Palestine Journal of Mathematics

Vol 15(1)(2026) , 747–753

On the distribution of the greatest $\mathrm{r^{th}}$ power common divisor of the elements in integral part sets

Pattira Ruengsinsub, Teerapat Srichan and Pinthira Tangsupphathawat

Communicated by: Ayman Badawi

MSC 2010 Classifications: 11L07, 11N69, 11P21.

Keywords and phrases: Beatty sequence, coprimality, the greatest r^{th} power common divisor, Piatetski-Shapiro sequence

This work was financially supported by National Research Council of Thailand (NRCT) and Kasetsart University N42A670565.

Corresponding Author: Teerapat Srichan

Abstract Let $r \geq 1$ be a fixed integer, the greatest r^{th} power common divisor of positive integers a and b is defined to be the largest positive integers d^r such that $d^r|a$ and $d^r|b$, which is denoted by $\gcd(a, b)_r$ and called the $r-$gcd of a and b. In this paper we study the distribution of the greatest r^{th} power common divisor of the elements in integral part sets.

1 Introduction and results

Let $\lfloor z \rfloor$ denote the integer part of a real z, Piatetski-Shapiro sequences are defined by

$$\mathbb{N}^c = \{\lfloor n^c \rfloor\}_{n\in\mathbb{N}},\ (c > 1, c \notin \mathbb{N}).$$

Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. The Beatty sequence of parameter α and β is defined by

$$\{\lfloor \alpha n + \beta \rfloor\}_{n\in\mathbb{N}}.$$

The natural density for the set of pairs of integers which are coprime is a classical result in number theory. In [3], Dirichlet asserts that the proportion of coprime pairs of integers in $\{1, \dots, n\}$,

$$\frac{1}{n^2}\#\{(n_1, n_2) \in \{1, \dots, n\}^2; \gcd(n_1, n_2) = 1\}$$

tends to $\frac{1}{\zeta(2)} = \frac{6}{\pi^2} \sim 0.608...$, where ζ is the Riemann zeta-function. The proof can be found in [6, Theorem 332]. In 1953, Watson [11] proved that for any given irrational number α, the positive integers n for which $(n, \lfloor \alpha n \rfloor) = 1$, have the natural density $\frac{6}{\pi^2}$. Later, Estermann [5] gave a different proof of a generalization of Watson's theorem. Afterwards, Erdös and Lorentz [4] established sufficient conditions for a differentiable function $f : [1, \infty) \to \mathbb{R}$ to satisfy $(n, f(n)) = 1$, have the natural density $\frac{6}{\pi^2}$. In 2022, Pimsert, Srichan and Tangsupphathawat [9] studied the asymptotic formulas for the number of the integral pairs $(\lfloor a^c \rfloor, \lfloor b^c \rfloor)$ that are coprime, $a, b \leq x$ and $1 < c < 2$. They proved that, as $x \to \infty$,

$$\sum_{\substack{a,b\leq x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)=1}} 1 = \frac{x^2}{\zeta(2)} + \begin{cases} O(x^{(c+4)/3}) & ; 1 < c \leq \frac{5}{4}, \\ O(x^{(c+1)/2}) & ; \frac{5}{4} < c < \frac{3}{2} \end{cases},$$

and for $1 < c_1 < c_2 < \frac{3}{2}$,

$$\sum_{\substack{a,b\leq x\\ \gcd(\lfloor a^{c_1}\rfloor,\lfloor b^{c_2}\rfloor)=1}} 1 = \frac{x^2}{\zeta(2)} + \begin{cases} O(x^{(c_2+4)/3}) & ; 1 < c_1 \leq \frac{5}{4}, \\ O(x^{1/2+(2c_1+c_2)/3}) & ; \frac{5}{4} < c_1 < \frac{3}{2}. \end{cases}$$

Very recently, Srichan [10] studied the coprimality of any pairs in other sequence. He established the natural density of the set

$$\{(n,m)\in\mathbb{N}^2 | n,m\le x; \gcd(\lfloor g_1(n)\rfloor,\lfloor g_2(n)\rfloor)=1\},$$

where $g_i(n) = n, n^c, \alpha n+\beta,\ i = 1,2$ and showed that the natural density of such sets also tends to $\frac{1}{\zeta(2)}$. It would be interesting to study the natural density of these sets with other greatest common divisors.

