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Developments in Fuzzy Operators using Boas-Buck Polynomials

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Abstract. By incorporating principles from fuzzy theory, this study extends and refines the main findings of the classical approximation theory. We define fuzzy Boas-Buck operators and prove that they are fuzzy linear and positive. We give their moments and central moments and thus using fuzzy Korovkin theorem, prove their convergence property. Further, we study their rate of convergence using the fuzzy modulus of continuity and fuzzy weighted modulus of continuity. Since, the fuzzy Boas-Buck operators are a very generalized form of operators, so we also give two of its particular examples, namely Laguerre and Charlier operators, and further talk about their convergence properties. This paper also presents special cases of the fuzzy Boas-Buck operators which includes the fuzzy Brenke, Sheffer, Appell and Szasz operators for which further studies are possible in fuzzy sense.

1 Introduction

In 1956, Ralph P. Boas and R. Creighton Buck [1, 2] considered the generalized Appell polynomials by means of generating function of the type

$$\mathcal{W}(t)\,\mathcal{P}(x\mathcal{Q}(t)) = \sum_{k=0}^{\infty} a_k(x)t^k, \tag{1.1}$$

where $\mathcal{W}$, $\mathcal{P}$ and $\mathcal{Q}$ are analytic functions

$$\begin{aligned} \mathcal{W}(t) &= \sum_{k=0}^{\infty} w_k t^k, \quad w_0 \neq 0, \\ \mathcal{P}(t) &= \sum_{k=0}^{\infty} p_k t^k, \quad p_k \neq 0, \\ \mathcal{Q}(t) &= \sum_{k=1}^{\infty} q_k t^k, \quad q_1 \neq 0. \end{aligned} \tag{1.2}$$

The generating function derived by Boas and Buck serves as a generalization of numerous other generating functions. This offers a framework for defining different types of polynomials as its special cases, listed in Table 1.

Our study will be confined to the Boas-Buck type polynomials that satisfy:

$$\begin{aligned} &(i)\ \mathcal{W}(1) \neq 0, \quad \mathcal{Q}'(1) = 1, \quad a_k(x) \geqslant 0,\ k = 0, 1, 2, \dots, \\ &(ii)\ \mathcal{P} : \mathbb{R} \to (0, \infty), \\ &(iii)\ \text{For } |t| < \rho \ (\text{where } \rho > 1),\ \text{the equations (1.1) and (1.2) are convergent.} \end{aligned} \tag{1.3}$$

With the constraints mentioned above in (1.3), Sucu et al. [3] presented the linear positive operators using Boas-Buck type polynomials in the subsequent manner:

$$\mathcal{B}_n f = \frac{1}{\mathcal{W}(1)\,\mathcal{P}(nx\mathcal{Q}(1))} \sum_{k=0}^{\infty} a_k(nx)\, f\left(\frac{k}{n}\right), \tag{1.4}$$

where $x \geqslant 0$ and $n \in \mathbb{N}$.

Table 1. Different generating functions obtained from Boas-Buck polynomials.

S.No.	Analytic Functions	Generating Function	Explicit polynomial
1.	$\mathcal{P}(t) = e^t$	Sheffer polynomials [4] $\mathcal{W}(t)e^{x\mathcal{Q}(t)} = \sum_{k=0}^{\infty} a_k(x)t^k$	–
2.	$\mathcal{Q}(t) = t$	Brenke polynomials [5] $\mathcal{W}(t)\mathcal{P}(xt) = \sum_{k=0}^{\infty} a_k(x)t^k$	$a_k(x) = \sum_{r=0}^{k} w_{k-r}p_r x^r$
3.	$\mathcal{P}(t) = e^t$ $\mathcal{Q}(t) = t$	Appell polynomials [6] $\mathcal{W}(t)e^{tx} = \sum_{k=0}^{\infty} a_k(x)t^k$	–
4.	$\mathcal{W}(t) = 1$ $\mathcal{P}(t) = e^t$ $\mathcal{Q}(t) = t$	Exponential polynomials [7] $e^{tx} = \sum_{k=0}^{\infty} a_k(x)t^k$	$a_k(x) = x^k/k!$

Above defined is the classical approach to approximate real continuous function on real domain. While classical approximation theory has been successfully applied in numerous domains and remains a valuable tool in many contexts (see [8, 9]), it may not always be the most suitable approach for dealing with uncertainty, complexity and qualitative information, where fuzzy approximation theory offers distinct advantages.

Fuzzy approximation theory is a specialized branch of mathematics that deals with representing and approximating vague or uncertain information using fuzzy sets and fuzzy logic. Unlike classical approximation theory, which focuses on precise mathematical representations, fuzzy approximation theory acknowledges and accounts for the inherent uncertainty and imprecision in real-world data and systems.

This field finds applications in various areas, including pattern recognition, control systems, decision-making, and artificial intelligence, where precise mathematical models may not adequately capture the complexity and variability of real-world phenomena. Fuzzy numbers have found significant applications in the study of convergence methods and approximation theory, particularly in the context of statistical convergence and related summability techniques for fuzzy-valued sequences and functions [10, 11, 12]. Moreover, fuzzy structures have been utilized in other mathematical frameworks, including fuzzy graph-based approaches to image segmentation [13]. By leveraging fuzzy approximation theory, researchers can develop robust and adaptable models and algorithms capable of handling uncertain and incomplete information effectively.

In this paper, we broaden the scope of previous research on real operators to fuzzy sense. Our study begins by defining the fuzzy Boas-Buck operators, based on the generating function (1.1) and assumptions (1.3), presented as follows:

$$\tilde{\mathcal{B}}_n \tilde{f} := \frac{1}{\mathcal{W}(1)\mathcal{P}(nx\mathcal{Q}(1))} \sum_{k=0}^{\infty}{}^{*} a_k(nx) \odot \tilde{f}\left(\frac{k}{n}\right), \tag{1.5}$$

where $\sum^*$ and $\odot$ represent fuzzy addition and multiplication, respectively, and $\tilde{f}$ is a fuzzy-valued function.

Distinct fuzzy operators emerge from different formulations of the defined fuzzy Boas-Buck operators (1.5), as detailed in Table 2. Researchers can further investigate the moments and convergence properties associated with these operators, employing analogous approaches within the fuzzy domain.

Table 2. Special cases of different fuzzy operators obtained from the fuzzy Boas-Buck operators.

S.No.	Analytic Functions	Fuzzy Operators
1.	$\mathcal{P}(t)=e^t$	Fuzzy Sheffer operators $\tilde{T}_n\tilde{f}=\frac{e^{-nx\mathcal{Q}(1)}}{\mathcal{W}(1)}\sum_{k=0}^{\infty}{}^{*}a_k(nx)\odot\tilde{f}\left(\frac{k}{n}\right)$
2.	$\mathcal{Q}(t)=t$	Fuzzy Brenke operators $\tilde{L}_n\tilde{f}=\frac{1}{\mathcal{W}(1)\mathcal{P}(nx)}\sum_{k=0}^{\infty}{}^{*}a_k(nx)\odot\tilde{f}\left(\frac{k}{n}\right)$
3.	$\mathcal{P}(t)=e^t$ $\mathcal{Q}(t)=t$	Fuzzy Appell operators $\tilde{P}_n\tilde{f}=\frac{e^{-nx}}{\mathcal{W}(1)}\sum_{k=0}^{\infty}{}^{*}a_k(nx)\odot\tilde{f}\left(\frac{k}{n}\right)$
4.	$\mathcal{W}(t)=1$ $\mathcal{P}(t)=e^t$ $\mathcal{Q}(t)=t$	Fuzzy Szasz operators $\tilde{S}_n\tilde{f}=e^{-nx}\sum_{k=0}^{\infty}{}^{*}\frac{(nx)^k}{k!}\odot\tilde{f}\left(\frac{k}{n}\right)$

Clearly, the Boas-Buck fuzzy operators represent the most general form among the operators mentioned above, thus we shall primarily focus on exploring the approximation properties of these operators. While the previously mentioned fuzzy analogs also warrant individual in-depth study, our discussion in this paper centers only on operators (1.5). The foundational concepts of fuzzy theory necessary for progressing through this paper are provided in Section 2. Section 3 is dedicated to examining the moments, approximation and convergence properties of the fuzzy Boas-Buck operators. By assigning particular values to the analytic functions $\mathcal{W}(t)$, $\mathcal{P}(t)$ and $\mathcal{Q}(t)$, several new linear positive fuzzy operators can be derived. This paper delves into two such instances in Section 4. And finally, the paper concludes in Section 5.

