determined by applying Lemma 2.5 provided by

$\Gamma(\mathbb{Z}_{{\xi_1}^F{\xi_2}^L}) = \gamma_{{\xi_1}^F{\xi_2}^L}[\overline{K}_{\phi(\xi_1^{F-1}\xi_2^L)},\ldots,\overline{K}_{\phi(\xi_1^\eta\xi_2^L)},\ldots,\overline{K}_{\phi\xi_2^L)},\overline{K}_{\phi(\xi_1^F\xi_2^{L-1})},\ldots,\overline{K}_{\phi(\xi_1^F\xi_2^\beta)},\ldots,$ $\overline{K}_{\phi(\xi_1^F)},\overline{K}_{\phi(\xi_1^{F-1}\xi_2^{T-1})},\ldots,\overline{K}_{\phi(\xi_1^{F-1}\xi_2^\beta)},\ldots,\overline{K}_{\phi(\xi_1^{F-1})},\ldots,\overline{K}_{\phi(\xi_1^\alpha\xi_2^{T-1})},\ldots,K_{\phi(\xi_1^\eta\xi_2^\beta)},\ldots,K_{\phi(\xi_1^\eta)},$ $K_{\phi(\xi_2^{L-1})},\ldots,K_{\phi(\xi_2^\beta)},\ldots,K_{\phi(\xi_2)}]$.

Now, we have to use Theorem 4.1, to calculate the D_α eigenvalues of $\Gamma(\mathbb{Z}_{\xi_1{}^F\xi_2{}^L})$ for this we have to calculate the values of $\varkappa' s$. The largest diameter of the ring's zero-divisor graph is 3, as is widely known. So that $\xi_1^\varepsilon \sim \xi_2^\varepsilon$ if and only if $n = \varepsilon = x$, otherwise $\xi_1^\varepsilon \sim \xi_\varepsilon^k \xi_2^n$, $k = \varepsilon \geq n$ and $\xi_2^x \sim \xi_1^n \xi_2^h, h + x \geq n$ and finally $\xi_1^k \xi_2^n \sim \xi_1^n \xi_2 h, k \geq 1, h \geq 1$. This means, $d(\xi_1^\varepsilon, \xi_2^x) = 3$, if $1 \leq x, \varepsilon \leq n-1$ in $\gamma_{\xi_1^F \xi_2^L}$. At most 2 is the distance between other vertices as well. Now by Theorem 4.1

$$\begin{aligned}\varkappa_1' =&2(\phi(\xi_1^{F-2}\xi_2^L) + \cdots + \phi(\xi_1^\eta \xi_2^L) + \cdots + \phi(\xi_2^L)) + 3(\phi(\xi_1^F \xi_2^{L-1}) + \cdots + \phi(\xi_1^F \xi_2^\beta) + \cdots + \\ &\phi(\xi_1^F)) + 2(\phi(\xi_1^{F-1}\xi_2^{L-1}) + \cdots + \phi(\xi_1^{F-1}\xi_2^\beta) + \cdots + \phi(\xi_1^{F-1})) + \cdots + 2(\phi(\xi_1^\eta \xi_2^{L-1}) + \cdots \\ &+ \phi(\xi_1^\eta \xi_2^\beta) + \cdots + \phi(\xi_1^\eta)) + 2(\phi(\xi_2^{L-1}) + \cdots + \phi(\xi_2^\beta) + \cdots + \phi(\xi_2)) - \phi(\xi_1),\end{aligned}$$

where by definition of $\varkappa_1'$, $\phi(\xi_1^{F-1}\xi_2^L)$ is removed and $\xi_1 \sim \xi_1^{F-1}\xi_2^L$, so we subtract $\phi(\xi_1)$. As $\sum_{s\setminus l} \phi(\xi_1) = l$, so order of $\Gamma(\mathbb{Z}_n)$ is $\varrho' = n - \phi(n) - 1 = \sum_{1,n\neq s\setminus l} \phi(n)$.

