EXPLORING THE FRAME SET PROPERTIES OF 2nd ORDER B-SPLINE

Sruthi Raghoothaman and Noufal Asharaf

Communicated by: Harikrishnan Panackal

Corresponding Author: Noufal Asharaf

Abstract. This work offers a comprehensive exploration of results related to the frame set properties of 2nd order B-spline B_2. While frame expansions necessitate a dual frame, finding the canonical dual frame for a Gabor frame can be challenging and computationally intensive. Therefore, we focus on constructing non-canonical dual generators. We present methods for constructing dual generators for B_2 for various shift parameters. We find a sequence of new non-frame points. Determining the frame set of the second-order B-spline requires equal focus on identifying both frame and non-frame points. The frame set is entirely contained within the region defined by $ab < 1$, but a significant portion of this region must be rigorously examined to confirm whether it belongs to the frame set or not. As part of this analysis, we identify a sequence of new non-frame points for B_2.

1 Introduction

Frame is a powerful and flexible tool for analyzing and synthesizing data in Hilbert spaces. They generalize the concept of an orthonormal basis but linear independence is not required. Hence this redundant system has numerous applications in signal processing, image compression, and quantum mechanics. A major application of frame theory is in Gabor analysis, which plays a central role in time-frequency analysis. For $a, b > 0$, the translation and modulation operators T_a and E_b on the Hilbert space $L^2(\mathbb{R})$ are defined by $T_a f(x) = f(x - a)$ and $E_b f(x) = e^{2\pi i b x} f(x)$, respectively. For $g \in L^2(\mathbb{R})$, and parameters $a, b > 0$, the sequence of functions $\mathcal{G}(g, a, b) := \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is called a Gabor system generated by g. A Gabor system enables simultaneous analysis of signals in both the time and frequency domains. Given $g \in L^2(\mathbb{R})$, and $a, b > 0$, the sequence of functions $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is said to be a frame for $L^2(\mathbb{R})$ if there exist constants $0 < A < B \leq \infty$ such that

$$A\|f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |<f, E_{mb} T_{na} g>|^2 \leq B\|f\|^2 \quad \forall f \in L^2(\mathbb{R}).$$

The constants A and B are referred to as the lower and upper frame bounds, respectively.. When at least the upper bound is true, we call the Gabor system $\mathcal{G}(g, a, b)$ as a Bessel sequence. It is said to be a tight frame when $A = B$. For $g \in L^2(\mathbb{R})$, suppose the Gabor system $\mathcal{G}(g, a, b)$

forms a frame for $L^2(\mathbb{R})$. Then, any function $f \in L^2(\mathbb{R})$ can be analyzed by the sequence $\{\langle f, E_{mb}T_{na}g\rangle\}_{m,n\in\mathbb{Z}}$. The map $C_g : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ which maps $f \to \{\langle f, E_{mb}T_{na}g\rangle\}_{m,n\in\mathbb{Z}}$ is called the analysis operator. Its adjoint C_g^* is called the synthesis operator. The synthesis operator maps the sequence $\{\langle f, E_{mb}T_{na}g\rangle\}_{m,n\in\mathbb{Z}} \to \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g$. The composition of the synthesis and analysis operators is the frame operator corresponding to the system $\mathcal{G}(g, a, b)$ and is given by

$$Sf := C_g^* C_g f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g.$$

The frame operator is always invertible and self-adjoint[3, 25]. Hence we can write

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle S^{-1}E_{mb}T_{na}g.$$

Since the frame operator as well as its inverse operator commute with the operator $E_{mb}T_{na}$ [3], we get

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}S^{-1}g.$$

$\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ for every frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. The function $S^{-1}g$ is called the canonical dual of g and $\mathcal{G}(S^{-1}g, a, b)$ is called the canonical dual frame of the frame $\mathcal{G}(g, a, b)$. In addition if $ab < 1$ then there exists infinitely many functions $h \in L^2(\mathbb{R})$ such that $\mathcal{G}(h, a, b)$ is a Bessel system and

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}h.$$

holds with unconditionally L^2-convergence. The function h is called a dual window. If $\mathcal{G}(h, a, b)$ itself is a frame for $L^2(\mathbb{R})$, then we call $\mathcal{G}(g, a, b)$ and $\mathcal{G}(h, a, b)$ a pair of dual Gabor frames. As we have seen, the three parameters that define a Gabor system are the window function g, the shift parameter a and modulation parameter b. In Gabor analysis, a central problem is identifying the specific sampling and modulation rates that ensure the time-frequency shifts of a chosen generator function form a frame. The set of all positive values of a and b in the plane for which a Gabor system $\mathcal{G}(g, a, b)$ forms a frame for $L^2(\mathbb{R})$ is said to be the frame set $\mathcal{F}_g$ for g. i.e., Frame set for g is the set

$$\mathcal{F}_g := \{(a, b) \in \mathbb{R}_+^2 | \mathcal{G}(g, a, b) \text{ forms a frame for } L^2(\mathbb{R})\}.$$

Finding the frame set for a given function g remains as one of the main issues in Gabor analysis. This task is notably difficult, and no universal method currently exists to address it. However, it is known that the different frame characteristics for $\mathcal{G}(g, a, b)$ are ensured by restricting the choice of parameters a and b, irrespective of the generator $g \in L^2(\mathbb{R})$ (Theorem 9.1.12, [3]). i.e., The system $\mathcal{G}(g, a, b)$ has a frame structure only when $ab < 1$ and forms an orthonormal basis if and only if $ab = 1$. A complete characterization of the frame set has been determined for only a few specific functions., particularly in 1992, Lyubarski [23] and Seip [26] gave the complete characterization about the frame set of Gaussian $e^{-\pi x^2}$. Janssen and Strohmer [19] reduced the case of the hyperbolic secant $\frac{1}{\cosh x}$ to the case of the Gaussian. The two exponential functions $e^{-|x|}$ and $e^{-x}\chi_{[0,\infty)}$ were analyzed by Janssen [16, 17]. A function $g \in L^2(\mathbb{R})$ is a totally positive function of type M, with $M \in \mathbb{N}$, if the associated Fourier transform takes the form $\hat{g}(\xi) = \prod_{k=1}^{M}(1 + 2\pi i\delta_k\xi)^{-1}$ were $\delta_k \neq \delta_l \in \mathbb{R}$ for $k \neq l$. All the above mentioned window functions g have the same frame set $\mathcal{F}_g = \{(a, b) \in \mathbb{R}_+^2 : ab < 1\}$.

In this work, we provide an in-depth analysis of the frame set for the second order B-spline B_2. B-splines are piecewise polynomials, meaning that they are composed of polynomial segments that are smoothly connected across their domain. The B-splines are defined inductively

as

$$B_1 = \chi_{[-\frac{1}{2}, \frac{1}{2}]} \text{ and } B_{N+1} = B_N * B_1.$$

Thus for any $N \in \mathbb{N}$, we have that

$$B_{N+1}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} B_N(x-t)dt.$$

The first three B-splines are shown in figure 1. B_N's are supported on $[\frac{-N}{2}, \frac{N}{2}]$, and are symmetric about zero [3]. For $N \geq 2$, all B_N's have the partition of unity property. i.e.,

$$\sum_{k \in \mathbb{Z}} B_N(x-k) = 1, \text{ for all } x \in \mathbb{R}.$$

The B-spline B_2 is given by

$$B_2(x) = \begin{cases} 1+x & \text{if } x \in [-1,0] \\ 1-x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} . \end{cases}$$
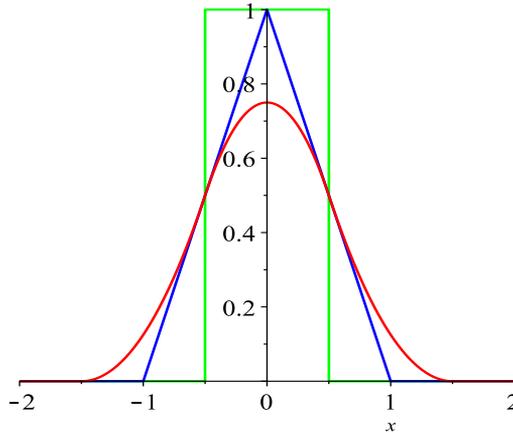

Figure 1: Plots of B-splines: B_1 (green), B_2 (blue), and B_3 (red).