2 Preliminaries

The concept of fuzzy sets was first introduced by Zadeh [14] to model uncertainty and vagueness in mathematical structures. Since then, fuzzy theory has been extensively developed and applied in various branches of mathematics, including approximation theory. In this paper, we adopt the standard notions of fuzzy numbers, fuzzy-valued functions, fuzzy continuity and fuzzy convergence as commonly used in the literature. Thus, in order to proceed with our paper, we must first lay the groundwork by introducing some basic definitions and concepts in fuzzy theory.

Definition 2.1. Let $\tilde{p}$ denote a fuzzy set with membership function $\tilde{p}(u):\mathbb{R}\to[0,1]$ such that:

(i) Normality: There exists $u_0\in\mathbb{R}$ such that $\tilde{p}(u_0)=1$,

(ii) Convexity: $\forall\ s,t\in\mathbb{R}$ and $\forall\ \gamma\in[0,1]$, $\tilde{p}\left(\gamma s+(1-\gamma)t\right)\geqslant\min\left\{\tilde{p}\left(s\right),\tilde{p}\left(t\right)\right\}$,

(iii) Upper semi-continuity: $\forall\ u_0\in\mathbb{R}$ and $\forall\ \varepsilon>0$, $\exists$ a neighbourhood of u_0 such that $\tilde{p}(u)\leqslant\tilde{p}(u_0)+\varepsilon$, for all u in the neighbourhood of u_0,

(iv) Boundedness of $\overline{supp\left(\tilde{p}\right)}$: Define $supp\left(\tilde{p}\right)=\{u\in\mathbb{R}:\tilde{p}\left(x\right)>0\}$ as the support of $\tilde{p}$. Then the closure of the support of $\tilde{p}$, i.e. $\overline{supp\left(\tilde{p}\right)}$ is bounded in $\mathbb{R}$,

then $\tilde{p}$ is called a fuzzy real number.

Let $\mathbb{F}$ represent the set of all fuzzy numbers $\tilde{p}$. Firstly, let us recall the ℓ-level cut of $\tilde{p}$, denoted by $[\tilde{p}]^{\ell}$. For $0\leqslant\ell\leqslant1$ and $\tilde{p}\in\mathbb{F}$ with membership function $\tilde{p}(u)$, the ℓ-level cut or simply ℓ-cut of $\tilde{p}$ is defined as

$$[\tilde{p}]^{\ell}=\begin{cases}\left\{u:\tilde{p}\left(u\right)\geqslant\ell\right\}, & \text{if } 0<\ell\leqslant1\\ \overline{\left\{u:\tilde{p}\left(u\right)>0\right\}}, & \text{if } \ell=0\end{cases}$$

where $\overline{A}$ denotes the closure of set A.

For each $\ell \in [0,1]$, the ℓ-cut of any fuzzy number is always a compact subset of $\mathbb{R}$ which can be denoted as $[\tilde{p}]^{\ell} = [p_{-}^{(\ell)}, p_{+}^{(\ell)}]$. It is apparent that, if $p_{-}^{(\ell)} = p_{+}^{(\ell)}$ then $\tilde{p}$ will reduce to a crisp real number.

Definition 2.2. [15] For any $\tilde{p}$ and $\tilde{q}$ belonging to the set $\mathbb{F}$, and for any α in $\mathbb{R}$, we uniquely define the sum $\tilde{p} \oplus \tilde{q}$ and the product with real scalars $\alpha \odot \tilde{p}$ in $\mathbb{F}$ by $\oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$,

$$(\tilde{p} \oplus \tilde{q})(u) = \sup_{v+w=u} \min \{\tilde{p}(v), \tilde{q}(w)\}$$

and by $\odot : \mathbb{R} \times \mathbb{F} \to \mathbb{F}$,

$$(\alpha \odot \tilde{p})(u) = \begin{cases} \tilde{o} & \text{if } \alpha = 0 \\ \tilde{p}\left(\frac{u}{\alpha}\right) & \text{if } \alpha \neq 0 \end{cases}$$

where $\tilde{o} : \mathbb{R} \to [0,1]$ is the characteristic function of $\{0\}$.

In the interval form, we can think of fuzzy addition and product as

$$\begin{aligned} [\tilde{p} \oplus \tilde{q}]^{\ell} &= [\tilde{p}]^{\ell} + [\tilde{q}]^{\ell}, & \forall \ell \in [0,1] \\ [\alpha \odot \tilde{p}]^{\ell} &= \alpha [\tilde{p}]^{\ell}, & \forall \ell \in [0,1] \end{aligned}$$

Here, $[\tilde{p}]^{\ell} + [\tilde{q}]^{\ell}$ represents the standard addition of intervals considered as real subsets, while $\alpha[\tilde{p}]^{\ell}$ denotes the standard multiplication between a scalar and a real subset.

Definition 2.3 (Distance between fuzzy numbers). [15] The distance between two fuzzy numbers $\tilde{p}$ and $\tilde{q}$ is given by $D : \mathbb{F} \times \mathbb{F} \to \mathbb{R}^{+} \cup \{0\}$ as

$$D(\tilde{p}, \tilde{q}) := \sup_{\ell \in [0,1]} \max \left\{ \left| p_{-}^{(\ell)} - q_{-}^{(\ell)} \right|, \left| p_{+}^{(\ell)} - q_{+}^{(\ell)} \right| \right\},$$

where $p_{-}^{(\ell)}$, $q_{-}^{(\ell)}$, $p_{+}^{(\ell)}$ and $q_{+}^{(\ell)}$ are the lower and upper bounds of $[\tilde{p}]^{\ell}$ and $[\tilde{q}]^{\ell}$.

The following properties hold for the distance between two fuzzy numbers:

(i) $D(\tilde{p} \oplus \tilde{q}, \tilde{r} \oplus \tilde{s}) \leqslant D(\tilde{p}, \tilde{r}) + D(\tilde{q}, \tilde{s}), \quad \forall\, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s} \in \mathbb{F}$

(ii) $D(a \odot \tilde{p}, a \odot \tilde{q}) = |a|\, D(\tilde{p}, \tilde{q}), \quad \forall\, \tilde{p}, \tilde{q} \in \mathbb{F},\ a \in \mathbb{R}$

(iii) $D(\tilde{r} \oplus \tilde{p}, \tilde{r} \oplus \tilde{q}) = D(\tilde{p}, \tilde{q}), \quad \forall\, \tilde{r}, \tilde{p}, \tilde{q} \in \mathbb{F}$

Define the usual norm on $\mathbb{F}$ as $\|\tilde{p}\|_{\mathcal{F}} = D(\tilde{o}, \tilde{p})$. Then we have the following properties:

(i) $\|\eta \odot \tilde{p}\|_{\mathcal{F}} = |\eta|\, \|\tilde{p}\|_{\mathcal{F}}$

(ii) $\|\tilde{p} \oplus \tilde{q}\|_{\mathcal{F}} \leqslant \|\tilde{p}\|_{\mathcal{F}} + \|\tilde{q}\|_{\mathcal{F}}$

(iii) $\|\tilde{p}\|_{\mathcal{F}} + \|\tilde{q}\|_{\mathcal{F}} \leqslant D(\tilde{p}, \tilde{q})$

(iv) $\tilde{p} = \tilde{o}$ iff $\|\tilde{p}\|_{\mathcal{F}} = 0$

The symbol $\precsim$ represents a partial order on $\mathbb{F}$ and is defined as follows:

$$\tilde{p} \precsim \tilde{q} \quad \text{iff} \quad p_{-}^{(\ell)} \leqslant q_{-}^{(\ell)}, \;\; p_{+}^{(\ell)} \leqslant q_{+}^{(\ell)},$$

for all fuzzy numbers $\tilde{p}$ and $\tilde{q}$ and $\ell \in [0,1]$. Herein, $\leqslant$ represents the partial order on the real number set.