Let $r \ge 1$ be a fixed integer, the greatest r^{th} power common divisor of positive integers a and b is defined to be the largest positive integers d^r such that $d^r|a$ and $d^r|b$, which is denoted by $\gcd(a,b)_r$ and called the $r-$gcd of a and b. Note that $\gcd(a,b)_1 = \gcd(a,b)$. We refer [8] for the related research on the greatest r^{th} power common divisor. Normally, we use the greatest common divisor function to count visible lattice points. In 2018, Harris and Omar [7] showed that a point $(k,n)\in\mathbb{Z}^2$ is visible along the curve $y = hx^r$ if and only if $\max\{d \ge 1 : d \mid k \text{ and } d^r \mid n\} = 1$, where $r\in\mathbb{N}$ and $h\in\mathbb{Q}$. They mention that the greatest r^{th} power common divisor provides a necessary criterion for the combined visibility of a lattice point along the curves $y = hx^r$ or $x = hy^r$. This indicates that the fundamental research on the greatest r^{th} power common divisor is important in future work.

The goal of this paper is to study the distribution of the greatest r^{th} power common divisor of the elements in integral part sets, for $r > 1$.

We obtain the following results.

Theorem 1.1. *Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta\in[0,\alpha)$. As $x\to\infty$, we have*

$$\sum_{\substack{a,b\le x\\ gcd(a,b)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O(x), \tag{1.1}$$

$$\sum_{\substack{a,b\le x\\ gcd(a,\lfloor b^c\rfloor)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O\big(x^{(c+4)/3}\big), \tag{1.2}$$

$$\sum_{\substack{a,b\le x\\ gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O(x^{(c+4)/3}), \tag{1.3}$$

$$\sum_{\substack{a,b\le x\\ gcd(\lfloor a^c\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O(x^{(c+4)/3}), \tag{1.4}$$

$$\sum_{\substack{a,b\le x\\ gcd(a,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O\Big(x^{1+1/2r}\log^{3-3/2r-\epsilon} x\Big), \tag{1.5}$$

$$\sum_{\substack{a,b\le x\\ gcd(\lfloor \alpha a+\beta\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O\Big(x^{1+1/2r}\log^{3-3/2r-\epsilon} x\Big). \tag{1.6}$$

2 Lemmas

Throughout this paper, implied constants in symbols O and $\ll$ may depend on parameters α,β,c,ϵ, but are absolute otherwise. For given function F and G, the notations $f \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \le C|G|$ holds with some constant $C > 0$.

The following Lemmas are the main ingredient in the our proofs.

Lemma 2.1. *([2]) For $1 < c < 2$. Let $x \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ such that $0 \leq a < q \leq x^c$. Then*

$$\sum_{\substack{n \leq x \\ \lfloor n^c \rfloor \equiv a \pmod q}} 1 = \frac{x}{q} + O\big(\min(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}})\big).$$

Lemma 2.2. *([1]) For $\alpha > 1$ irrational and with bounded partial quotients, $\beta \in [0, \alpha)$, and positive integer $d \geq 2, 0 \leq a < d$, we have*

$$\sum_{\substack{n \leq x \\ \lfloor \alpha n + \beta \rfloor \equiv a (\bmod d)}} 1 = \frac{x}{d} + O(d \log^3 x), \quad \textit{as } x \to \infty.$$

For growing difference d the result is non-trivial provided $d \ll \sqrt{x} \log^{-3/2-\varepsilon} x$, for $\varepsilon > 0$.