B-spline generators are frequently employed in applications and have a number of beneficial characteristics, such as being piecewise polynomial and having good time-frequency localization. Since B-splines are nonzero only over a finite interval, they exhibit strong time localization, which enables efficient computation and minimizes overlap between shifted copies. This property is crucial for constructing localized bases in time-frequency representations such as Gabor frames. Moreover, B-splines of order greater than two yield smoother window functions, leading to more regular and continuous time-frequency representations while maintaining a balanced trade-off between time and frequency localization. Their recursive definition further enhances computational efficiency, allowing straightforward implementation in both continuous and discrete settings, which is particularly advantageous for real-time signal analysis and processing. In addition, B-splines are highly flexible, capable of approximating various window shapes and adapting to different signal characteristics. Owing to these properties, they find broad applications in denoising, feature extraction, and time-frequency-based signal reconstruction.

Despite their wide applicability and desirable mathematical properties, the theoretical understanding of the frame sets associated with B-splines remains incomplete. Apart from the first B-spline, there is limited information available about the frame sets for other B-splines. Dai and Sun [6] conducted an in-depth study and provided a complete description of the frame set specifically for the first B-spline B_1. The following theorem, established by Dai and Sun, precisely characterizes the frame set for the first-order B-spline B_1 and provides necessary and sufficient

conditions under which the corresponding Gabor system forms a frame.

Theorem 1.1. *[6] Let $a, b > 0$. Then the following statements hold.*

(i) *If $a > 1$, then $\mathcal{G}(\chi_{[0,1)}, a, \frac{1}{b})$ does not form a frame.*

(ii) *If $a = 1$, then $\mathcal{G}(\chi_{[0,1)}, a, \frac{1}{b})$ forms a frame if and only if $b \geq 1$.*

(iii) *If $a < 1$ and $b \leq a$, then $\mathcal{G}(\chi_{[0,1)}, \frac{1}{b}))$ does not form a frame.*

(iv) *If $a < 1$ and $b \geq 1$, then $\mathcal{G}(\chi_{[0,1)}, \frac{1}{b})$ forms a frame.*

The studies about $\mathcal{F}_{B_N}$ for $N \geq 2$ were first started in [7, 8, 15]. A key finding was that $(a, b) \notin \mathcal{F}_{B_N}$ for any integer $b \geq 2$. In [3] while identifying a class of functions a new frame region is found for B_N. In [12] Grochenig conjectured that $\mathcal{F}_{B_N} = \{(a, b) \in \mathbb{R}_+^2 : ab < 1, a < N, b \neq 2, 3, ...\}$ for $N \geq 2$. This B-spline conjecture is later disproved by Kamilla Haahr Nielsen and Jakob Lemvig [21] by showing counter examples.

The remaining of this article is structured as follows: A brief review of the existing results concerning the frame and non-frame points of B_2 is given in section 2. The methods currently in use for locating non-frame points using the Zak transorm and Zibulski-Zeevi matrix are described in Section 3. This allowed us to identify a few new known-frame points, which are described in Section 4.

2 Finding frame points by dual function construction

If we can construct a dual generator $h \in L^2(\mathbb{R})$ for B_2 corresponding to a given pair $(a, b) \in \mathbb{R}_+^2$ then we can say that $(a, b) \in \mathcal{F}_{B_N}$. So we will focus on the area in the plane in which we can construct dual generators for B_2. The key idea of dual generator construction lies in the well-known duality condition for Gabor frames [Theorem 9.3.5, [3]]. This states that two Bessel sequences $\mathcal{G}(g, a, b)$ and $\mathcal{G}(h, a, b)$ form dual frames if and only if

$$\sum_{k \in \mathbb{Z}} \overline{g(x - \frac{n}{b} - ka)} h(x - ka) = b\delta_{n,0}, \quad a.e. \ x \in [0, a]. \tag{2.1}$$

where δ is the delta function $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. It is well known that for a bounded compactly supported function $g \in L^2(\mathbb{R})$, the corresponding Gabor system $\mathcal{G}(g, a, b)$ is a Bessel sequence for any choice of a and b. Using this fact a compactly supported dual generator can be constructed for B_2 [21] in the region

$$\Sigma_2 = \left\{ (a, b) \in \mathbb{R}_+^2 : 0 < a < 2, \ ab < 1, \ b \leq \frac{2}{2 + a} \right\}.$$

Since $B_2(x) > 0$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$, we can define the function,

$$h(x) = \begin{cases} \frac{b}{B_2(x)}, & \text{if} \quad x \in [-\frac{a}{2}, \frac{a}{2}] \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Clearly, this function is bounded and has compact support. Therefore, $\mathcal{G}(h, a, b)$ is a Bessel sequence. Since supp $h = [-\frac{a}{2}, \frac{a}{2}]$, $h(x - ka) = 0$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $k \neq 0$. Hence in equation 2.1, only the term corresponding to $k = 0$ becomes non-zero. i.e., equation 2.1 reduces to

$$h(x)B_2(x - \frac{n}{b}) = \delta_{n,0}b, \quad x \in [-\frac{a}{2}, \frac{a}{2}], \quad n \in \mathbb{Z}$$

For $n = 0$, we get

$$h(x)B_2(x) = \frac{b}{B_2(x)}B_2(x) = b, \quad x \in [-\frac{a}{2}, \frac{a}{2}]$$

For $n = 1$, we get

$$h(x)B_2(x - \frac{1}{b}) = 0, \quad x \in [-\frac{a}{2}, \frac{a}{2}].$$

Since supp $h = [-\frac{a}{2}, \frac{a}{2}]$, supp $B_2(\cdot - \frac{1}{b}) = [\frac{1}{b} - 1, \frac{1}{b} + 1]$ and $\frac{a}{2} \leq \frac{1}{b} - 1$, we get supp $h \cdot$ supp $B_2(\cdot - \frac{1}{b}) = \emptyset$.

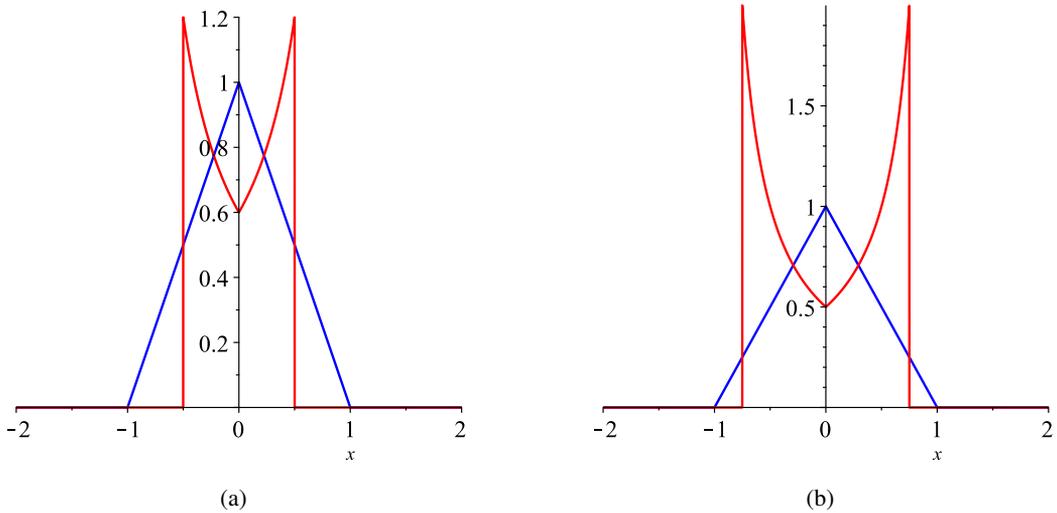Hence the equations are satisfied. Thus the function h is a dual generator for B_2. Dual function

Figure 2: B-spline B_2 (blue) and its dual window h (red) for different parameter choices: (a) $a = 1, b = \frac{3}{5}$; (b) $a = \frac{3}{2}, b = \frac{1}{2}$.