Definition 2.4 (Fuzzy-Valued Function). A fuzzy-valued function, or simply a fuzzy function, $\tilde{h}$ from A to B is defined as a mapping that maps each element a in A to a fuzzy set $H(a)$ in B. That is,

$$\tilde{h} : A \to \mathcal{H}(B),$$

where A is the domain, B is the output space, and $\mathcal{H}(B)$ denotes the set of all fuzzy subsets of B acting as the co-domain of $\tilde{h}$.

A fuzzy function $\tilde{h}$ that takes inputs from the real interval $[0, \infty)$ and maps them to the fuzzy field $\mathbb{F}$, can be represented as

$$\left[\tilde{h}\,(u)\right]^{\ell} = \left[h_{-}^{(\ell)}\,(u)\,, h_{+}^{(\ell)}\,(u)\right],$$

for every $u \in [0, \infty]$ and ℓ within the range $[0, 1]$, where $h_{-}^{(\ell)}\,(u)$ and $h_{+}^{(\ell)}\,(u)$ represent the greatest lower bound and least upper bound of the interval $\left[\tilde{h}\,(u)\right]^{\ell}$, respectively. Furthermore, $h_{-}^{(\ell)}$ and $h_{+}^{(\ell)}$ are real-valued functions defined over the interval $[0, \infty)$.

For each a in A, the function $\tilde{h}$ assigns a fuzzy set $H(a)$ in B. To represent the fuzziness, each $H(a)$ is typically described by a membership function, denoted as $\mu_{H(a)}(b)$, which gives the degree of membership of each element b in B to the fuzzy set $H(a)$.

Example 2.5. To better understand the concept of a fuzzy valued function, let us take an example. Let $\tilde{h} : [0, 1] \to \mathbb{R}_{\mathcal{F}}$ such that

$$\tilde{h}\,(x) = -x \sin 3\pi x - \cos^2 4x,$$

where each $\tilde{h}(x)$ is a triangular fuzzy number with the membership function:

$$\mu_h\left(t; \tilde{h}(x) - 2, \tilde{h}(x), \tilde{h}(x) + 2\right) = \max\left(\min\left(\frac{t - \tilde{h}(x) + 2}{2}, \frac{\tilde{h}(x) + 2 - t}{2}\right), 0\right).$$

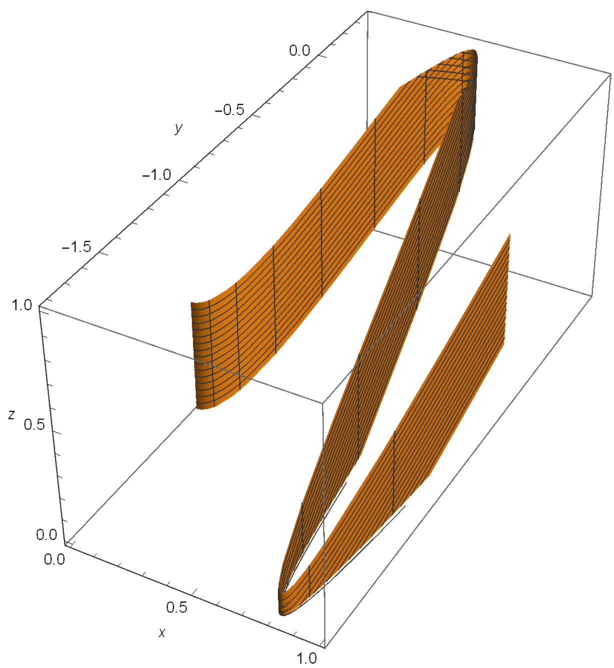
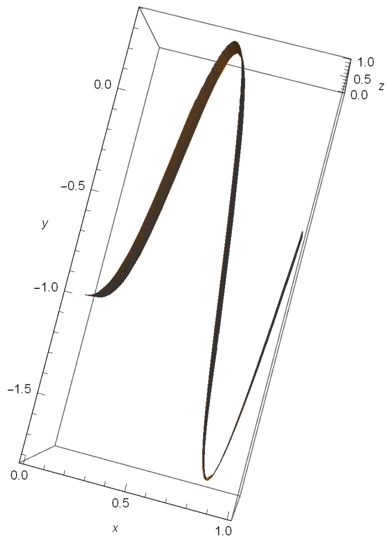


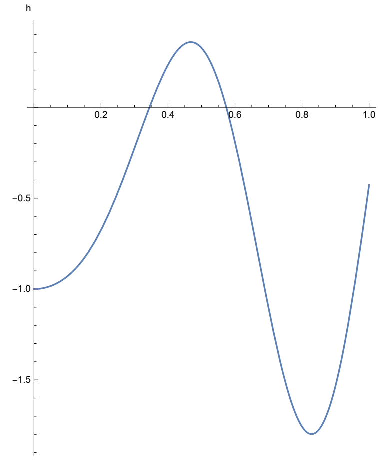
Figure 1. Graph of fuzzy number-valued function $\tilde{h}\,(x) = -x \sin 3\pi x - \cos^2 4x$ with triangular membership function.

The graph of a fuzzy valued function is a 3-dimensional graph, as the Z-axis corresponds to the membership value corresponding to each $\tilde{h}(x)$. Figure 1 represents a function whose range is the triangular fuzzy number. Let us change this membership function and see what changes we get in our graph. Suppose each point in the range is a trapezoidal fuzzy number. Then the membership value corresponding to each $\tilde{h}(x)$ will be,

$$\mu_h\left(t; \tilde{h}(x) - 1.5, \tilde{h}(x) - 0.5, \tilde{h}(x) + \frac{1}{2}, \tilde{h}(x) + 1.5\right)$$
$$= \max\left(\min\left(t - \tilde{h}(x) + 1.5, 1, \tilde{h}(x) + 1.5 - t\right), 0\right).$$



(a) Aerial view of fuzzy-valued function



(b) Crisp function

Figure 2. Association of a fuzzy-valued function with its corresponding crisp function.

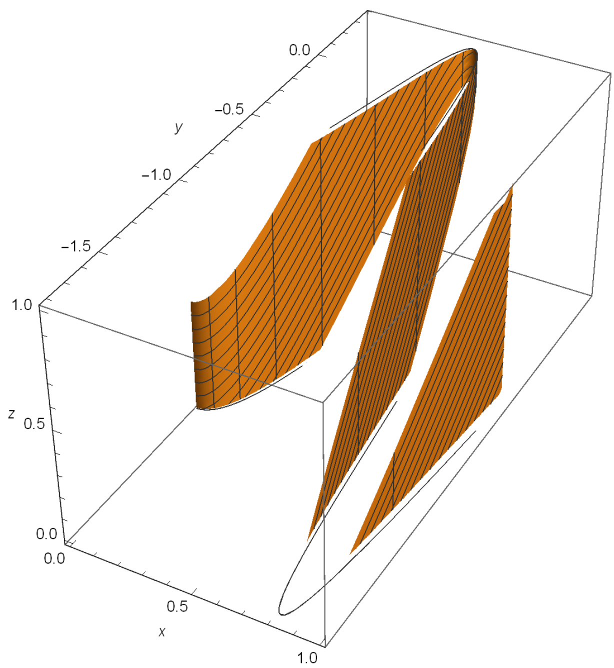


Figure 3. Graph of fuzzy number-valued function $\tilde{h}(x) = -x \sin 3\pi x - \cos^2 4x$ with trapezoidal membership function.