By Theorem 4.1, and using $\sum_{i=1}^{m} \phi(q^i) = q^m - 1$ and $\phi(st) = \phi(s)\phi(t)$ if and only if $(s,t) = 1$, we simplify the form of $\varkappa_1'$ as:

$$\begin{aligned}\varkappa_1' &= 2(\varrho' - \phi(\xi_1^{F-1}\xi_2^L)) + (\phi(\xi_1^F \xi_2^{L-1}) + \cdots + \phi(\xi_1^F \xi_2^\beta) + \cdots + \phi(\xi_1^F)) - \phi(\xi_1) \\ &= 2(\varrho' - \phi(\xi_1^{F-1}\xi_2^L)) + \phi(\xi_1^F)\big(\phi(\xi_2^{L-1}) + \cdots + \phi(\xi_2^\beta) + \cdots + \phi(\xi_2) + 1\big) - \phi(\xi_1) \\ &= 2(\varrho' - \phi(\xi_1^{F-1}\xi_2^L)) + \phi(\xi_1^F)\xi_2^{L-1} - \phi(\xi_1).\end{aligned}$$

Now, by using Theorem 4.1 D_α eigenvalue,

$$\begin{aligned}\lambda_1 &= 2\alpha m_1 + \alpha\varkappa_1' - 2 \\ &= 2\alpha(\phi(\xi_1^{F-1}\xi_2^L)) + 2\alpha(\varrho' - \phi(\xi_1^{F-1}\xi_2^L)) + \alpha\phi(\xi_1^F)\xi_2^{L-1} - \alpha\phi(\xi_1) - 2 \\ &= 2\alpha\varrho' + \alpha\phi(\xi_1^F)\xi_2^{L-1} - \alpha\phi(\xi_1) - 2,\end{aligned}$$

is the D_α eigenvalue with multiplicity $\phi(\xi_1^{F-1}\xi_2^L) - 1$. Other $\varkappa_\varepsilon'$'s are given by

$$\begin{aligned}\varkappa_\varepsilon' &= 2(\varrho' - \phi(\xi_1^{F-x}\xi_2^L)) + \phi(\xi_1^F)\xi_2^{L-1} - (\xi_1^x - 1),\ for\ \varepsilon = x = 2, \ldots, \alpha, \ldots, F-1, \\ \varkappa_F' &= 2(\varrho' - \phi(\xi_2^L)) + \phi(\xi_1^F)(\xi_2^{L-1} - 1) - (\xi_1^L - 1), \\ \varkappa_\varepsilon' &= 2(\varrho' - \phi(\xi_1^F \xi_2^{L-x})) + \phi(\xi_1^L)\xi_1^{F-1} - (\xi_2^x - 1),\ for\ \ \varepsilon = F+1, \ldots, F+L-1 \\ &\quad and\ \ x = 1, \ldots, \beta, \ldots, L-1, \\ \varkappa_{F+L}' &= 2(\varrho' - \phi(\xi_1^F)) + \phi(\xi_2^L)(\xi_1^{F-1} - 1) - (\xi_2^L - 1), \\ \varkappa_\varepsilon' &= 2(\varrho' - \phi(\xi_1^{F-1}\xi_2^{L-x})) - (\xi_1 \xi_2^x - 1),\ for\ \ \varepsilon = F+L+1, \ldots, F+2L \\ &\quad and\ \ x = 1, \ldots, \beta, \ldots, T, \\ &\ \vdots \\ \varkappa_\varepsilon' &= 2(\varrho' - \phi(\xi_1^\alpha \xi_2^{L-x})) - (\xi_1^\alpha \xi_2^x - 1),\ for\ \ \varepsilon = F+\alpha L+1, \ldots, F+\alpha L+\beta-1 \\ &\quad and\ \ x = 1, \ldots, \beta-1, \\ \varkappa_\varepsilon' &= 2\varrho' - \phi(\xi_1^\alpha \xi_2^{L-x}) - (\xi_1^\alpha \xi_2^x - 1),\ for\ \ \varepsilon = F+\alpha L+\beta, \ldots, F+(\alpha+1)L \\ &\quad and\ \ x = \beta, \ldots, L, \\ &\ \vdots \\ \varkappa_\varepsilon' &= 2(\varrho' - \phi(\xi_2^{L-x})) - (\xi_1^F \xi_2^x - 1),\ for\ \ \varepsilon = F+FL+1, \ldots, F+FL+\beta-1 \\ &\quad and\ \ x = 1, \ldots, \beta-1, \\ \varkappa_\varepsilon' &= 2\varrho' - \phi(\xi_2^{L-x}) - (\xi_1^F \xi_2^x - 1),\ for\ \ \varepsilon = F+FL+\beta, \ldots, F+FL+L-1 \\ &\quad and\ \ x = \beta, \ldots, L-1.\end{aligned}$$