in equation (2.2) with $a = 1$, $b = \frac{3}{5}$ and $a = \frac{3}{2}$, $b = \frac{1}{2}$ is shown in figure 2. It is an interesting fact that dual function is not unique. For $0 < b < 1$, we can find another dual for B_2 using the technique presented by Jakob Lemvig and Kamilla Haahr Nielsen [22] for a specific class of functions. For each $n \in \mathbb{N}$, the class V_+^n consists of all $f \in C^n(\mathbb{R})$ which are supported on $[-1, 1]$ and $|f| > 0$ in $(-1, 1)$. Clearly $B_2 \in V_+^1$. This approach can be applied to B_2, to obtain a different dual generator in the region $0 < b < 1$ and $a = 1$. Since supp $B_2 = [-1, 1]$, we can simplify the system of equations 2.1. For $n = 0$,

$$B_2(x - 1)h(x - 1) + B_2(x)h(x) = b, \quad x \in [-\frac{1}{2}, \frac{1}{2}], \qquad (2.3)$$

For $n \in \mathbb{Z} \setminus \{0\}$,

$$B_2(x - \frac{n}{b} - 1)h(x - 1) + B_2(x - \frac{n}{b})h(x) = 0, \quad x \in [\frac{n}{b}, \frac{n}{b} + 1] \qquad (2.4)$$

To solve these equations, we start by solving the equation for $n = 0$. Define h as follows on the interval $[-1, 1]$

$$h(x) = \begin{cases} B_2(x + 1)z(x + 1) + \dfrac{b}{B_2(x) + B_2(x + 1)}, & x \in [-1, 0] \\ -B_2(x - 1)z(x) + \dfrac{b}{B_2(x - 1) + B_2(x)}, & x \in [0, 1] \end{cases}$$

where z denotes a real valued measurable function on $[0, 1]$. Since h depends on the function z, we will denote h_z in the remaining discussion. Now consider the equation corresponding to $n = 1$, i.e.,

$$B_2(x - \frac{1}{b} - 1)h(x - 1) + B_2(x - \frac{1}{b})h_z(x) = 0, \quad x \in [\frac{1}{b}, \frac{1}{b} + 1] \qquad (2.5)$$

This equation shows that we need to define h_z on $[\frac{1}{b} - 1, \frac{1}{b} + 1]$. Since $0 < b < 1$, we know that $\frac{1}{b} - 1 > 0$, hence $[\frac{1}{b} - 1, \frac{1}{b} + 1] \subset [0, \frac{1}{b} + 1]$. We have $1 \leq \frac{1}{b} - 1$, if and only if $b \leq \frac{1}{2}$ then $[\frac{1}{b} - 1, \frac{1}{b} + 1] \cap [0, 1] = \emptyset$ and the equation 2.5 is satisfied. Similar argument shows that the equation 2.4 is satisfied for $n = -1$. For $|n| > 1$, $[\frac{n}{b} - 1, \frac{n}{b} + 1] \cap [-1, 1] = \emptyset$. Hence, the

characterizing equations for $|n| > 1$ are satisfied. Since all characterizing equations are satisfied, h_z is a dual function if and only if $\mathcal{G}(h_z, a, b)$ is a Bessel sequence. Now consider $b > \frac{1}{2}$, since supp $h_z = [-1, 1]$, the equation 2.4 is trivially satisfied for $x \in [2, \frac{1}{b} + 1]$. For $x \in [\frac{1}{b}, 2]$,

$$B_2(x - \frac{1}{b} - 1)\left[-B_2(x - 2)z(x - 1) + \frac{b}{B_2(x - 2) + B_2(x - 1)}\right] + B_2(x - \frac{1}{b})h_z(x) = 0.$$

Since $B_2(x - \frac{1}{b}) \neq 0$ for $x \in (\frac{1}{b}, 2)$, we get

$$h_z(x) = -\frac{B_2(x - \frac{1}{b} - 1)}{B_2(x - \frac{1}{b})}\left[-B_2(x - 2)z(x - 1) + \frac{b}{B_2(x - 2) + B_2(x - 1)}\right],$$

Using a similar approach for the equation corresponding to $n = -1$ we can define h_z on $[-2, -\frac{1}{b}]$. Hence h_z can be defined as follows

$$h_z(x) = \begin{cases} -\dfrac{B_2(x + \frac{1}{b} + 1)}{B_2(x + \frac{1}{b})}\left[B_2(x + 2)z(x + 2) + \dfrac{b}{B_2(x + 1) + B_2(x + 2)}\right], & x \in [-2, -\frac{1}{b}] \\ B_2(x + 1)z(x + 1) + \dfrac{b}{B_2(x) + B_2(x + 1)}, & x \in [-1, 0] \\ B_2(x - 1)z(x) + \dfrac{b}{B_2(x - 1) + B_2(x)}, & x \in [0, 1] \\ -\dfrac{B_2(x - \frac{1}{b} - 1)}{B_2(x - \frac{1}{b})}\left[-B_2(x - 2)z(x - 1) + \dfrac{b}{B_2(x - 2) + B_2(x - 1)}\right], & x \in [\frac{1}{b}, 2] \end{cases}$$

If the characteristic equations for $n = \pm 2$ are satisfied, then we are done. Otherwise, we extend the definition of h_z on $[-3, -\frac{2}{b}]$ and $[\frac{2}{b}, 3]$ such that the characterizing equations for $n = \pm 2$ are satisfied. This process continues as long as

$$\frac{n}{b} - 1 < n \iff n < \frac{1}{1 - b}.$$

Let $n_{\max}$ be the largest such n. i.e., $n_{\max} = \max\left\{n \in \mathbb{Z}_{\geq 0} : n < \frac{b}{1-b}\right\}$. It is clear that $n_{\max} \geq 1$ when $1/2 < b < 1$. For each $n \in \{1, \ldots, n_{\max}\}$, we get the following auxiliary functions:

$$\eta_n(x) = (-1)^n \prod_{j=1}^{n} \frac{B_2(x + n + 1 + j(\frac{1}{b} - 1))}{B_2(x + n + j(\frac{1}{b} - 1))}[-B_2(x + n + 1)z(x + n + 1) + b],$$

for $x \in [-n - 1, -\frac{n}{b}]$ and

$$\gamma_n(x) = (-1)^n \prod_{j=1}^{n} \frac{B_2(x - n - 1 - j(\frac{1}{b} - 1))}{B_2(x - n - j(\frac{1}{b} - 1))}[B_2(x - n - 1)z(x - n) + b]$$

for $x \in [\frac{n}{b}, n + 1]$. Then we define $h_z : \mathbb{R} \to \mathbb{R}$,

$$h_z(x) = \begin{cases} \eta_n(x), & x \in [-n - 1, -\frac{n}{b}], \ n = 1, 2 \ldots, n_{\max} \\ -B_2(x + 1)z(x + 1) + b, & x \in [-1, 0), \\ B_2(x - 1)z(x) + b, & x \in [0, 1], \\ \gamma_n(x), & x \in [\frac{n}{b}, n + 1], \ n = 1, 2 \ldots, n_{\max} \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.1. For $b = \frac{3}{5}$, we have $n_{\max} = 1$. We take z as a linear polynomial $z(x) = c_0 + c_1 x$, and we solve for c_0 and c_1 (Lemma 3.6 in [22]) and obtain as follows:

$$z(0) = b \text{ and } z(1) = -b$$

Thus $z(x) = \frac{3}{5}x - \frac{6}{5}$. The dual function h_z is then obtained as follows:

$$
h_z(x) = \begin{cases}
\dfrac{B_2(x+1+\frac{1}{b})}{B_2(x+\frac{1}{b})}\left[B_2(x+2)z(x+2) - b\right], & x \in [-2, -\frac{1}{b}], \\
-B_2(x+1)z(x+1) + b, & x \in [-1, 0), \\
B_2(x-1)z(x) + b, & x \in [0,1], \\
-\dfrac{B_2(x-1-\frac{1}{b})}{B_2(x-\frac{1}{b})}\left[B_2(x-2)z(x-1) + b\right], & x \in [\frac{1}{b}, 2], \\
0, & \text{otherwise.}
\end{cases}
$$