The aerial view will remain the same; only the degree of uncertainty, given by the membership function μ_h, changes along the z axis.

Definition 2.6 (Distance between fuzzy valued functions). Consider two fuzzy-valued functions, $\tilde{h}$ and $\tilde{g}$, defined on a real interval $U \subseteq \mathbb{R}$. We define the distance between $\tilde{h}$ and $\tilde{g}$ as,

$$D^*\left(\tilde{h}, \tilde{g}\right) := \sup_{u \in U} D\left(\tilde{h}(u), \tilde{g}(u)\right).$$

We say $\tilde{h}$ is crisp when $\tilde{h}(u)$ assumes crisp values for all u in its domain. We define $\tilde{h}$ to be a fuzzy continuous function when,

$$\lim_{u \to u_0} D\left(\tilde{h}(u), \tilde{h}(u_0)\right) = 0,$$

holds for any $u_0 \in U$.

Represented by $C\,[0,\infty)$ and $C_{\mathcal{F}}\,[0,\infty)$ are the sets of continuous and fuzzy continuous functions on the interval $[0,\infty)$. If $\tilde{h}$ is a fuzzy continuous function over $[0,\infty)$, then the corresponding functions $h_-^{(r)}$ and $h_+^{(r)}$ become real-valued continuous functions defined on the same interval. The operations of addition and scalar multiplication in $C_{\mathcal{F}}\,[0,\infty)$ are defined as follows:

$$\begin{aligned}(a \odot \tilde{g})\,(u) &= a \odot \tilde{g}\,(u)\\ \left(\tilde{h} \oplus \tilde{g}\right)(u) &= \tilde{h}\,(u) \oplus \tilde{g}\,(u)\,,\end{aligned}$$

for every $u \in [0,\infty)$, $a \in \mathbb{R}$ and $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}\,[0,\infty)$. Moreover, a fuzzy-valued function $\tilde{0}$ exists, defined on the interval $[0,\infty)$, satisfying the condition $\tilde{0}(u) = \tilde{o}$ for every $u \in [0,\infty)$ where $\tilde{o}$ represents the neutral element with respect to the operation $\oplus$ in $\mathbb{F}$. We can also define the norm of $\tilde{h}$ as,

$$\|\tilde{h}\|_{\mathcal{F}} = \sup_{0 \leqslant u < \infty} D\left(\tilde{o}, \tilde{h}\,(u)\right).$$

Definition 2.7 (Fuzzy weighted modulus of continuity). Consider a continuous fuzzy-valued function $\tilde{h}$. We can define the weighted fuzzy modulus of continuity for $\tilde{h}$ as:

$$\Omega^{\mathcal{F}}\left(\tilde{h};\delta\right) := \sup_{\substack{u,v \geqslant 0\\ |u-v| \leqslant \delta}} \frac{1}{\left(1+(u-v)^2\right)(1+u^2)} D\left(\tilde{h}\,(u)\,, \tilde{h}\,(v)\right).$$

Definition 2.8 (Fuzzy modulus of continuity). For a continuous fuzzy-valued function $\tilde{h}$ mapping from the interval $[0,\infty)$ to $\mathbb{F}$, the first modulus of continuity is defined as follows:

$$\omega_1^{\mathcal{F}}\left(\tilde{h};\delta\right) := \sup_{\substack{u,v \in [0,\infty)\\ |u-v| \leqslant \delta}} D\left(\tilde{h}\,(u)\,, \tilde{h}\,(v)\right).$$

In the case of operators $\tilde{\mathcal{L}}$ from $C_{\mathcal{F}}[0,\infty)$ to itself, their representation takes the form,

$$\left[\tilde{\mathcal{L}}(\tilde{h})\,(u)\right]^{\ell} = \left[\left(\tilde{\mathcal{L}}(\tilde{h})\,(u)\right)_-^{(\ell)}, \left(\tilde{\mathcal{L}}(\tilde{h})\,(u)\right)_+^{(\ell)}\right].$$

Suppose $\tilde{\mathcal{L}}$ is an operator mapping the space of all continuous fuzzy-valued functions $C_{\mathcal{F}}[0,\infty)$ to itself, with the condition

$$\widetilde{\mathcal{L}}\left(a \odot \tilde{h} \oplus b \odot \tilde{g}\right) = a \odot \widetilde{\mathcal{L}}(\tilde{h}) \oplus b \odot \widetilde{\mathcal{L}}(\tilde{g}), \tag{2.1}$$

for any $a, b \in \mathbb{R}$ and $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[0,\infty)$. In this case, $\tilde{\mathcal{L}}$ is identified as a fuzzy linear operator. Consider $\tilde{\mathcal{L}}$ mapping $C_{\mathcal{F}}[0,\infty)$ to itself such that for any $\tilde{h},\ \tilde{g} \in C_{\mathcal{F}}[0,\infty)$, $\tilde{\mathcal{L}}$ is linear and

$$\tilde{h} \precsim \tilde{g} \Rightarrow \tilde{\mathcal{L}}(\tilde{h}) \precsim \tilde{\mathcal{L}}(\tilde{g}).$$

Then $\tilde{\mathcal{L}}$ termed as a linear positive fuzzy operator.

3 Fuzzy Boas-Buck Operators

In this section, first we relate the fuzzy Boas-Buck operators to the classical ones as defined by Sucu et al. [3] and then prove that the proposed operators are fuzzy positive linear operators. Further, the approximation properties of the fuzzy Boas-Buck operators are proved using the fuzzy Korovkin theorem and fuzzy weighted modulus of continuity.

Lemma 3.1. *For the fuzzy operators $\tilde{\mathcal{B}}_n$, we have*

$$\left(\tilde{\mathcal{B}}_n \tilde{h}\right)_{\pm}^{(\ell)} = \mathcal{B}_n\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

where $\mathcal{B}_n$ are the classical Boas-Buck operators in real sense as defined by (1.4).

Proof. For fuzzy operators (1.5) and fuzzy function $\tilde{h}$, we have

$$\begin{aligned}
\left(\tilde{\mathcal{B}}_n \tilde{h}\right)_{\pm}^{(\ell)} &= \left(\frac{1}{\mathcal{W}(1)\,\mathcal{P}(nx\mathcal{Q}(1))} \sum_{k=0}^{\infty}{}^{*} a_k(nx) \odot \tilde{h}\left(\frac{k}{n}\right)\right)_{\pm}^{(\ell)} \\
&= \frac{1}{\mathcal{W}(1)\,\mathcal{P}(nx\mathcal{Q}(1))} \sum_{k=0}^{\infty} a_k(nx) \left(\tilde{h}\left(\frac{k}{n}\right)\right)_{\pm}^{(\ell)} \\
&= \mathcal{B}_n\left(\tilde{h}_{\pm}^{(\ell)}\right).
\end{aligned}$$

□

Lemma 3.2. *The fuzzy Boas-Buck operators are characterized as linear positive fuzzy operators.*

Proof. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,\infty)$. Using the linearity of the real Boas-Buck operators and Lemma 3.1 we have,

$$\begin{aligned}
\left(\tilde{\mathcal{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right)_{\pm}^{(\ell)} &= \mathcal{B}_n\left(\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)_{\pm}^{(\ell)}\right) \\
&= \mathcal{B}_n\left(\left(a\odot\tilde{h}\right)_{\pm}^{(\ell)} + \left(b\odot\tilde{g}\right)_{\pm}^{(\ell)}\right) \\
&= \mathcal{B}_n\left(a\tilde{h}_{\pm}^{(\ell)} + b\tilde{g}_{\pm}^{(\ell)}\right) \\
&= a\mathcal{B}_n\left(\tilde{h}_{\pm}^{(\ell)}\right) + b\mathcal{B}_n\left(\tilde{g}_{\pm}^{(\ell)}\right) \\
&= a\left(\tilde{\mathcal{B}}_n(\tilde{h})\right)_{\pm}^{(\ell)} + b\left(\tilde{\mathcal{B}}_n(\tilde{g})\right)_{\pm}^{(\ell)}
\end{aligned} \tag{3.1}$$

for every $x \in [0,\infty)$ and $\ell \in [0,1]$.