Next, we utilize these $\varkappa'_\varepsilon$ values and Theorem 4.1, the D_α-spectrum of $\Gamma(\mathbb{Z}_{\xi_1^F\xi_2^L})$ consists of the eigenvalues

$$\lambda_\varepsilon = 2\alpha\varrho' + \alpha\phi(\xi_1^F)\xi_2^{L-1} - \alpha(\xi_1^x - 1) - 2,\ for\ \varepsilon = x = 1, 2, \ldots, \eta, \ldots, F-1,$$
$$\lambda_S = 2\alpha\varrho' + \alpha\phi(\xi_1^F)(\xi_2^{L-1} - 1) - \alpha(\xi_1^L - 1) - 2,$$
$$\lambda_\varepsilon = 2\alpha\varrho' + \alpha\xi_1^{F-1}\phi(\xi_2^L) - \alpha(\xi_2^x - 1) - 2,\ for\ x = 1, 2, \ldots, L-1,\ \text{and}\ \varepsilon = F+1, \ldots, F+L-1,$$
$$\lambda_{F+L} = 2\alpha\varrho' + \alpha\phi(\xi_2^L)(\xi_1^{F-1} - 1) - \alpha(\xi_2^L - 1) - 2,$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1\xi_2^x - 1) - 2,\ for\ x = 1, 2, \ldots, L,\ and\ \varepsilon = F+L+1, \ldots, F+2L,$$
$$\vdots$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^\eta\xi_2^x - 1) - 2,\ for\ x = 1, 2, \ldots, \beta-1,\ and\ \varepsilon = F+\beta L+1, \ldots, F+\beta L+\beta-1,$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^\eta\xi_2^x - 1) - 1,\ for\ x = \beta, \ldots, L,\ and\ \varepsilon = F+\beta L+\beta, \ldots, F+(\beta+1)L,$$
$$\vdots$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^F\xi_2^x - 1) - 2,\ for\ x = 1, 2, \ldots, \beta-1,\ and\ \varepsilon = F+FL+1, \ldots, \beta-1,$$
$$\lambda_\varepsilon = 2\alpha\varrho' - \alpha(\xi_1^F\xi_2^x - 1) - 1,\ for\ x = \beta, \ldots, L,\ and\ \varepsilon = F+FL+\beta, \ldots, F+FL+L-1,$$

with multiplicities $\phi(\xi_1^{F-1}\xi_2^L) - 1, \phi(\xi_1^F\xi_2^{L-x}) - 1, \phi(\xi_1^{F-\varepsilon}\xi_2^{L-x}) - 1, \ldots, \phi(\xi_1^\beta\xi_2^{L-x}) - 1, \ldots, \phi(\xi_2^{L-x}) - 1$, respectively, where $\varepsilon = 1, \ldots, F$ and $x = 1, \ldots, L$. D_α-eigenvalues that remain for the graph $\Gamma(\mathbb{Z}_{\xi_1^F\xi_2^L})$ are eigenvalues of the matrix (4.1). □

The following results are an obvious outcome of the Theorem 4.3.

Corollary 4.4. *Let $n = \xi_1^{2\eta}$, where $\xi_1 > 2$ is prime and η is a positive integer. Then the D_α-spectrum of $\Gamma(\mathbb{Z}_n)$ consists of eigenvalue λ_ε with multiplicity $\phi(p^{2\eta-\varepsilon})-1$, for $\varepsilon = 1, 2, \ldots, \eta-1$, the eigenvalue λ_j with multiplicity $\phi(p^{2\alpha-j}) - 1$, for $j = \eta, \eta+1, \ldots, 2\eta-2, 2\eta-1$, and*

$$\lambda_\varepsilon = \begin{cases} \alpha(2m_\varepsilon + \varkappa') - 2, & for\ \ \varepsilon = 1, 2, \ldots, \eta-1, \\ \alpha(m_\varepsilon + \varkappa') - 1 & for\ \ \varepsilon = \eta, \eta+1, \eta+2, \ldots, 2\eta-1. \end{cases}$$

Where m_ε are order of $\Gamma(A_{\xi_1^\varepsilon})$ for $\varepsilon = 1, 2, 3, \ldots, 2\eta-2, 2\eta-1$ and $\varkappa' = \sum_{j=1 j\neq\varepsilon}^{2\eta-1} m_j d_{\varepsilon,j}$. The eigenvalues of the matrix (4.1) are the other D_α eigenvalue of $\Gamma(\mathbb{Z}_n)$.