If we take $b = \frac{4}{5}$, then $n_{\max} = 3$, $z(x) = \frac{4}{5} - \frac{8}{5}x$ and h_z will be

$$
h_z(x) = \begin{cases}
-\dfrac{B_2(x+1+\frac{3}{b})B_2(x+2+\frac{2}{b})B_2(x+3+\frac{1}{b})}{B_2(x+\frac{3}{b})B_2(x+1+\frac{2}{b})B_2(x+2+\frac{1}{b})}\left[-B_2(x+4)z(x+4) + b\right], & x \in [-4, -\frac{3}{b}], \\
\dfrac{B_2(x+1+\frac{2}{b})B_2(x+2+\frac{1}{b})}{B_2(x+\frac{2}{b})B_2(x+1+\frac{1}{b})}\left[-B_2(x+3)z(x+3) + b\right], & x \in [-3, -\frac{2}{b}], \\
-\dfrac{B_2(x+1+\frac{1}{b})}{B_2(x+\frac{1}{b})}\left[-B_2(x+2)z(x+2) + b\right], & x \in [-2, -\frac{1}{b}], \\
-B_2(x+1)z(x+1) + b, & x \in [-1, 0), \\
B_2(x-1)z(x) + b, & x \in [0,1], \\
-\dfrac{B_2(x-1-\frac{1}{b})}{B_2(x-\frac{1}{b})}\left[B_2(x-2)z(x-1) + b\right], & x \in [\frac{1}{b}, 2], \\
\dfrac{B_2(x-1-\frac{2}{b})B_2(x-2-\frac{1}{b})}{B_2(x-\frac{2}{b})B_2(x-1-\frac{1}{b})}\left[B_2(x-3)z(x-2) + b\right], & x \in [\frac{2}{b}, 3], \\
-\dfrac{B_2(x-1-\frac{3}{b})B_2(x-2-\frac{2}{b})B_2(x-3-\frac{1}{b})}{B_2(x-\frac{3}{b})B_2(x-1-\frac{2}{b})B_2(x-2-\frac{1}{b})}\left[B_2(x-4)z(x-3) + b\right], & x \in [-4, -\frac{3}{b}], \\
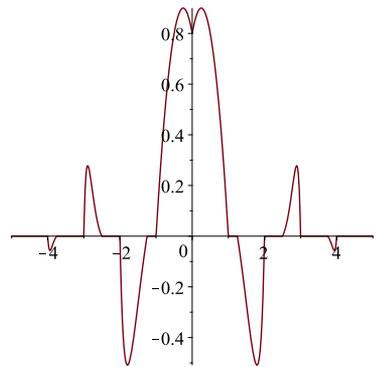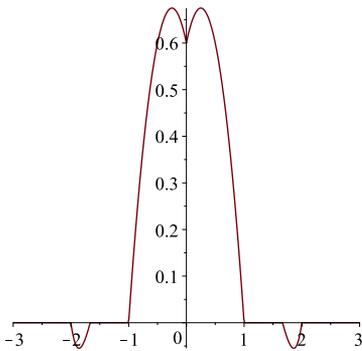0, & \text{otherwise.}
\end{cases}
$$

Figure 3: Dual window h_z of the B-spline B_2 for a fixed parameter $a = 1$ and different values of b: (a) $b = \frac{3}{5}$ and (b) $b = \frac{4}{5}$.

This construction gives us a method to find alternate duals in the region $a = 1$ and $0 < b < 1$. Next we will look into the extension of the frame region $\sum_2$. This region is found in [5], for a certain class of functions $V_{N,a}$. For each $N \in \mathbb{N}$ and $a > 0$, the class $V_{N,a}$ consists of all real valued continuous functions f on $\mathbb{R}$ with support $[-\frac{N}{2}, \frac{N}{2}]$ such that f is symmetric about the origin, f is strictly increasing on $[-\frac{N}{2}, 0]$ and satisfies the condition: If $a < \frac{N}{3}$, then $\Delta_a^2 f(x) \geq 0$, $x \in [-\frac{N}{2}, -\frac{N}{4} + \frac{3a}{4}]$; if $a \geq \frac{N}{3}$, then $\Delta_a^2 f(x) \geq 0$, $x \in [-\frac{N}{2}, 0] \cup \{-\frac{N}{4} + \frac{3a}{4}\}$, where

$$\Delta_a f(x) = f(x) - f(x-a) \text{ and } \Delta_a^2 f(x) = f(x) - 2f(x-a) + f(x-2a).$$

Proposition 3.1 of [5] ensures that all symmetric splines B_N belong to the class $V_{N,a}$. Hence, $B_2 \in V_{2,a}$. Based on these properties, the following result, established in [5], identifies an

extended frame region for B_2 and provides precise conditions under which the corresponding Gabor system forms a frame along with a compactly supported dual window.

Theorem 2.2. *[5] Let $0 < a < 2$, and $\frac{2}{2+a} < b \leq \frac{4}{2+3a}$. Then $\mathcal{G}(B_2, a, b)$ constitutes a frame for $L^2(\mathbb{R})$ and a unique dual function $h \in L^2(\mathbb{R})$ can be found such that supp $h \subseteq [-\frac{3a}{2}, \frac{3a}{2}]$.*

Here also we use equation 2.1 to prove the existence of the dual in the mentioned region. Since we have set supp $h \subseteq [-\frac{3a}{2}, \frac{3a}{2}]$, $h(x \pm ka) = 0$ for $k \neq \pm 1, 0$ in equation 2.1. Thus it reduces to

$$\sum_{k=-1}^{1} B_2(x - \frac{n}{b} - ka)h(x - ka) = b\delta_{n,0}, \quad a.e. \quad x \in [0, a].$$

Thus we can represent this system of equations in terms of the matrix equation

$$\begin{pmatrix} h(x-a) \\ h(x) \\ h(x+a) \end{pmatrix} = \begin{pmatrix} B_2(x+\frac{1}{b}-a) & B_2(x+\frac{1}{b}) & B_2(x+\frac{1}{b}+a) \\ B_2(x-a) & B_2(x) & B_2(x+a) \\ B_2(x-\frac{1}{b}-a) & B_2(x-\frac{1}{b}) & B_2(x-\frac{1}{b}+a) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \quad \text{a. e. } x \in [-\frac{a}{2}, \frac{a}{2}].$$

$$(2.6)$$

For $0 < a < 2$ and $\frac{2}{2+a} < b \leq \frac{4}{2+3a}$, the determinant of the matrix in the right hand side is non-zero. So the dual function h can be constructed from solving the above matrix equation and hence this method allows us to say that

$$\left\{ (a,b) \in (\mathbb{R})_+^2 : ab < 1, 0 < a < 2, \frac{2}{2+a} < b \leq \frac{4}{2+3a} \right\} \quad \subset \quad \mathcal{F}_{B_2}.$$

By using similar technique new frame regions $\Gamma_3 \cup \Lambda$ were found in [2], where

$$\Gamma_3 = \left\{ (a,b) \in \mathbb{R}_+^2 : a \in \left(0, \frac{2}{9}\right] \cup \left(\frac{2}{7}, \frac{1}{2}\right), b \in \left(\frac{4}{2+3a}, \frac{2}{1+a}\right] \right\}$$

$$\Lambda_3 = \left\{ (a,b) \in \mathbb{R}_+^2 : a \in \left[\frac{1}{2}, \frac{4}{5}\right], b \in \left(\frac{2}{7}, \frac{1}{2}\right), b \in \left(\frac{4}{2+3a}, \frac{6}{2+5a}\right], b > 1 \right\}$$

For $m \geq 4$

$$\Lambda_m = \left\{ (a,b) \in \mathbb{R}_+^2 : a \in \left[\frac{m-3}{m-2}, \frac{2(m-1)}{2m-1}\right], b \in \left(\frac{2(m-1)}{2+(2m-3)a}, \min_a \left(\frac{2m}{2+(2m-1)a}, \frac{2}{1+a}\right)\right], b > 1 \right\}$$

and $\Lambda := \cup_{m=3}^{\infty} \Lambda_m$.

The authors' approach is constructive and hinges on demonstrating the existence of a dual window for the Gabor system generated by the B-spline. They leverage the fact that, for a compactly supported function, the duality condition between two Gabor systems simplifies to a finite set of linear equations. This system is then analyzed through a matrix formulation. The core of their method involves proving that for all parameters (a, b) within the newly defined regions Γ_3 and Λ, the associated matrix is always invertible. They achieve this by meticulously showing that the determinant of this matrix remains strictly positive across the entire parameter range. This is done through a sophisticated induction argument for the Λ regions and a detailed case-by-case analysis of the matrix's structure for the Γ_3 region, ultimately guaranteeing the existence of a suitable dual window.