Using the above equation (3.1), we obtain

$$\begin{aligned}
&\left[\tilde{\mathcal{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right]^{\ell} \\
&= \left[\left(\tilde{\mathcal{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right)_{-}^{(\ell)}, \left(\tilde{\mathcal{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right)_{+}^{(\ell)}\right] \\
&= \left[a\left(\tilde{\mathcal{B}}_n(\tilde{h})\right)_{-}^{(\ell)} + b\left(\tilde{\mathcal{B}}_n(\tilde{g})\right)_{-}^{(\ell)}, a\left(\tilde{\mathcal{B}}_n(\tilde{h})\right)_{+}^{(\ell)} + b\left(\tilde{\mathcal{B}}_n(\tilde{g})\right)_{+}^{(\ell)}\right] \\
&= \left[a\left(\tilde{\mathcal{B}}_n(\tilde{h})\right)_{-}^{(\ell)}, a\left(\tilde{\mathcal{B}}_n(\tilde{h})\right)_{+}^{(\ell)}\right] + \left[b\left(\tilde{\mathcal{B}}_n(\tilde{g})\right)_{-}^{(\ell)}, b\left(\tilde{\mathcal{B}}_n(\tilde{g})\right)_{+}^{(\ell)}\right] \\
&= a\left[\tilde{\mathcal{B}}_n(\tilde{h})\right]^{\ell} + b\left[\tilde{\mathcal{B}}_n(\tilde{g})\right]^{\ell} \\
&= \left[a\odot\tilde{\mathcal{B}}_n(\tilde{h})\right]^{\ell} + \left[b\odot\tilde{\mathcal{B}}_n(\tilde{g})\right]^{\ell} \\
&= \left[a\odot\tilde{\mathcal{B}}_n(\tilde{h})\oplus b\odot\tilde{\mathcal{B}}_n(\tilde{g})\right]^{\ell}; \qquad \forall\, x\in[0,\infty),\ \ell\in[0,1].
\end{aligned}$$

This implies that for $\tilde{h}, \tilde{g} \in C_{\mathcal{F}}[0,\infty)$,

$$\tilde{\mathcal{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right) = a\odot\tilde{\mathcal{B}}_n(\tilde{h})\oplus b\odot\tilde{\mathcal{B}}_n(\tilde{g}).$$

Thus, the fuzzy Boas-Buck operators are fuzzy linear operators. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,\infty)$ with $\tilde{h} \precsim \tilde{g}$. Then, $\tilde{h}_{-}^{(\ell)} \leqslant \tilde{g}_{-}^{(\ell)}$ and $\tilde{h}_{+}^{(\ell)} \leqslant \tilde{g}_{+}^{(\ell)}$, where $\leqslant$ is a partial order on $C[0,\infty)$.

Since $\tilde{h}_-^{(\ell)}$, $\tilde{h}_+^{(\ell)}$, $\tilde{g}_-^{(\ell)}$ and $\tilde{g}_+^{(\ell)}$ are continuous real functions on $[0,\infty)$ then by the positivity of $\mathcal{B}_n$, we have

$$\begin{aligned} \mathcal{B}_n\left(\tilde{h}_\pm^{(\ell)}\right) &\leqslant \mathcal{B}_n\left(\tilde{g}_\pm^{(\ell)}\right), && \ell\in[0,1]\\ \Rightarrow \left(\tilde{\mathcal{B}}_n(\tilde{h})\right)_\pm^{(\ell)} &\leqslant \left(\tilde{\mathcal{B}}_n(\tilde{g})\right)_\pm^{(\ell)}, && \ell\in[0,1]\\ \Rightarrow \qquad \tilde{\mathcal{B}}_n(\tilde{h}) &\precsim \tilde{\mathcal{B}}_n(\tilde{g}), && \ell\in[0,1],\ x\in[0,\infty). \end{aligned}$$

This gives the positivity of $\tilde{\mathcal{B}}_n$. □

Theorem 3.3. *[3] The moments of the classical sequence of operators* (1.4) *are given as follows:*

(i) $\mathcal{B}_n(1)=1$

(ii) $\mathcal{B}_n(t)=\frac{\mathcal{P}'(nx\mathcal{Q}(1))}{\mathcal{P}(nx\mathcal{Q}(1))}x+\frac{\mathcal{W}'(1)}{n\mathcal{W}(1)}$

(iii) $\mathcal{B}_n(t^2)=\frac{\mathcal{P}''(nx\mathcal{Q}(1))}{\mathcal{P}(nx\mathcal{Q}(1))}x^2+\frac{2\mathcal{W}'(1)+\left(1+\mathcal{Q}'(1)\right)\mathcal{W}(1)}{n\mathcal{W}(1)\mathcal{P}(nx\mathcal{Q}(1))}\mathcal{P}'(nx\mathcal{Q}(1))\,x+\frac{\mathcal{W}''(1)+\mathcal{W}'(1)}{n^2\mathcal{W}(1)}$

Assume that $\lim\limits_{y\to\infty}\frac{\mathcal{P}'(y)}{\mathcal{P}(y)}=1$ and $\lim\limits_{y\to\infty}\frac{\mathcal{P}''(y)}{\mathcal{P}(y)}=1$. Then, using fuzzy Korovkin theorem given by G. A. Anastassiou in 2010 (see [16]), $D^*\left(\tilde{\mathcal{B}}_n\tilde{h},\tilde{h}(x)\right)\to 0$, as $n\to\infty$. Thus, $\tilde{\mathcal{B}}_n\tilde{h}$ are a sequence of fuzzy positive linear operators converging to $\tilde{h}$ as $n\to\infty$.

Lemma 3.4. *Let us denote* $b_{n,k}(x)=\frac{a_k(nx)}{\mathcal{W}(1)\mathcal{P}(nx\mathcal{Q}(1))}$*. Then by Theorem 3.3, we have*

(i) $\sum\limits_{k=0}^{\infty}b_{n,k}(x)=1$,

(ii) $\sum\limits_{k=0}^{\infty}(t-x)^2b_{n,k}(x)=\frac{\mathcal{W}''(1)+\mathcal{W}'(1)}{n^2\mathcal{W}(1)}+\frac{\mathcal{P}''(nx\mathcal{Q}(1))-2\mathcal{P}'(nx\mathcal{Q}(1))+\mathcal{P}(nx\mathcal{Q}(1))}{\mathcal{P}(nx\mathcal{Q}(1))}x^2$
$+\frac{\mathcal{W}(1)\left(1+\mathcal{Q}''(1)\right)\mathcal{P}'(nx\mathcal{Q}(1))+2\mathcal{W}'(1)\left(\mathcal{P}'(nx\mathcal{Q}(1))-\mathcal{P}(nx\mathcal{Q}(1))\right)}{n\mathcal{W}(1)\mathcal{P}(nx\mathcal{Q}(1))}$.