3 Proofs

Proof of Theorem 1.1. To prove (1.1), we write

$$\begin{aligned}
\sum_{\substack{a,b \leq x \\ \gcd(a,b)_r = 1}} 1 &= \sum_{a,b \leq x} \sum_{\substack{d^r | a \\ d^r | b}} \mu(d) = \sum_{d \leq x^{1/r}} \mu(d) \sum_{\substack{a \leq x \\ a \equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b \leq x \\ b \equiv 0 \pmod{d^r}}} 1 \\
&= \sum_{d \leq x^{1/r}} \mu(d) \big(\frac{x}{d^r} + O(1)\big)\big(\frac{x}{d^r} + O(1)\big) \\
&= x^2 \sum_{d \leq x^{1/r}} \frac{\mu(d)}{d^{2r}} + O(\sum_{d \leq x^{1/r}} \frac{x}{d^r}) + O(\sum_{d \leq x^{1/r}} 1) \\
&= x^2 \sum_{d \leq x^{1/r}} \frac{\mu(d)}{d^{2r}} + O(x) + O(x^{1/r}). \qquad (3.1)
\end{aligned}$$

We note that, for $r > 1$,

$$\begin{aligned}
\sum_{d \leq x^{1/r}} \frac{\mu(d)}{d^{2r}} &= \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{2r}} + \sum_{d > x^{1/r}} \frac{\mu(d)}{d^{2r}} \\
&= \frac{1}{\zeta(2r)} + O(x^{1/r - 2}). \qquad (3.2)
\end{aligned}$$

The proof of (1.1) follows from (3.1) and (3.2).

We now prove (1.2). We do this by considering the value of c in the two cases. In view of

Lemma 2.1, for $1 < c \leq 3/2$, we have

$$\begin{aligned}\sum_{\substack{a,b\leq x\\ \gcd(a,\lfloor b^c\rfloor)_r=1}} 1 &= \sum_{a,b\leq x}\sum_{\substack{d^r|a\\ d^r|\lfloor b^c\rfloor}}\mu(d) = \sum_{d\leq x^{1/r}}\mu(d)\sum_{\substack{a\leq x\\ a\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\leq x\\ \lfloor b^c\rfloor\equiv 0 \pmod{d^r}}} 1\\
&= \sum_{d\leq x^{1/r}}\mu(d)\Big(\frac{x}{d^r}+O(1)\Big)\Big(\frac{x}{d^r}+O(\min(\frac{x^c}{d^r},\frac{x^{(c+1)/3}}{d^{r/3}}))\Big)\\
&= \sum_{d\leq x^{c/r-1/2r}}\mu(d)\Big(\frac{x}{d^r}+O(1)\Big)\Big(\frac{x}{d^r}+O(\frac{x^{(c+1)/3}}{d^{r/3}})\Big)\\
&\quad+\sum_{x^{c/r-1/2r}<d\leq x^{1/r}}\mu(d)\Big(\frac{x}{d^r}+O(1)\Big)\Big(\frac{x}{d^r}+O(\frac{x^c}{d^r})\Big)\\
&= \sum_{d\leq x^{c/r-1/2r}}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x}{d^r})+O(\frac{x^{(c+1)/3}}{d^{r/3}})+O(\frac{x^{(c+4)/3}}{d^{4r/3}})\Big)\\
&\quad+\sum_{x^{c/r-1/2r}<d\leq x^{1/r}}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x}{d^r})+O(\frac{x^{c+1}}{d^{2r}})+O(\frac{x^c}{d^r})\Big). \qquad (3.3)\end{aligned}$$