The remaining sections need some knowledge about Zak transform and Zibulski-Zeevi matrix: For a fixed parameter $\lambda > 0$, the Zak transform of a function $g \in L^2(\mathbb{R})$ is defined as a function of two real variables:

$$(Z_\lambda g)(x, t) = \sqrt{\lambda} \sum_{k \in \mathbb{Z}} g(\lambda(x - k))e^{2\pi i k t} \text{ a.e., } x, t \in \mathbb{R}.$$

It is clear that Zak transform is 1-periodic in ν and it is quasi-periodic in x.

$$Z_\lambda g(x+1, t) = e^{2\pi i t} Z_\lambda g(x, t), \quad Z_\lambda g(x, t+1) = e^{2\pi i t} Z_\lambda g(x, t) \text{ for a. e. } x, t \in \mathbb{R}$$

Z_λ is a unitary map of $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$. The Zak transform plays a crucial role in simplifying the analysis of Gabor systems, especially when the lattice parameters satisfy $ab = 1$. By transforming the problem into a quasi-periodic and compact domain, it allows one to study frame properties through the behavior of $Z_\lambda g(x,t)$. In particular, the completeness and frame conditions of a Gabor system can be characterized directly in terms of the Zak transform, as stated in the following proposition.

Proposition 2.3. [5] *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ with $ab = 1$. The following statement is true:*

(i) $\mathcal{G}(g,a,b)$ *is complete in $L^2(\mathbb{R})$ if and only if $Z_a g \neq 0$, a.e.,*

(ii) $\mathcal{G}(g,a,b)$ *is a frame for $L^2(\mathbb{R})$ with frame bounds A, B if and only if $A \leq |Z_a g|^2 \leq B$*

For $f \in W(\mathbb{R})$, the Zak transform of f is bounded on $\mathbb{R}^2$. Here, $W(\mathbb{R})$, known as the Wiener space, consists of functions in $L^\infty(\mathbb{R})$ that have a finite W-norm defined by

$$||f||_W = \sum_{k \in \mathbb{N}} \underset{x \in \mathbb{R}}{\text{ess sup}} |f(x+k)|.$$

If $f \in W(\mathbb{R}) \cup C^0(\mathbb{R})$, then according to [3, Lemma 8.2.1(c)], the Zak transform $Z_\lambda f$ remains continuous. Since all B-splines belong to the Wiener space, $Z_a B_2$ is continuous.

For rational values of ab, a connection of the Zak transform $Z_a g$ with the frame bounds of the Gabor system $\mathcal{G}(g,a,b)$ was given by Zibulski and Zeevi [27]. Using the definition of Zak transform, we can define the Zibulski-Zeevi matrix as follows:

Consider Gabor system $\mathcal{G}(g,a,b)$ with rational oversampling $ab \in \mathbb{Q}$, $ab = \frac{p}{q}$, with p and q being relatively prime. For $g \in \mathbb{R}$, we consider the column vectors $\phi_l^g(x,t) \in \mathbb{C}^p$ for $l \in \{0,1,...,q-1\}$ by

$$\phi_l^g(x,t) = \left(p^{-\frac{1}{2}}(Z_{\frac{1}{b}}g)(x - l\frac{p}{q}, t + \frac{k}{p}) \right)_{k=0}^{p-1} \quad a.e. \ x, t \in \mathbb{R} \tag{2.7}$$

and column vectors $\psi_l^g(x,t) \in \mathbb{C}^p$ for $l \in \{0,1,...,q-1\}$ by

$$\psi_l^g(x,t) = \left(b^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} g(x + aqn + al + \frac{k}{b}) e^{-2\pi i a q n t} \right)_{k=0}^{p-1} \quad a.e. \ x, t \in \mathbb{R} \tag{2.8}$$

The $p \times q$ matrix defined by $\Phi^g(x,t) = [\phi_l^g(x,t)]_{l=0}^{q-1}$ is known as Zibulski-Zeevi matrix and the $p \times q$ matrix $\Psi^g(x,t) = [\psi_l^g(x,t)]_{l=0}^{q-1}$ is a variant of the Zibulski-Zeevi matrix. This matrix has transformed the infinite dimensional Gabor system $\mathcal{G}(g,a,b)$ into a finite dimensional vector system. These matrices can be used to determine the frame properties of the corresponding Gabor system.

The following theorem shows that the frame property of $\mathcal{G}(g,a,b)$ is equivalent to the frame property of the corresponding finite-dimensional vector sets derived from the Zibulski-Zeevi matrices.

Theorem 2.4. [21] *Let $A, B > 0$ and let $g \in L^2(\mathbb{R})$. Let $a, b > 0$ such that $ab \in \mathbb{Q}$. Then the following statements are equivalent:*

(i) $\mathcal{G}(g,a,b)$ *is a Gabor frame for $L^2(\mathbb{R})$ with bounds A and B,*

(ii) $\{\phi_l^g(x,t)\}_{l=0}^q$ *is a frame for $\mathbb{C}^p$ with uniform bounds A and B for a.e. $(x,t) \in [0, \frac{1}{p}) \times [0,1)$.*

(iii) $\{\psi_l^g(x,t)\}_{l=0}^q$ *is a frame for $\mathbb{C}^p$ with uniform bounds A and B for a.e. $(x,t) \in [0, \frac{a}{p}) \times [0, \frac{1}{a})$.*

The theorem says that the Gabor system $\mathcal{G}(g,a,b)$ is a frame with (optimal) bounds A and B for $L^2(\mathbb{R})$ if and only if $\sqrt{A}$ is the infimum over a.e. $(x,t) \in \mathbb{R}^2$ of the smallest singular value of $\Phi^g(x,t)$ (or $\Psi^g(x,t)$) and $\sqrt{B}$ is the supremum over a.e. $(x,t) \in \mathbb{R}^2$ of the largest singular value of $\Phi^g(x,t)$ (or $\Psi^g(x,t)$). In other words a rationally oversampled system $\mathcal{G}(g,a,b)$ forms a frame for $L^2(\mathbb{R})$ with frame bounds $A, B > 0$ if and only if

$$AI \leq \Phi_g(x,t)(\Phi_g^*(x,t))^* \leq BI \quad a.e.(x,t) \in [0,1]^2 \tag{2.9}$$

The finite system will have a dimension of p and we aim to determine whether the q columns of the Zibulski-Zeevi matrix form a frame for $\mathbb{C}^p$. We can determine this by checking whether equation 2.9 holds for some $A, B > 0$. If we assume that the singular values are given as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ then this is equivalent to verifying that $\sigma_p \geq \sqrt{A}$ and $\sigma_1 \leq \sqrt{B}$ for a.e. $(x, t) \in [0, 1]^2$.

3 Non- frame points of B_2

In this section we give an overview of some results regarding non frame points of B_2. For $N \geq 2$, we can see that $\mathcal{G}(B_N, a, b)$ is not a frame if $a > 2$ [24]. Consider $f = \chi_{[1, \frac{a}{2}]} \in L^2(\mathbb{R})$. Then $\langle f, E_{mb}T_{na}B_2 \rangle = \int_{-\infty}^{\infty} f(x)\overline{E_{mb}T_{na}\,B_2(x)}dx = \int_1^{\frac{a}{2}} e^{-2\pi imbx}B_2(x-na)dx = 0$. Thus the lower frame bound in the frame inequality will be violated and $\mathcal{G}(B_N, a, b)$ with $a > 2$ cannot be a frame. In [7] it is shown that $\mathcal{G}(B_2, a, b)$ is not a frame when $b = 2, 3, 4 \ldots$. To establish this result rigorously, we make use of the following general property that relates a Gabor system's frame bounds to the energy distribution of the generator and its Fourier transform.