Theorem 3.5. *If* $\tilde{h}\in C_{\mathcal{F}}[0,\infty)$*, then for* $n\geqslant 1$

$$D^*\left(\tilde{\mathcal{B}}_n\tilde{h},\tilde{h}(x)\right)\leqslant 2\omega_1^{\mathcal{F}}\left(\tilde{h};\sqrt{\varphi_{2,n}(x)}\right),$$

where $\omega_1^{\mathcal{F}}$ *is the fuzzy first order modulus of continuity and* $\varphi_{m,n}(x)=\mathcal{B}_n\left((t-x)^m;x\right)$.

Proof. Let $b_{n,k}(x)=\frac{a_k(nx)}{\mathcal{W}(1)\mathcal{P}(nx\mathcal{Q}(1))}$, then we can write,

$$\tilde{h}(x)=\left[\sum_{k=0}^{\infty}b_{n,k}(x)\right]\odot\tilde{h}(x)=\sum_{k=0}^{\infty}{}^*\left(b_{n,k}(x)\odot\tilde{h}(x)\right).$$

Thus,

$$\begin{aligned} D^*\left(\tilde{\mathcal{B}}_n\tilde{h},\tilde{h}(x)\right) &= D^*\left(\sum_{k=0}^{\infty}{}^*b_{n,k}(x)\odot\tilde{h}\left(\frac{k}{n}\right),\sum_{k=0}^{\infty}{}^*b_{n,k}(x)\odot\tilde{h}(x)\right)\\ &\leqslant \sum_{k=0}^{\infty}b_{n,k}(x)\,D^*\left(\tilde{h}\left(\frac{k}{n}\right),\tilde{h}(x)\right)\\ &\leqslant \sum_{k=0}^{\infty}b_{n,k}(x)\,\omega_1^{\mathcal{F}}\left(\tilde{h};|k/n-x|\right)\\ &\leqslant \sum_{k=0}^{\infty}b_{n,k}(x)\left[1+\frac{|k/n-x|}{\delta}\right]\omega_1^{\mathcal{F}}\left(\tilde{h};\delta\right)\\ &\leqslant \omega_1^{\mathcal{F}}\left(\tilde{h};\delta\right)+\frac{\sqrt{\varphi_{2,n}(x)}}{\delta}\omega_1^{\mathcal{F}}\left(\tilde{h};\delta\right). \end{aligned}$$

Substituting $\delta=\sqrt{\varphi_{2,n}(x)}$, we achieve our result. □

Lemma 3.6. *For fuzzy valued function $\tilde{h}$ and weighted modulus of continuity $\Omega^{\mathcal{F}}$,*

$$D\left(\tilde{h}\left(t\right),\tilde{h}\left(x\right)\right)\leqslant 4\left[1+\frac{\left(t-x\right)^{4}}{\delta^{4}}\right]\left(1+\delta^{2}\right)^{2}\left(1+x^{2}\right)\Omega^{\mathcal{F}}\left(\tilde{h};\delta\right).$$

Proof. For a fuzzy-valued function $\tilde{h}$, taking values from the real line, we can say that

$$\begin{aligned}D\left(\tilde{h}\left(t\right),\tilde{h}\left(x\right)\right)&\leqslant\left(1+\left(t-x\right)^{2}\right)\left(1+x^{2}\right)\Omega^{\mathcal{F}}\left(\tilde{h};\left|t-x\right|\right)\\&\leqslant 2\left(1+\frac{\left|t-x\right|}{\delta}\right)\left(1+\left(t-x\right)^{2}\right)\left(1+x^{2}\right)\left(1+\delta^{2}\right)\Omega^{\mathcal{F}}\left(\tilde{h};\delta\right)\\&\leqslant 4\left[1+\frac{\left(t-x\right)^{4}}{\delta^{4}}\right]\left(1+\delta^{2}\right)^{2}\left(1+x^{2}\right)\Omega^{\mathcal{F}}\left(\tilde{h};\delta\right).\end{aligned}$$

□

Theorem 3.7. *Let $\tilde{h}(x)$ is a continuous, bounded and twice differential fuzzy function, defined for all $x\in[0,\infty)$. Then for the fuzzy Boas-Buck type operators, we have the following result:*

$$D^{*}\left(\widetilde{\mathcal{B}}_{n}\tilde{h},\tilde{h}\left(x\right)\right)\leqslant\varphi_{1,n}\left\|\tilde{h}'\right\|+\frac{1}{2}\varphi_{2,n}\left\|\tilde{h}''\right\|+16\varphi_{2,n}\left(1+x^{2}\right)\Omega^{\mathcal{F}}\left(\tilde{h}'',\sqrt[4]{\frac{\varphi_{6,n}}{\varphi_{2,n}}}\right)$$

Proof. Consider,

$$D^{*}\left(\tilde{\mathcal{B}}_{n}\tilde{h}\left(\frac{k}{n}\right),\tilde{h}\left(x\right)\right)\leqslant\sum_{k=0}^{\infty}b_{n,k}\left(x\right)D\left(\tilde{h}\left(\frac{k}{n}\right),\tilde{h}\left(x\right)\right).$$

Now,

$$D\left(\tilde{h}\left(\frac{k}{n}\right),\tilde{h}\left(x\right)\right)=\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{h}\left(\frac{k}{n}\right)_{-}^{(\alpha)}-\tilde{h}\left(x\right)_{-}^{(\alpha)}\right|,\left|\tilde{h}\left(\frac{k}{n}\right)_{+}^{(\alpha)}-\tilde{h}\left(x\right)_{+}^{(\alpha)}\right|\right\}.$$

From Taylor's theorem,

$$h\left(u\right)=h\left(x\right)+\left(u-x\right)h'\left(x\right)+\frac{\left(u-x\right)^{2}}{2!}h''\left(x\right)+\frac{\left(u-x\right)^{2}}{2!}\varepsilon\left(u,x\right),$$

where $\varepsilon\left(u,x\right)=h''\left(\xi\right)-h''\left(x\right)\to 0$ as $u\to x$. Then, we can say that

$$\tilde{h}\left(u\right)_{\pm}^{(\ell)}=\tilde{h}\left(x\right)_{\pm}^{(\ell)}+\left(u-x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)}+\frac{\left(u-x\right)^{2}}{2!}\tilde{h}''\left(x\right)_{\pm}^{(\ell)}+\frac{\left(u-x\right)^{2}}{2!}\varepsilon\left(u,x\right),$$

where $\varepsilon\left(u,x\right)=D\left(\tilde{h}''\left(\xi\right),\tilde{h}''\left(x\right)\right)\to 0$ as $u\to x$, with $x<\xi<u$. Thus,

$$\begin{aligned}&\left[\tilde{h}\left(u\right)\right]^{\ell}-\left[\tilde{h}\left(x\right)\right]^{\ell}\\&=\left(u-x\right)\left[\tilde{h}'\left(x\right)\right]^{\ell}+\frac{\left(u-x\right)^{2}}{2!}\left[\tilde{h}''\left(x\right)\right]^{\ell}+\frac{\left(u-x\right)^{2}}{2!}\varepsilon\left(u,x\right)\\\Rightarrow&\left[\tilde{h}\left(u\right)_{-}^{(\ell)}-\tilde{h}\left(x\right)_{-}^{(\ell)},\tilde{h}\left(u\right)_{+}^{(\ell)}-\tilde{h}\left(x\right)_{+}^{(\ell)}\right]\\&=\Bigg[\left(u-x\right)\tilde{h}'\left(x\right)_{-}^{(\ell)}+\frac{\left(u-x\right)^{2}}{2!}\tilde{h}''\left(x\right)_{-}^{(\ell)}+\frac{\left(u-x\right)^{2}}{2!}\varepsilon\left(u,x\right)_{-}^{(\ell)},\\&\qquad\left(u-x\right)\tilde{h}'\left(x\right)_{+}^{(\ell)}+\frac{\left(u-x\right)^{2}}{2!}\tilde{h}''\left(x\right)_{+}^{(\ell)}+\frac{\left(u-x\right)^{2}}{2!}\varepsilon\left(u,x\right)_{+}^{(\ell)}\Bigg]\\&=\left[A,B\right].\end{aligned}$$