We note that $d \leq x^{c/r-1/2r} < x^{1/r} < x^{c/r+1/r}$, then $\frac{x^{(c+4)/3}}{d^{4r/3}} > \frac{x}{d^r}$ and $\frac{x^{(c+4)/3}}{d^{4r/3}} > \frac{x^{(c+1)/3}}{d^{r/3}}$. Thus, the O-terms of the first sum in (3.3) are dominated by $O\Big(\frac{x^{(c+4)/3}}{d^{4r/3}}\Big)$. For the second sum in (3.3), we note that $\frac{x^{c+1}}{d^{2r}} > \frac{x^c}{d^r} > \frac{x}{d^r}$, when $d < x^{1/r}$. Thus the O-terms of the second sum in (3.3) are dominated by $O\Big(\frac{x^{c+1}}{d^{2r}}\Big)$. Then, we have

$$\begin{aligned}\sum_{\substack{a,b\leq x\\ \gcd(a,\lfloor b^c\rfloor)_r=1}} 1 &= \sum_{d\leq x^{c/r-1/2r}}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x^{(c+4)/3}}{d^{4r/3}})\Big)+\sum_{x^{c/r-1/2r}<d\leq x^{1/r}}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x^{c+1}}{d^{2r}})\Big)\\
&= \sum_{d\leq x^{1/r}}\mu(d)\frac{x^2}{d^{2r}}+O(x^{(c+4)/3})+O(x^{2-1/2r+c/r-c}).\end{aligned}$$

From $c > 1$, we have $\frac{c+4}{3} > 2-\frac{1}{2r}+\frac{c}{r}-c$. Thus

$$\sum_{\substack{a,b\leq x\\ \gcd(a,\lfloor b^c\rfloor)_r=1}} 1 = \sum_{d\leq x^{1/r}}\mu(d)\frac{x^2}{d^{2r}}+O(x^{(c+4)/3}). \qquad (3.4)$$

Thus, for the case $1 < c \leq \frac{3}{2}$, the proof of (1.2) follows from (3.2), (3.4) and the inequality $\frac{c+4}{3} > \frac{1}{r}$ for $r \geq 2$.

For $3/2 < c < 2$, the case $x^{c/r-1/2r} < d \leq x^{1/r}$ is impossible. From Lemma 2.1, we have

$$\begin{aligned}\sum_{\substack{a,b\leq x\\ \gcd(a,\lfloor b^c\rfloor)_r=1}} 1 &= \sum_{a,b\leq x}\sum_{\substack{d^r|a\\ d^r|\lfloor b^c\rfloor}}\mu(d) = \sum_{d\leq x^{1/r}}\mu(d)\sum_{\substack{a\leq x\\ a\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\leq x\\ \lfloor b^c\rfloor\equiv 0 \pmod{d^r}}} 1\\
&= \sum_{d\leq x^{1/r}}\mu(d)\Big(\frac{x}{d^r}+O(1)\Big)\Big(\frac{x}{d^r}+O(\min(\frac{x^c}{d^r},\frac{x^{(c+1)/3}}{d^{r/3}}))\Big)\\
&= \sum_{d\leq x^{1/r}}\mu(d)\Big(\frac{x}{d^r}+O(1)\Big)\Big(\frac{x}{d^r}+O(\frac{x^{(c+1)/3}}{d^{r/3}})\Big)\\
&= \sum_{d\leq x^{1/r}}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x}{d^r})+O(\frac{x^{(c+1)/3}}{d^{r/3}})+O(\frac{x^{(c+4)/3}}{d^{4r/3}})\Big).\end{aligned}$$

We note that, for $d \le x^{1/r}$, $\frac{x}{d^r} \le \frac{x^{(c+4)/3}}{d^{4r/3}}$ and $\frac{x^{(c+1)/3}}{d^{r/3}} \le \frac{x^{(c+4)/3}}{d^{4r/3}}$. Thus, in view of (3.2), we have

$$\sum_{\substack{a,b\le x\\ \gcd(a,\lfloor b^c\rfloor)_r=1}} 1 = \frac{x^2}{\zeta(2r)} + O(x^{(c+4)/3}).$$

This completes the proof of (1.2).