Proposition 3.1 (Proposition 9.1.2, [14]). *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Suppose that $\mathcal{G}(g, a, b)$ is a frame with frame bounds A and B, then*

$$bA \leq \sum_{n\in\mathbb{Z}} |g(x - na)|^2 \leq bB, \ \ a.e \ \ x \in \mathbb{R}$$

and

$$aA \leq \sum_{n\in\mathbb{Z}} |\hat{g}(\gamma - nb)|^2 \leq aB, \ \ a.e \ \ x \in \mathbb{R}$$

Now, for $b \in \mathbb{N} \setminus \{1\}$, and $n \in \mathbb{Z}$, $\hat{B}_2(1 - nb) = \left(\frac{\sin(\pi(1-nb))}{\pi(1-nb)}\right)^2 = 0$. So, the lower bound is not satisfied. Since no other non-frame $(a, b)-$ values for the B-spline of order $N \geq 2$ are known, Grochenig conjectured that [12]: $\mathcal{F}_{B_N} = \{(a, b) \in \mathbb{R}_+^2 / ab < 1, a < N, b \neq 2, 3, 4, ...\}$. But this conjecture disproved in [21] by showing the following counter example. Consider $ab = \frac{5}{6}$ and $b \in [\frac{7}{3}, \frac{8}{3}]$. Since $ab = \frac{5}{6}$, the Zibulski-Zeevi matrix is of size $\frac{5}{6}$. At $(x, t) = (0, 0)$,

$$\Psi^{B_2}(0,0) = \left[b^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} B_2(6an + al + \frac{k}{b}),\right]_{k=0,..,4;l=0,..,5}. \tag{3.1}$$

$ab = \frac{5}{6}$ and $b \in [\frac{7}{3}, \frac{8}{3}]$ can be equivalently expressed as $\frac{1}{b} = \frac{6a}{5}$ for $a \in [\frac{5}{16}, \frac{5}{14}]$. We interpret the entries in $\Psi^{B_2}(0,0)$ as $6a-$ periodizations of B_2 at sampling locations $al + \frac{k}{b}$, specifically, at $al + \frac{6}{5}ak \mod 6a$. The 2nd and 5th row of $\Psi^{B_2}(0,0)$ are sampled at locations

$$k = 1 : \frac{6}{5}a, \ \frac{11}{5}a, \ -\frac{14}{5}a, \ -\frac{9}{5}a, \ -\frac{4}{5}a, \ \frac{1}{5}a, \quad \mod 6a$$

and

$$k = 4 : -\frac{6}{5}a, \ -\frac{1}{5}a, \ \frac{4}{5}a, \ \frac{9}{5}a, \ \frac{14}{5}a, \ -\frac{11}{5}a, \quad \mod 6a$$

By definition of B_2, we directly see that

$$R_2 - R_5 = [0 \ -2a \ -2a \ 0 \ 2a \ 2a]$$

where R_i denotes the ith row of $\Psi^{B_2}(0,0)$. Similarly, we observe that,

$$R_3 - R_4 = [0 \ -a \ -a \ 0 \ a \ a]$$

for $a \in [\frac{5}{16}, \frac{5}{14}]$. This implies that $\Psi^{B_2}(0,0)$ does not have full rank, indicates that the smallest singular values is zero. Hence the lower frame bound of $\mathcal{G}(B_2, a, \frac{5}{6})$ is zero for $a \in [\frac{5}{16}, \frac{5}{14}]$. $\frac{7}{3} \leq b \leq \frac{8}{3}$ and $a = \frac{5}{6b}$. At the point $a = \frac{1}{3}$ and $b = \frac{5}{2}$, we observe that the Zibulski-Zeevi

matrix does not have full rank, leading to a violation of the lower frame bound in the equation 2.9. Since $ab = \frac{5}{6}$, the Zibulski-Zeevi type matrix $\Psi^{B_2}(x,t)$ is of size 5×6. At $(x,t) = (0,0)$. It is noted that the lower frame condition must be satisfied for almost every point $(x,t) \in [0,1]^2$. Given that the Zak transform of B_2 is continuous, each entry of the Zibulski-Zeevi matrix will also be continuous. Consequently, the singular values of the matrix depend continuously on these entries. Hence, if we prove that the rank is not complete in one point, then $\sigma_p = 0$, and will violate the lower frame bound. Here the Zibulski-Zeevi matrix is

$$\Phi^{B_2}(0,0) = 5^{-\frac{1}{2}} \left((Z_{\frac{2}{5}} B_2)(-l\frac{5}{6}, \frac{k}{5}) \right)_{k=0,..,4;l=0,..,5}$$

The reduced row echelon form of $\Phi^{B_2}(0,0)$ is
$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

We observe that the Zibulski-Zeevi matrix does not have full rank at the point $(x,t) = (0,0)$. Therefore, the lower frame bound is violated, indicating that the Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{5}{2})$ is not a frame. These computations suggest that the frame property of $\mathcal{G}(B_2, a, b)$ fails for certain rational values of the lattice parameters. Motivated by such observations, the following conjecture was proposed in [21].

Conjecture 3.2. [21] The Gabor system $\mathcal{G}(g, a, b)$ is not a frame for

$$a_0 = \frac{1}{2m+1}, \ b_0 = \frac{2k+1}{2}, \ k, m \in \mathbb{N}, \ k > m, \ a_0 b_0 < 1.$$

Furthermore, the Gabor system $\mathcal{G}(B_2, a, b)$ is not a frame along the hyperbolas

$$ab = \frac{2k+1}{2(2m+1)}, \ \text{for } b \in \left[b_0 - a_0\frac{k-m}{2}, b_0 + a_0\frac{k-m}{2} \right]$$

The first part of the conjecture is later proved by Riya Ghosh and Antony Selvan [10]. This result provides a deeper understanding of the structure of non-frame regions for B_2. It highlights that the failure of the frame property is not arbitrary but follows specific geometric patterns in the (a, b)-plane, particularly along certain hyperbolic curves. These insights have guided subsequent investigations into identifying new non-frame points and determining the exact boundary of the frame set for B_2.

4 New non-frame points

Building upon the non-frame points identified in [21], we employ a similar analytical approach to investigate additional non-frame points for B_2. The points presented in this section lie outside the set described in the previous conjecture, thereby extending the known regions where the Gabor system generated by B_2 fails to form a frame.

Theorem 4.1. *The Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{8}{3})$ is not a frame.*

Proof. We will show that the Zibulski-Zeevi matrix does not have full rank and thus the lower frame bound in equation 2.9 is violated. It is stated that the lower frame condition needs to hold for points $(x,t) \in [0,1]^2$ a.e. Since $B_2 \in W(\mathbb{R}) \cup C^0(\mathbb{R})$, each entry of the Zibulski-Zeevi matrix will be continuous and the singular values of the matrix depend continuously on the entries. Hence, if we prove that the rank is not full in at least one point, then we can conclude that the lower frame bound will be zero. Now, consider the Zibulski-Zeevi matrix for the Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{8}{3})$ at the point $(0,0)$. Since $ab = \frac{1}{3} \cdot \frac{8}{3} = \frac{8}{9}$ we get

$$\Psi^{B_2}(0,0) = \left[\left(\frac{8}{3} \right)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} B_2 \left(9an + al + \frac{9a}{8}k \right) \right]_{k=0,\cdots,7;l=0,\cdots8} . \tag{4.1}$$

Using Gauss-Jordan elimination, the reduced row echelon form of $\Phi^{B_2}(0,0)$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the Zibulski-Zeevi matrix does not have full rank at the point $(0,0)$. Hence by equation 2.9, the lower frame bound will be zero and the Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{8}{3})$ is not a frame. $\qquad\square$

This result shows that the Gabor system with B_2 loses its frame property at certain specific values of (a,b) and not by chance. The point $(a,b) = \left(\frac{1}{3}, \frac{8}{3}\right)$ is an example of a new non-frame point that does not belong to the earlier conjectured set. Finding such points helps us better understand where the frame property fails and contributes to a more precise characterization of the frame set for B_2. In figure 4(a) we have plotted the value of the smallest singular value for $b = \frac{8}{3}$ and $0 < a < \frac{1}{b}$.