Since $\varepsilon(u,x)$ is crisp, using Lemma 3.6,

$$\begin{aligned}\varepsilon(u,x) &= D\left(\tilde{h}''(\xi), \tilde{h}''(x)\right) \text{ with } x<\xi<u \\ &\leqslant 4\left[1+\frac{(u-x)^4}{\delta^4}\right]\left(1+\delta^2\right)^2\left(1+x^2\right)\Omega^F\left(\tilde{h};\delta\right).\end{aligned}$$

Considering $\delta \leq 1$,

$$(u-x)^2\varepsilon(u,x) \leqslant 16\left(1+x^2\right)\left[(u-x)^2+\frac{(u-x)^6}{\delta^4}\right]\Omega^F\left(\tilde{h};\delta\right).$$

Thus, we have

$$\begin{aligned}D^*\left(\tilde{\mathcal{B}}_n\tilde{h}, \tilde{h}(x)\right) &\leqslant \sum_{k=0}^{\infty} b_{n,k}(x)\, D\left(\tilde{h}\left(\frac{k}{n}\right), \tilde{h}(x)\right) \\ &= \sum_{k=0}^{n} b_{n,k}(x) \sup_{\ell\in[0,1]} \max\{A, B\} \\ &= \sup_{\ell\in[0,1]} \max\left\{\sum_{k=0}^{n} A b_{n,k}(x), \sum_{k=0}^{n} B b_{n,k}(x)\right\} \\ &\leqslant \sup_{\ell\in[0,1]} \max\left\{\varphi_{1,n}(x)\tilde{h}'(x)_-^{(\ell)} + \frac{1}{2}\varphi_{2,n}(x)\tilde{h}''(x)_-^{(\ell)}\right. \\ &\qquad + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathcal{F}}\left(\tilde{h}''; \sqrt[4]{\frac{\varphi_{6,n}(x)}{\varphi_{2,n}(x)}}\right), \varphi_{1,n}(x)\tilde{h}'(x)_+^{(\ell)} \\ &\qquad \left. + \frac{1}{2}\varphi_{2,n}(x)\tilde{h}''(x)_+^{(\ell)} + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathcal{F}}\left(\tilde{h}''; \sqrt[4]{\frac{\varphi_{6,n}(x)}{\varphi_{2,n}(x)}}\right)\right\} \\ &\leqslant \sup_{\ell\in[0,1]}\left[\varphi_{1,n}(x)\max\left\{\tilde{h}'(x)_-^{(\ell)}, \tilde{h}'(x)_+^{(\ell)}\right\}\right. \\ &\qquad \left. + \frac{1}{2}\varphi_{2,n}(x)\max\left\{\tilde{h}''(x)_-^{(\ell)}, \tilde{h}''(x)_+^{(\ell)}\right\}\right] \\ &\qquad + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathcal{F}}\left(\tilde{h}''; \sqrt[4]{\frac{\varphi_{6,n}(x)}{\varphi_{2,n}(x)}}\right) \\ &= \varphi_{1,n}(x)\left\|\tilde{h}'(x)\right\|_{\mathcal{F}} + \frac{1}{2}\varphi_{2,n}(x)\left\|\tilde{h}''(x)\right\|_{\mathcal{F}} \\ &\qquad + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathcal{F}}\left(\tilde{h}'', \sqrt[4]{\frac{\varphi_{6,n}}{\varphi_{2,n}}}\right),\end{aligned}$$

where $\tilde{o}$ is the zero element. And thus, we arrive at our desired result. □

4 Particular examples

The Boas-Buck polynomials are a very general form of a special class of operators formed using the generating function (1.1). For particular values of $\mathcal{W}(t), \mathcal{P}(t)$ and $\mathcal{Q}(t)$ one can get different fuzzy linear operators. Mentioned below are two particular examples of the Boas-Buck fuzzy operators.

4.1 Fuzzy Laguerre Operators

The Laguerre polynomials were first introduced in 1960 by Rainville [17] by means of its generating function. Later, these polynomials were explored by Gurland et al. [18] and Gupta [19] where the moments of these operators were found using the moment generating function and some direct convergence results were proved. Here, we consider the fuzzy analog of the Laguerre operators. Taking $\mathcal{W}(t) = (1-t)^{-\alpha-1}, \mathcal{P}(t) = e^t$ and $\mathcal{Q}(t) = \frac{-t}{1-t}$, for $\alpha > -1$ we get

$$\left(\tilde{G}_n^\alpha \tilde{h}\right)(x) = e^{-nx/2} 2^{-\alpha-1} \sum_{k=0}^{\infty}{}^{*} 2^{-k} L_k^\alpha\left(\frac{-nx}{2}\right) \odot \tilde{h}\left(\frac{k}{n}\right),$$

$$\text{where} \quad L_k^\alpha(-x) = \frac{(\alpha+1)_k}{k!} {}_1F_1(-k;\alpha+1;-x) = \sum_{s=0}^{k} \frac{(\alpha+k)!}{(k-s)!(\alpha+s)!s!} x^s.$$

Remark 4.1. $\left(\tilde{G}_n^\alpha \tilde{h}\right)(x)$ are fuzzy positive linear operators with $\left(\tilde{G}_n^\alpha(\tilde{h})\right)_\pm^{(\ell)} = G_n^\alpha\left(\tilde{h}_\pm^{(\ell)}\right)$, where G_n^α are the real classical Laguerre operators [17].

Theorem 4.2. *If $\tilde{h} \in C_{\mathcal{F}}[0,\infty)$, then for $n \geqslant 1$*

$$D^*\left(\tilde{G}_n^\alpha \tilde{h}, \tilde{h}(x)\right) \leq \left(1 + \sqrt{\alpha^2 + 4\alpha + 3}\right) \omega_1^{\mathcal{F}}\left(\tilde{h}; \frac{1}{n}\right),$$

where $\omega_1^{\mathcal{F}}$ is the fuzzy modulus of continuity.

Proof. Let $g_{n,k}(x) = e^{-nx/2} 2^{-\alpha-k-1} L_k^\alpha\left(\frac{-nx}{2}\right)$. Then, we have

$$\begin{aligned} D^*\left(\tilde{G}_n^\alpha \tilde{h}, \tilde{h}(x)\right) &\leq \sum_{k=0}^{\infty} g_{n,k}(x) D^*\left(\tilde{h}\left(\frac{k}{n}\right), \tilde{h}(x)\right) \\ &\leq \sum_{k=0}^{\infty} g_{n,k}(x) \omega_1^{\mathcal{F}}\left(\tilde{h}; \left|\frac{k}{n} - x\right|\right) \\ &\leq \sum_{k=0}^{\infty} g_{n,k}(x) \left[1 + \frac{|k/n - x|}{\delta}\right] \omega_1^{\mathcal{F}}\left(\tilde{h}; \delta\right) \\ &= \omega_1^{\mathcal{F}}\left(\tilde{h}; \delta\right) + \frac{\omega_1^{\mathcal{F}}\left(\tilde{h}; \delta\right)}{\delta} \sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right| g_{n,k}(x), \end{aligned}$$

and,

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 g_{n,k}(x) &= \frac{\alpha^2 + 4\alpha + 3}{n^2} + \frac{3x}{n} \\ &\leq \frac{\alpha^2 + 4\alpha + 3}{n^2}. \end{aligned}$$

Thus,

$$\sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right| g_{n,k}(x) \leq \frac{\sqrt{\alpha^2 + 4\alpha + 3}}{n}.$$