Next we will prove (1.3).

$$\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)_r=1}} 1 = \sum_{a,b\le x} \sum_{\substack{d^r|\lfloor a^c\rfloor\\ d^r|\lfloor b^c\rfloor}} \mu(d) = \sum_{d\le x^{c/r}} \mu(d) \sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\le x\\ \lfloor b^c\rfloor\equiv 0 \pmod{d^r}}} 1.$$

In view of Lemma 2.1, we have

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)_r=1}} 1 &= \sum_{d\le x^{c/r}} \mu(d)\left(\frac{x}{d^r} + O(\min(\frac{x^c}{d^r}, \frac{x^{(c+1)/3}}{d^{r/3}}))\right)^2\\
&= \sum_{d\le x^{c/r-1/2r}} \mu(d)\left(\frac{x}{d^r} + O(\frac{x^{(c+1)/3}}{d^{r/3}})\right)^2 + \sum_{x^{c/r-1/2r}<d\le x^{c/r}} \mu(d)\left(\frac{x}{d^r} + O(\frac{x^c}{d^r})\right)^2\\
&= \sum_{d\le x^{c/r-1/2r}} \mu(d)\left(\frac{x^2}{d^{2r}} + O(\frac{x^{(2c+2)/3}}{d^{2r/3}}) + O(\frac{x^{(c+4)/3}}{d^{4r/3}})\right)\\
&\quad + \sum_{x^{c/r-1/2r}<d\le x^{c/r}} \mu(d)\left(\frac{x^2}{d^{2r}} + O(\frac{x^{c+1}}{d^{2r}}) + O(\frac{x^{2c}}{d^{2r}})\right). \qquad (3.5)
\end{aligned}$$

We note that $x^{(2-c)/2r} < x^{c/r-1/2r}$. Thus, for $d \le x^{c/r-1/2r}$, the O-terms in the first sum of (3.5) are dominated by $O(\frac{x^{(c+4)/3}}{d^{4r/3}})$ and in the second sum of (3.5) we have $x^{2c} > x^{1+c}$. Thus, we have

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor b^c\rfloor)_r=1}} 1 &= \sum_{d\le x^{c/r-1/2r}} \mu(d)\left(\frac{x^2}{d^{2r}} + O(\frac{x^{(c+4)/3}}{d^{4r/3}})\right)\\
&\quad + \sum_{x^{c/r-1/2r}<d\le x^{c/r}} \mu(d)\left(\frac{x^2}{d^{2r}} + O(\frac{x^{2c}}{d^{2r}})\right)\\
&= \sum_{d\le x^{c/r}} \mu(d)\frac{x^2}{d^{2r}} + O(x^{(c+4)/3}) + O(x^{c/r+1-1/2r}). \qquad (3.6)
\end{aligned}$$

Then (1.3) follows from (3.2), (3.6) and the inequality $\frac{c+4}{3} > \frac{c}{r} + 1 - \frac{1}{2r}$, for $r \ge 2, 1 < c < 2$.

Next we prove (1.4). For $x > 1$, we write

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 &= \sum_{a,b\le x} \sum_{\substack{d^r|\lfloor a^c\rfloor\\ d^r|\lfloor \alpha b+\beta\rfloor}} \mu(d)\\
&= \sum_{d\le(\alpha x+\beta)^{1/r}} \mu(d) \sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0 \pmod{d^r}}} 1.
\end{aligned}$$

In view of Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 &= \sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon} x} \mu(d)\left(\frac{x}{d^r} + O(\min(\frac{x^c}{d^r}, \frac{x^{(c+1)/3}}{d^{r/3}}))\right)\left(\frac{x}{d^r} + O(d^r\log^3 x)\right)\\
&\quad + \sum_{x^{1/2r}\log^{-3/2r-\epsilon} x<d\le(\alpha x+\beta)^{1/r}} \mu(d) \sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0 \pmod{d^r}}} 1.
\end{aligned}$$