(a) $b = \frac{8}{3}$

(b) $b = \frac{11}{4}$

(c) $b = \frac{14}{5}$
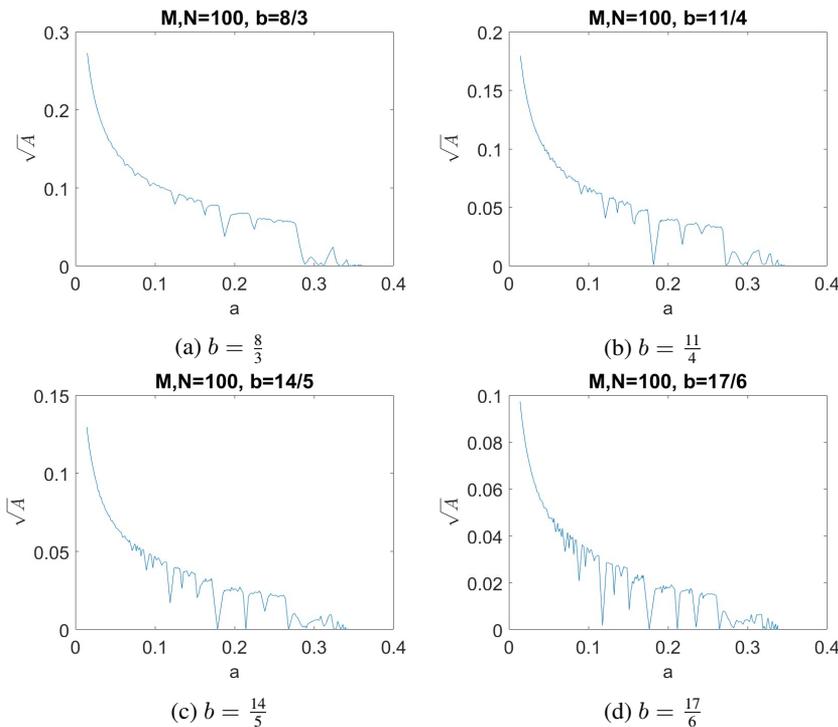
(d) $b = \frac{17}{6}$

Figure 4: Plots of the value of $\sqrt{A}$ based on the Zibulski-Zeevi matrix for $b = \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, \frac{17}{6}$

We have fixed the (x,t) grid sizes as 100×100. The smallest singular value of the Φ^{B_2} at each grid point is extracted, and the global minimum for each rational value of ab is stored. The relationship between these rational numbers and the lower bounds of the singular values is plotted, indicating how the smallest singular value changes with different values. As we approach the point $a = \frac{1}{3}$, we note that the minimum singular value σ_p on the (x,t) grid declines in a smooth manner and approaches zero. Similar plots are drawn for $b = \frac{11}{4}$ and $b = \frac{14}{5}$ in figure 4(b), (c) and (d) respectively. In each case we see that $\sqrt{A}$ goes to zero as $a = \frac{1}{3}$. Numerical computations show that $\mathcal{G}(B_2, a, b)$ is not a frame for $a = \frac{1}{k}$ and $b = k - \frac{1}{n+1}$, $k = 3, 4, \ldots$ and $n = 1, 2, \ldots$. We can not say that the Gabor system $\mathcal{G}(B_2, a, b)$ is not a frame along the corresponding hyperbolas, in contrast to the case in Conjecture 3.2. In figure 5 it seems like $\sqrt{A}$ goes to zero for

$b \in [2.64, 2.7]$. To check the exact situation, consider the Zibulski-Zeevi matrix in 4.1,

$$\Psi^{B_2}(0,0) = \begin{bmatrix} B_2(0) & B_2(a) & B_2(2a) & B_2(3a) & B_2(4a) & B_2(-4a) & B_2(3a) & B_2(2a) & B_2(a) \\ B_2(\frac{9a}{8}) & B_2(\frac{17a}{8}) & B_2(\frac{25a}{8}) & B_2(\frac{33a}{8}) & B_2(\frac{-31a}{8}) & B_2(\frac{-23a}{8}) & B_2(\frac{-15a}{8}) & B_2(\frac{-7a}{8}) & B_2(\frac{a}{8}) \\ B_2(\frac{9a}{4}) & B_2(\frac{13a}{4}) & B_2(\frac{17a}{4}) & B_2(\frac{-15a}{4}) & B_2(\frac{-11a}{4}) & B_2(\frac{-7a}{4}) & B_2(\frac{-3a}{4}) & B_2(\frac{a}{4}) & B_2(\frac{5a}{4}) \\ B_2(\frac{27a}{8}) & B_2(\frac{35a}{8}) & B_2(\frac{-29a}{8}) & B_2(\frac{-21a}{8}) & B_2(\frac{-13a}{8}) & B_2(\frac{-5a}{8}) & B_2(\frac{3a}{8}) & B_2(\frac{11a}{8}) & B_2(\frac{19a}{8}) \\ B_2(\frac{9a}{2}) & B_2(\frac{-7a}{2}) & B_2(\frac{-5a}{2}) & B_2(\frac{-3a}{2}) & B_2(\frac{-a}{2}) & B_2(\frac{a}{2}) & B_2(\frac{3a}{2}) & B_2(\frac{5a}{2}) & B_2(\frac{7a}{2}) \\ B_2(\frac{-27a}{8}) & B_2(\frac{-19a}{8}) & B_2(\frac{-11a}{8}) & B_2(\frac{-3a}{8}) & B_2(\frac{5a}{8}) & B_2(\frac{13a}{8}) & B_2(\frac{21a}{8}) & B_2(\frac{29a}{8}) & B_2(\frac{-35a}{8}) \\ B_2(\frac{-9a}{4}) & B_2(\frac{-5a}{4}) & B_2(\frac{-a}{4}) & B_2(\frac{3a}{4}) & B_2(\frac{7a}{4}) & B_2(\frac{11a}{4}) & B_2(\frac{15a}{4}) & B_2(\frac{19a}{4}) & B_2(\frac{23a}{4}) \\ B_2(\frac{-9a}{8}) & B_2(\frac{-a}{8}) & B_2(\frac{7a}{8}) & B_2(\frac{15a}{8}) & B_2(\frac{23a}{8}) & B_2(\frac{31a}{8}) & B_2(\frac{-33a}{8}) & B_2(\frac{-25a}{8}) & B_2(\frac{-17a}{8}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-a & 1-2a & 1-3a & 0 & 0 & 1-3a & 1-2a & 1-a \\ 1-\frac{9a}{8} & 1-\frac{17a}{8} & 0 & 0 & 0 & 1-\frac{23a}{8} & 1-\frac{15a}{8} & 1-\frac{7a}{8} & 1-\frac{a}{8} \\ 1-\frac{9a}{4} & 0 & 0 & 0 & 1-\frac{11a}{4} & 1-\frac{7a}{4} & 1-\frac{3a}{4} & 1-\frac{a}{4} & 1-\frac{5a}{4} \\ 0 & 0 & 0 & 1-\frac{21a}{8} & 1-\frac{13a}{8} & 1-\frac{5a}{8} & 1-\frac{3a}{8} & 1-\frac{11a}{8} & 1-\frac{19a}{8} \\ 0 & 0 & 1-\frac{5a}{2} & 1-\frac{3a}{2} & 1-\frac{a}{2} & 1-\frac{a}{2} & 1-\frac{3a}{2} & 1-\frac{5a}{2} & 0 \\ 0 & 1-\frac{19a}{8} & 1-\frac{11a}{8} & 1-\frac{3a}{8} & 1-\frac{5a}{8} & 1-\frac{13a}{8} & 1-\frac{21a}{8} & 0 & 0 \\ 1-\frac{9a}{4} & 1-\frac{5a}{4} & 1-\frac{a}{4} & 1-\frac{3a}{4} & 1-\frac{7a}{4} & 1-\frac{11a}{4} & 0 & 0 & 0 \\ 1-\frac{9a}{8} & 1-\frac{a}{8} & 1-\frac{7a}{8} & 1-\frac{15a}{8} & 1-\frac{23a}{8} & 0 & 0 & 0 & 1-\frac{17a}{8} \end{bmatrix}$$