If we take $L = 0$ and $\eta = 1$ in Theorem 4.3, we have following result.

Corollary 4.5. *The D_α spectrum of $\Gamma(\mathbb{Z}_n)$, if $n = \xi_1^2$ is*

$$\left\{ \begin{matrix} \xi_1 - 2 & \alpha(\xi_1 - 1) - 1 \\ 1 & \xi_1 - 2 \end{matrix} \right\}.$$

If we take $L = 0$ and $\eta = 2$ in Theorem 4.3 the result follows.

Corollary 4.6. *The D_α spectrum of $\Gamma(\mathbb{Z}_n)$, if $n = \xi_1^4$ is*

$$\left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \phi(\xi_1^3) - 1 & \phi(\xi_1^2) - 1 & \phi(\xi_1) - 1 \end{matrix} \right\},$$

where $\lambda_1 = \alpha(2\xi_1^3 - \xi_1 - 1) - 2, \lambda_2 = 2\alpha\xi_1^3 - \alpha\xi_1^2 - \alpha - 1$ *and* $\lambda_3 = \alpha(\xi_1^3 - 1) - 1$. *The other* D_α*-eigenvalues of* $\Gamma(\mathbb{Z}_n)$ *are the eigenvalues of following matrix*

$$\begin{bmatrix} 2\alpha\xi_1^2 - \alpha(\xi_1+1) + 2(m_1 - 1) & 2\delta\sqrt{\phi(\xi_1^3)\phi(\xi_1^2)} & \delta\sqrt{\phi(\xi_1^3)\phi(\xi_1)} \\ 2\delta\sqrt{\phi(\xi_1^3)\phi(\xi_1^2)} & \alpha(\xi_1 - 1)(2\xi_1^2 + 1) + m_2 - 1 & \delta\sqrt{\phi(\xi_1^2)\phi(\xi_1)} \\ \delta\sqrt{\phi(\xi_1^3)\phi(\xi_1)} & \delta\sqrt{\phi(\xi_1^2)\phi(\xi_1)} & \alpha\xi_1(\xi_1^2 - 1) + m_3 - 1 \end{bmatrix}$$

where $\delta = 1 - \alpha$.

Theorem 4.7. *For distinct primes* ξ_1 *and* ξ_2*, if* $n = {\xi_1}^2\xi_2$ *then, the* D_α *spectrum of the graph* $\Gamma(\mathbb{Z}_n)$ *is given by*

$$\left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \phi(\xi_1\xi_2) - 1 & \phi(\xi_1^2) - 1 & \phi(\xi_1) - 1 & \phi(\xi_2) - 1 \end{matrix} \right\}.$$

Where $\lambda_1 = \alpha(2\xi_1\xi_2 + 3\xi_1^2 - 4\xi_1 - 1) - 2, \lambda_2 = \alpha(2\xi_1^2 + 3\xi_1\xi_2 - 3\xi_1 - 2\xi_2) - 2, \lambda_3 = \alpha(\xi_1\xi_2 + 2\xi_1^2 - 2\xi_1 - 1) - 1$ *and* $\lambda_4 = \alpha(2\xi_1\xi_2 + \xi_1^2 - 2\xi_1 - 1) - 2$ *and the matrix given in 4.4 , can be used to determine the remaining four eigenvalues.*

Proof. Let $n = \xi_1^2\xi_2$, where ξ_1 and ξ_2 are distinct primes. Since $\xi_1, \xi_2, \xi_1\xi_2, \xi_1^2$ are the proper divisors of n such that $\xi_1 \sim \xi_1\xi_2 \sim \xi_1^2 \sim \xi_1$. Therefore, by Lemma 2.5 and Corollary 2.4, we have

$$\begin{aligned} \Gamma(\mathbb{Z}_{\xi_1^2\xi_2}) &= \gamma_{\xi_1^2\xi_2}[\Gamma(A_{\xi_1}), \Gamma(A_{\xi_2}), \Gamma(A_{\xi_1\xi_2}), \Gamma(A_{\xi_1^2})] \\ &= \gamma_{\xi_1^2\xi_2}[\overline{K}_{\phi(\xi_1\xi_2)}, \overline{K}_{\phi(\xi_1^2)}, K_{\phi(\xi_1)}, \overline{K}_{\phi(\xi_2)}]. \end{aligned}$$