From $x^{1/2r}\log^{-3/2r-\epsilon}x < x^{c/r-1/2r}$, we have

$$\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x} \mu(d)\Big(\frac{x}{d^r}+O(\frac{x^{(c+1)/3}}{d^{r/3}})\Big)\Big(\frac{x}{d^r}+O(d^r\log^3 x)\Big)$$
$$+O\Big(\sum_{x^{1/2r}\log^{-3/2r-\epsilon}x<d\le(\alpha x+\beta)^{1/r}} \frac{x^2}{d^{2r}}\Big)$$
$$=\sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x} \mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x^{(c+4)/3}}{d^{4r/3}})+O(x\log^3 x)+O(x^{(c+1)/3}d^{2r/3}\log^3 x)\Big)$$
$$+O\Big(x^{1+1/2r}\log^{3-3/2r+\epsilon}x\Big).$$

It is easy to see that, for $d\le x^{1/2r}\log^{-3/2r-\epsilon}x$, $\frac{x^{(c+4)/3}}{d^{4r/3}}\ge x\log^3 x$, $\frac{x^{(c+4)/3}}{d^{4r/3}}\ge x^{(c+1)/3}d^{2r/3}\log^3 x$. Thus, in view of (3.2), we have

$$\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x} \mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x^{(c+4)/3}}{d^{4r/3}})\Big)+O\Big(x^{1+1/2r}\log^{3-3/2r+\epsilon}x\Big)$$
$$=\frac{x^2}{\zeta(2r)}+O(x^{(c+4)/3})+O\Big(x^{1+1/2r}\log^{3-3/2r+\epsilon}x\Big).$$

The proof of (1.4) follows from $\frac{c+1}{3}>\frac{1}{4}\ge\frac{1}{2r}$.

We now prove (1.5). In view of Lemma 2.2, we have

$$\sum_{\substack{a,b\le x\\ \gcd(a,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \sum_{a,b\le x}\sum_{\substack{d^r|a\\ d^r||\alpha b+\beta\rfloor}}\mu(d) = \sum_{d\le x^{1/r}}\mu(d)\sum_{\substack{a\le x\\ a\equiv 0 \pmod{d^r}}}1\sum_{\substack{b\le x\\ \lfloor\alpha b+\beta\rfloor\equiv 0 \pmod{d^r}}}1$$
$$=\sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x}\mu(d)\Big(\frac{x}{d^r}+O(1)\Big)\Big(\frac{x}{d^r}+O(d^r\log^3 x)\Big)$$
$$+\sum_{x^{1/2r}\log^{-3/2r-\epsilon}x<d\le x^{1/r}}\mu(d)\Big(\sum_{\substack{a\le x\\ a\equiv 0 \pmod{d^r}}}1\sum_{\substack{b\le x\\ \lfloor\alpha b+\beta\rfloor\equiv 0 \pmod{d^r}}}1\Big)$$
$$=\sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(\frac{x}{d^r})+O(x\log^3 x)+O(d^r\log^3 x)\Big)$$
$$+\sum_{x^{1/2r}\log^{-3/2r-\epsilon}x<d\le x^{1/r}}O\Big(\frac{x^2}{d^{2r}}\Big).$$

We note that, for $d\le x^{1/2r}\log^{-3/2r-\epsilon}x$, $x\log^3 x\ge\frac{x}{d^r}$ and $x\log^3 x\ge d^r\log^3 x$. Thus,

$$\sum_{\substack{a,b\le x\\ \gcd(a,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x}\mu(d)\Big(\frac{x^2}{d^{2r}}+O(x\log^3 x)\Big)+O\Big(x^{1/2r}\log^{3-3/2r+\epsilon}x\Big)$$
$$=x^2\sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon}x}\frac{\mu(d)}{d^{2r}}+O\Big(x^{1+1/2r}\log^{3-3/2r-\epsilon}x\Big).$$

In view of (3.2), the proof of (1.5) completes.