Choosing $\delta = 1/n$, we arrive at the required result. □

4.2 Fuzzy Charlier Operators

Charlier polynomials [20] represent a significant family of orthogonal polynomials defined on the set of non-negative real numbers. They arise as a particular instance of the Boas-Buck operators characterized by $\mathcal{W}(t) = e^t$, $\mathcal{P}(t) = e^t$ and $\mathcal{Q}(t) = \ln\left(1 - \frac{t}{a}\right)$. Extensive research has been dedicated to explore the properties and applications of real classical Charlier operators. One can refer to [21, 22, 23, 24, 25, 26, 27, 28, 29] for the various mathematical contexts of these operators. We have extended these notations of Charlier operators into fuzzy approximation theory by defining the fuzzy analog of Charlier operators as follows:

$$\tilde{T}_n^a \tilde{h} = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty}{}^{*} \frac{C_k^{(a)}\left(-(a-1)nx\right)}{k!} \odot \tilde{h}\left(\frac{k}{n}\right),$$

$$\text{where } C_k^{(a)}(-u) = \sum_{r=0}^{k} \binom{k}{r} (u)_r \left(\frac{1}{a}\right)^r = \sum_{r=0}^{k} \binom{k}{r} \frac{\Gamma(u+r)}{\Gamma(u)} \left(\frac{1}{a}\right)^r.$$

Remark 4.3. $\left(\tilde{T}_n^a \tilde{h}\right)(x)$ are fuzzy positive linear operators with $\left(\tilde{T}_n^a(\tilde{h})\right)_{\pm}^{(\ell)} = T_n^a \tilde{h}\left(\tilde{h}_{\pm}^{(\ell)}\right)$, where T_n^a are the real classical Charlier operators [20].

Theorem 4.4. *If $\tilde{h} \in C_{\mathcal{F}}[0,\infty)$, then for $n \geqslant 1$*

$$D^*\left(\tilde{T}_n^a \tilde{h}, \tilde{h}(x)\right) \leq \left(1 + \sqrt{x\left(1+\frac{1}{a-1}\right)+\frac{2}{n}}\right) \omega_1^{\mathcal{F}}\left(\tilde{h}; \frac{1}{\sqrt{n}}\right),$$

where $\omega_1^{\mathcal{F}}$ is the fuzzy modulus of continuity.

Proof. The proof of this theorem follows from the moments of Charlier operators and the properties of fuzzy modulus of continuity. □

5 Conclusion

In conclusion, this study contributes significantly to the field of approximation by incorporating the fundamentals from fuzzy theory. We introduced the fuzzy Boas-Buck operators and subsequently proved their fuzzy linearity and positivity. Through the derivation of their moments and central moments, along with the application of fuzzy Korovkin theorem, we established the convergence properties of operators (1.5) to any continuous fuzzy-valued function. We also examined the convergence rate using fuzzy modulus of continuity and fuzzy weighted modulus of continuity. Moreover, this paper includes particular examples of fuzzy Boas-Buck operators by employing particular values to the analytic functions (1.2), namely fuzzy Laguerre and Charlier operators. This enhances the understanding of fuzzy Boas-Buck operators and provides a flexible framework that can be extended to other families of fuzzy approximation operators, such as Baskakov, Jain and Kantorovich-type operators. The approach presented here may also be adapted to study convergence, multivariate generalizations and approximation in weighted function spaces. Furthermore, the proposed operators may find potential applications in the numerical schemes of fuzzy differential and integral equations.

References

[1] R. P. Boas Jr and R. C. Buck, *Polynomials defined by generating relations*, The American Mathematical Monthly, **63** 626–632, (1956).

[2] R. P. Boas and R. C. Buck, *Polynomial expansions of analytic functions*, Springer Science & Business Media, (2013).

[3] S. Sucu, G. İçöz and S. Varma, *On Some Extensions of Szasz Operators Including Boas-Buck-Type Polynomials*, Abstract and Applied Analysis, **2012** 680340, (2012).

[4] M. Ismail, *On a generalization of Szász operators*, Mathematica (Cluj), **39** 259–267, (1974).

[5] T. S. Chihara, *An introduction to orthogonal polynomials*, Courier Corporation, (2011).

[6] A. Jakimovski and D. Leviatan, *Generalized Szász operators for the approximation in the infinite interval*, Mathematica (Cluj), **11** 97–103, (1969).

[7] O. Szasz, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bur. Standards, **45** 239–245, (1950).

[8] M. H. Lafta and E. S. Bhaya, *$L_p(D)$ Approximation by a Szasz Operator Type*, Palestine Journal of Mathematics, **13** (2024).

[9] A. Kumar and others, *Approximation of functions by Stancu variant of genuine Gupta-Srivastava operators*, Palestine Journal of Mathematics, **10** (2021).

[10] B. Hazarika, A. Alotaibi and S. A. Mohiuddine, *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Computing, **24** 6613–6622, (2020).

[11] İ.Kisi, R. Akbıyık and M. Gürdal, *Some insights into weighted ideal statistical convergence of fuzzy variables in credibility spaces*, Palestine Journal of Mathematics, **14** 438–449, (2025).

[12] S. A. Mohiuddine, A. Asiri and B. Hazarika, *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, International Journal of General Systems, **48** 492–506, (2019).

[13] P. Karthick, S. Mohiuddine, K. Tamilvanan, S. Narayanamoorthy and S. Maheswari, *Investigations of color image segmentation based on connectivity measure, shape priority and normalized fuzzy graph cut*, Applied Soft Computing, **139** 110239, (2023).

[14] L. A. Zadeh, *Fuzzy sets*, Information and control, **8** 338–353, (1965).

[15] S. G. Gal, *Approximation theory in fuzzy setting*, in Handbook of analytic computational methods in applied mathematics, 617–666, Chapman and Hall/CRC, (2019).

[16] G. A. Anastassiou, *Fuzzy mathematics: Approximation theory*, Springer, (2010).

[17] I. Rainville and E. S. Functions, *Chelsea Publishing Company*, Bronz, New York, **211** (1960).

[18] J. Gurland, E. E. Chen and F. M. Hernandez, *A new discrete distribution involving Laguerre polynomials*, Communications in Statistics-Theory and Methods, **12** 1987–2004, (1983).

[19] V. Gupta, *New operators based on Laguerre polynomials*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **118** 19, (2024).

[20] J. Labelle and Y. N. Yeh, *The combinatorics of Laguerre, Charlier, and Hermite polynomials*, Studies in Applied Mathematics, **80** 25–36, (1989).

[21] A. Kajla and P. N. Agrawal, *Approximation properties of Szász type operators based on Charlier polynomials*, Turkish Journal of Mathematics, **39** 990–1003, (2015).

[22] A. Kajla and P. N. Agrawal, *Szász-Kantorovich type operators based on Charlier polynomials*, Kyungpook Mathematical Journal, **56** 877–897, (2016).

[23] A. Kajla and P. N. Agrawal, *Szász-Durrmeyer type operators based on Charlier polynomials*, Applied Mathematics and Computation, **268** 1001–1014, (2015).

[24] S. Mohiuddine, A. Kajla, S. Deshwal and A. Alotaibi, *Approximation by Szász-Durrmeyer type operators involving Charlier polynomials*, TWMS Journal of Pure and Applied Mathematics, **14** 214–231, (2023).

[25] W. Van Assche, *Difference equations for multiple Charlier and Meixner polynomials*, Proceedings of the Sixth International Conference on Difference Equations, CRC, Boca Raton, FL, 549–557, (2004).

[26] N. Özmen and E. Erkuş-Duman, *On the Poisson-Charlier polynomials*, Serdica Mathematical Journal, **41** 457p–470p, (2015).

[27] S. Varma and F. Taşdelen, *Szász type operators involving Charlier polynomials*, Mathematical and Computer Modelling, **56** 118–122, (2012).

[28] M. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Cambridge university press, (2005).

[29] B. Rui and R. Wong, *Uniform asymptotic expansion of Charlier polynomials*, Methods and Applications of Analysis, **1** 294–313, (1994).

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