The rank of this matrix was found to be 8. Consequently, this result does not allow us to conclude that $\mathcal{G}(B_2, a, b)$ fails to form a frame for $b \in [2.64, 2.7]$ and $ab = \frac{8}{9}$. We attempted the same by decreasing the length of $[2.64, 2.7]$. We plot the same for $ab = \frac{14}{15}$ in Figure 5(b) and do the similar computations, but no conclusion can be made from this. The same situation occurs when ab is $\frac{n}{n+1}$ with even n. Odd n leads to the parameter values mentioned in Conjecture 3.2.
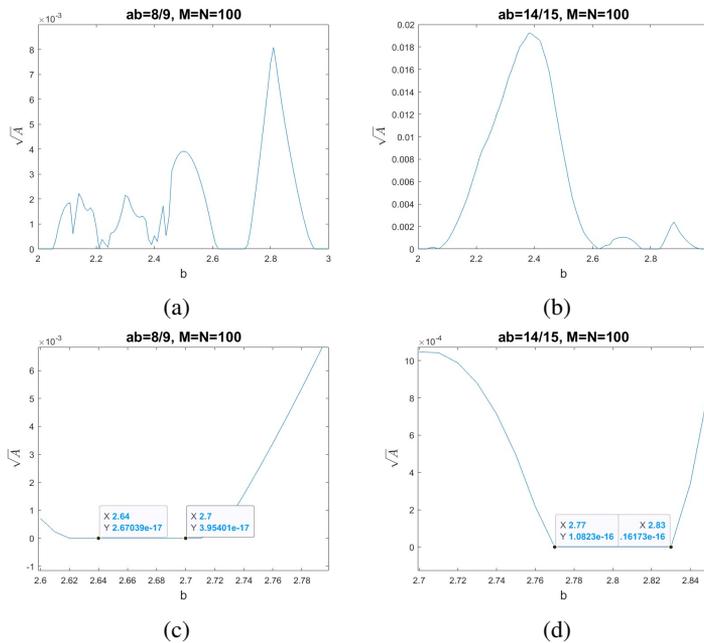

Figure 5: Plots of the value of $\sqrt{A}$ for $ab = \frac{8}{9}, \frac{14}{15}$

Figure 6 shows the known frame set for B_2 including non-frame areas. The new non-frame points are also included.

5 Conclusion

We present a comprehensive study on the construction of dual frames for the second-order B-spline B_2 corresponding to various values of the time and frequency parameters a and b, which naturally leads to an exploration of its frame set. In this work, both known frame and non-
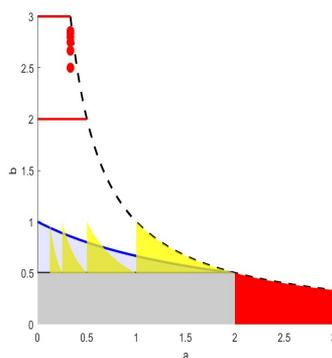
Figure 6: The red region denotes the non-frame region in which the red dots show the newly found points. All other colours indicate the frame property. The grey region is from [9], and the yellow region is the result from [4]. The blue region is from [21].

frame points are examined, and new non-frame points are identified through the analysis of the Zak transform and the singular value behavior of the Zibulski-Zeevi matrix. In particular, we establish that $(a, b) = \left(\frac{8}{3}, \frac{1}{3}\right)$ is a new non-frame point that does not belong to the set of non-frame points predicted by Grochenig's conjecture [21]. Furthermore, our computations indicate that $\mathcal{G}(B_2, a, b)$ fails to form a frame for parameters of the form $a = \frac{1}{k}$ and $b = k - \frac{1}{n+1}$, $k = 3, 4, \ldots$ and $n = 1, 2, \ldots$. However, numerical evidence suggests that, unlike the prediction in Grochenig's conjecture [21], the system $\mathcal{G}(B_2, a, b)$ cannot be conclusively said to be non-frame along the hyperbola $ab = \frac{n}{n+1}$ for even values of n. These observations motivate further investigation into the regions illustrated in Figure 5(c) and (d), to determine whether they belong to the frame set $\mathcal{F}(B_2)$ or not.

Acknowledgements

Statements and Declarations

Competing Interests: All authors declare that they have no conflict of interest to disclose.
Data Availability: Data sharing is not applicable to this article as no new data were created or analyzed in this study.
Author Contributions: Both authors contributed equally to this work.

References

[1] Aldroubi, A., Gröchenig, K.: "Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces." Journal of Fourier Analysis and Applications 6 (2000): 93-103.

[2] Atindehou, A. G. D., et al.: "On the frame set of the second-order B-spline." Applied and Computational Harmonic Analysis 62 (2023): 237-250.

[3] Christensen, O.: An introduction to frames and Riesz bases. Vol. 7. Boston: Birkhäuser, 2003.

[4] Christensen, O., Kim, H.O., Kim, R.Y.: "On Gabor frames generated by sign-changing windows and B-splines." Applied and Computational Harmonic Analysis 39.3 (2015): 534-544.

[5] Christensen, O., Kim, H. O., Kim, R. Y.: "On the Gabor frame set for compactly supported continuous functions." Journal of Inequalities and Applications 2016.1 (2016): 1-17.

[6] Dai, Xin-Rong, Sun, Q.: The abc-problem for Gabor systems. Vol. 244. No. 1152. American Mathematical Society, (2016).

[7] Del Prete, V.: "Estimates, decay properties, and computation of the dual function for Gabor frames." Journal of Fourier Analysis and Applications 5 (1999): 545-562.

[8] Del Prete, V.: "On a necessary condition for B-spline Gabor frames." Ricerche di matematica 59 (2010): 161-164.

[9] Daubechies, I., Grossmann, A., Meyer, Y.: "Painless nonorthogonal expansions." Journal of Mathematical Physics 27.5 (1986): 1271-1283.

[10] Ghosh, R., Selvan, A.A.: "Obstructions for Gabor frames of the second order B-spline." arXiv preprint arXiv:2310.01141 (2023).

[11] Gröchenig, K.: Foundations of time-frequency analysis. Springer Science and Business Media, (2013).

[12] Gröchenig, K.: "The mystery of Gabor frames." Journal of Fourier Analysis and Applications 20.4 (2014): 865-895.

[13] Gröchenig, K.: "Partitions of unity and new obstructions for Gabor frames." arXiv preprint arXiv:1507.08432 (2015). interpolation in the Bargmann-Fock space I." (1992): 91-106.

[14] Gröchenig, K., Stöckler, J.: "Gabor frames and totally positive functions." (2013): 1003-1031.

[15] Grochenig, K., et al.: "Note on B-splines, wavelet scaling functions, and Gabor frames." IEEE transactions on information theory 49.12 (2003): 3318-3320.

[16] Janssen, A. J.: "Some Weyl-Heisenberg frame bound calculations." Indagationes Mathematicae 7.2 (1996): 165-183.

[17] Janssen, A.J.: "On generating tight Gabor frames at critical density." Journal of Fourier Analysis and Applications 9 (2003): 175-214.

[18] Janssen, A. J.: "Zak transforms with few zeros and the tie." Advances in Gabor analysis (2003): 31-70.

[19] Janssen, A.J., Strohmer.T.: "Hyperbolic secants yield Gabor frames." Applied and Computational Harmonic Analysis 12.2 (2002): 259-267.

[20] Kloos, Tobias, Stöckler, J.: "Zak transforms and Gabor frames of totally positive functions and exponential B-splines." Journal of approximation theory 184 (2014): 209-237.

[21] Lemvig, J., Nielsen, K. H.: "Counterexamples to the B-spline conjecture for Gabor frames." Journal of Fourier Analysis and Applications 22.6 (2016): 1440-1451.

[22] Lemvig, J., Nielsen, K.H.: "Gabor windows supported on [- 1, 1] and construction of compactly supported dual windows with optimal smoothness." Journal of Approximation Theory 249 (2020): 105304.

[23] Lyubarskii, Y. I.: "Frames in the Bargmann space of entire functions." Adv. Soviet Math 429 (1992): 107-113.

[24] Nielsen, K.H.: "The Frame Set of Gabor Systems with B-spline Generators." (2015).

[25] Okoudjou, K.A.; "An invitation to Gabor analysis." Notices of the American Mathematical Society 66.6 (2019).

[26] Seip, K.: "Density theorems for sampling and

[27] Zibulski, M., Zeevi, Y.Y.: "Analysis of multiwindow Gabor-type schemes by frame methods." Applied and Computational Harmonic Analysis 4.2 (1997): 188-221.

Author information

Sruthi Raghoothaman, Department of Mathematics, Cochin University of Science and Technology, Kochi, 682022, Kerala, India.
E-mail: sruthir@cusat.ac.in

Noufal Asharaf, Department of Mathematics, Cochin University of Science and Technology, Kochi, 682022, Kerala, India.
E-mail: noufal@cusat.ac.in