Also, $m_1 = \phi(\xi_1\xi_2), m_2 = \phi(\xi_1^2), m_3 = \phi(\xi_1)$ and $m_4 = \phi(\xi_2)$. Now, by Theorem 4.1, the values of $\varkappa_i'$ is

$$(\varkappa_1', \varkappa_2', \varkappa_3', \varkappa_4') = (3m_2 + m_3 + 2m_4, 3m_1 + 2m_3 + m_4, m_1 + 2m_2 + m_4, 2m_1 + m_2 + m_3).$$

Again by using Theorem 4.1, the D_α eigenvalues are,

$$\begin{aligned} \lambda_1 &= \alpha(2m_1 + \varkappa_1') - 2 \\ &= \alpha(2\xi_1\xi_2 + 3\xi_1^2 - 4\xi_1 - 1) - 2, \\ \lambda_2 &= \alpha(2m_2 + \varkappa_2') - 2 \\ &= \alpha(2\xi_1^2 + 3\xi_1\xi_2 - 3\xi_1 - 2\xi_2) - 2, \\ \lambda_3 &= \alpha(m_3 + \varkappa_3') - 1 \\ &= \alpha(\xi_1\xi_2 + 2\xi_1^2 - 2\xi_1 - 1) - 1, \\ \lambda_4 &= \alpha(2m_4 + \varkappa_4') - 2 \\ &= \alpha(2\xi_1\xi_2 + \xi_1^2 - 2\xi_1 - 1) - 2. \end{aligned}$$

Therefore, the D_α spectrum of $\Gamma(\mathbb{Z}_{\xi_1^2\xi_2})$ is

$$\left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \phi(\xi_1\xi_2) - 1 & \phi(\xi_1^2) - 1 & \phi(\xi_1) - 1 & \phi(\xi_2) - 1 \end{matrix} \right\}.$$

Additionally, the remaining four eigenvalues are provided by the matrix below

$$\mathcal{K} = \begin{bmatrix} A & 3\delta\sqrt{\phi(\xi_1\xi_2)\phi(\xi_1^2)} & \delta\sqrt{\phi(\xi_1\xi_2)\phi(\xi_1)} & 2\delta\sqrt{\phi(\xi_1\xi_2)\phi(\xi_2)} \\ 3\delta\sqrt{\phi(\xi_1^2)\phi(\xi_1\xi_2)} & B & 2\delta\sqrt{\phi(\xi_1^2)\phi(\xi_1)} & \delta\sqrt{\phi(\xi_1^2)\phi(\xi_2)} \\ \delta\sqrt{\phi(\xi_1)\phi(\xi_1\xi_2)} & 2\delta\sqrt{\phi(\xi_1)\phi(\xi_1^2)} & C & \delta\sqrt{\phi(\xi_2)\phi(\xi_1)} \\ 2\delta\sqrt{\phi(\xi_2)\phi(\xi_1\xi_2)} & \delta\sqrt{\phi(\xi_1^2)\phi(\xi_2)} & \delta\sqrt{\phi(\xi_1)\phi(\xi_2)} & D \end{bmatrix}. \quad (4.4)$$

Where $A = \alpha(3\xi_1^2 - 2(\xi_1 - \xi_2) - 3) + 2(m_1 - 1), B = \alpha(3\xi_1\xi_2 - \xi_1 - 4\xi_2) + 2(m_2 - 1), C = \alpha(\xi_1\xi_2 - 3\xi_1 + 2\xi_1^2) + (m_3 - 1), D = \alpha(2\xi_1\xi_2 - 2\xi_1 - 2\xi_2 + \xi_1^2 + 1) + 2(m_4 - 1)$ and $\delta = 1 - \alpha$. □

Conclusion and further work

Our result presents the D_α spectrum of the zero-divisor graph of the integers modulo n obtained through the generalized join of its induced subgraphs. Further one can calculate D_α spectrum for co-zero divisor graph, unit graph, co-maximal graph and many more such graphs.

Conflict of interest

All authors made equal contribution and among the authors there is no conflict of interest.

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