We now prove (1.6). Similarly to the proof of (1.5), in view of Lemma 2.2, we write

$$\sum_{\substack{a,b\le x\\ \gcd(\lfloor \alpha a+\beta\rfloor,\lfloor \alpha b+\beta\rfloor)_r=1}} 1 = \sum_{a,b\le x}\sum_{\substack{d^r|\lfloor \alpha a+\beta\rfloor\\ d^r|\lfloor \alpha b+\beta\rfloor}}\mu(d) = \sum_{d\le(\alpha x+\beta)^{1/r}}\mu(d)\sum_{\substack{a\le x\\ \lfloor \alpha a+\beta\rfloor\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0 \pmod{d^r}}} 1$$

$$= \sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon} x}\mu(d)\left(\frac{x}{d^r}+O(d^r\log^3 x)\right)^2$$

$$+ \sum_{x^{1/2r}\log^{-3/2r-\epsilon} x<d\le(\alpha x+\beta)^{1/r}}\mu(d)\sum_{\substack{a\le x\\ \lfloor \alpha a+\beta\rfloor\equiv 0 \pmod{d^r}}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0 \pmod{d^r}}} 1$$

$$= \sum_{d\le x^{1/2r}\log^{-3/2r-\epsilon} x}\mu(d)\left(\frac{x^2}{d^{2r}}+O(x\log^3 x)+O(d^{2r}\log^6 x)\right)$$

$$+ \sum_{x^{1/2r}\log^{-3/2r-\epsilon} x<d\le x^{1/r}} O\left(\frac{x^2}{d^{2r}}\right).$$

Using the same calculation as in the proof of (1.5), the equation (1.6) follows. □

Acknowledgments

This work was financially supported by National Research Council of Thailand (NRCT) and Kasetsart University N42A670565.

References

[1] A. V. Begunts and D. V. Goryashin, On the values of Beatty sequence in an arithmetic progression, *Chebyshevskii Sb.*, **21(1)**(2020), 343–346.

[2] J. M. Deshouillers, A remark on cube-free numbers in Segal-Piatetski-Shapiro sequences, *Hardy-Ramanujan J.*, **41**(2018), 127–132.

[3] G. L. Dirichlet, Über die Bestimmung der mittleren Werthe in der Zahlentheorie, *Abh. König. Preuss. Akad. Wiss.*, (1849), 69–83.

[4] P. Eröds and G. G. Lorentz, On the probability that "n and $g(n)$ are relatively prime", *Acta Arith.*, **5**(1959), 35–44.

[5] T. Estermann, On the number of primitive lattice points in a paralleiogram, *Can. J.*, **5**(1953), 456–459.

[6] G. H. Hardy and E. M.Wright, An Introduction to the Theory of Numbers, *Probab. Theory Relat. Fields*, Oxford University Press, Oxford, 2008.

[7] P. E. Harris and M. Omar, Lattice point visibility on power functions. *Integers* 2018, 18, 1–7.

[8] V. L. Klee, A generalization of Euler's ϕ-function. *Am. Math. Mon.* 1948, **55**, 358–359.

[9] W. Pimsert, T. Srichan and P. Tungsupphathawat, Coprimality of integers in Piatetski-Shapiro sequences, *Czech. Math. J.*, **73**(2023), 197–212.

[10] T. Srichan, On the distribution of the greatest common divisor of the elements in integral part sets, *Acta et Comment. Univ. Tartu. de Math.*, **27**(2023), 47–57.

[11] G. L. Watson, On integers n relatively prime to $(n, \lfloor \alpha n\rfloor)$,*Canad. J. Math.*, **5**(1953), 451–455.

Author information

Pattira Ruengsinsub, Department of Mathematics, Faculty of Science, Kasetsart University, Thailand.
E-mail: `fscipar@ku.ac.th`

Teerapat Srichan, Department of Mathematics, Faculty of Science, Kasetsart University, Thailand.
E-mail: `fscitrp@ku.ac.th`

Pinthira Tangsupphathawat, Department of Mathematics, Faculty of Science and Technology, Phranakhon Rajabhat University, Bangkok, 10220, Thailand.
E-mail: `t.pinthira@hotmail.com`
Received: 2025-05-05
Accepted: 2025-11-07