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The Horizontally Deformed Sasaki Metric and Its Geometric Properties

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Abstract. This paper presents the horizontally deformed Sasaki metric on the tangent bundle of a Riemannian manifold, extending the family of invariant metrics in this setting. We first examine the geometric properties of the tangent bundle endowed with this metric, including its Levi-Civita connection and various curvature characteristics, such as the curvature tensors, Ricci curvature, sectional curvature, and scalar curvature. Additionally, we investigate a hypersurface within the tangent bundle, analyzing its geometric structure along with the associated Levi-Civita connection and curvature tensors in full detail.

1 Introduction

The study of Riemannian metrics on tangent bundles has long been a central topic in differential geometry, with the classical Sasaki metric serving as a foundational example [11]. While this metric has been extensively studied (see [3, 4, 9, 13, 16]), recent developments have explored deformations that introduce richer geometric structures while preserving key invariance properties (see [5, 7, 10, 14, 15, 17, 19]), for some related study see [6, 8, 18]. In addition to these deformations, we introduce the horizontally deformed Sasaki metric as a deformation (in the horizontal bundle), offering new avenues for investigating the interplay between the base manifold and its tangent bundle.

In this paper, we introduce and systematically analyze the horizontally deformed Sasaki metric, expanding the class of invariant metrics on tangent bundles of Riemannian manifolds. Our work is motivated by the need to understand how such deformations influence curvature properties, connections, and induced geometric structures-questions with potential implications for both theoretical and applied differential geometry.

We begin by deriving the Levi-Civita connection for the tangent bundle equipped with this metric, followed by a comprehensive study of its curvature properties , including the Riemannian curvature tensor, Ricci curvature, sectional curvature, and scalar curvature. These results reveal how horizontal deformations alter the geometry of the tangent bundle compared to the classical Sasaki case.

Furthermore, we investigate the geometry of hypersurfaces embedded in the tangent bundle, characterizing their Levi-Civita connections. This analysis not only extends known results but also provides a framework for future studies of submanifolds in deformed metric settings.

2 Preliminary results

Given a Riemannian manifold (M^m, g), its tangent bundle TM and the natural projection $\pi : TM \to M$ of TM onto M. A local chart $(U, x^h)_{h=1,\dots,m}$ on M induces on TM a local chart $(\pi^{-1}(U), x^h, x^{\bar{h}} = v^h)_{h=1,\dots,m}$, where $\bar{h} = 1 + m, \dots, 2m$ and (v^h) is the Cartesian coordinates in each tangent space T_xM at $x \in M$ with respect to the natural base $\{\frac{\partial}{\partial x^h}|_x\}$, x being an arbitrary point in U whose coordinates are (x^h). Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im_0^1(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,v)}TM = V_{(x,v)}TM \oplus H_{(x,v)}TM, \tag{2.1}$$

of the tangent bundle to TM at all $(x,v) \in TM$ into the vertical subspace

$$V_{(x,v)}TM = \ker(d\pi_{(x,v)}) = \{\alpha^i \partial_{\bar{i}}|_{(x,v)},\ \alpha^i \in \mathbb{R}\},$$

and the horizontal subspace

$$H_{(x,v)}TM = \{\alpha^i \partial_i|_{(x,v)} - \alpha^i v^j \Gamma_{ij}^k \partial_{\bar{k}}|_{(x,v)},\ \alpha^i \in \mathbb{R}\},$$

for all $(x,v) \in TM$, where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} = \dfrac{\partial}{\partial x^{\bar{i}}}$. The operators V and H denote the vertical and horizontal projections on TM induced by the connection ∇ see [3].

Let $Z = Z^i \partial_i$ be a vector field on M. The vertical and horizontal lifts of Z are defined by

$$\begin{aligned} {}^VZ &= Z^i \partial_{\bar{i}}, \\ {}^HZ &= Z^i(\partial_i - v^j \Gamma_{ij}^k \partial_{\bar{k}}). \end{aligned}$$

We have ${}^H\partial_i = \partial_i - v^j \Gamma_{ij}^k \partial_{\bar{k}}$ and ${}^V\partial_i = \partial_{\bar{i}}$, then $({}^H\partial_i, {}^V\partial_i)_{i=1,\dots,m}$ is a local adapted frame on TTM.

The vertical distribution ${}^Vv$ on TM is defined by

$${}^Vv = v^{i\,V}\partial_i = v^i \partial_{\bar{i}},$$

which is also known as the canonical or Liouville vector field on TM see [12].

Let X, Y and Z be any vector fields on M, then we have

$$\begin{cases} {}^HZ(g(X,v)) = g(\nabla_Z X, v), \\ {}^VZ(g(X,v)) = g(X,Z), \\ {}^HZ(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \\ {}^VZ(g(X,Y)) = 0, \end{cases} \tag{2.2}$$

for any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\begin{cases} {}^HZ(f(r)) = 0, \\ {}^VZ(f(r)) = 2f'(r)g(Z,v), \end{cases} \tag{2.3}$$

where $r = g(v,v)$ [1, 12].

The Lie bracket relations between horizontal and vertical vector fields are given by (see [3, 13]):

$$\begin{cases} [{}^HZ, {}^HY] = {}^H[Z,Y] - {}^V(\mathrm{R}(Z,Y)v), \\ [{}^HZ, {}^VY] = {}^V(\nabla_Z Y), \\ [{}^VZ, {}^VY] = 0, \end{cases} \tag{2.4}$$

for all vector fields Y and Z on M.

3 Horizontally deformed Sasaki metric

Definition 3.1. Given a Riemannian manifold (M^m, g) and TM be its tangent bundle. We define the horizontally deformed Sasaki metric g^f on the tangent bundle TM by

$$\begin{aligned} g^f({}^HX, {}^HY) &= f(r)g(X,Y), \\ g^f({}^VX, {}^HY) &= g^f({}^HX, {}^VY) = 0, \\ g^f({}^VX, {}^VY) &= g(X,Y), \end{aligned}$$

for all vector fields X and Y on M, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth positive function satisfying $f(r) \neq 0$ and $r = g(v, v)$.

In the following (TM, g^f) denotes the tangent bundle of (M^m, g) equipped with the horizontally deformed Sasaki metric g^f

Lemma 3.2. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f), we have the following:*

$$\begin{aligned}(1)\ {}^HZ\big(g^f({}^HX, {}^HY)\big) &= g^f({}^H(\nabla_Z X), {}^HY) + g^f({}^HX, {}^H(\nabla_Z Y)),\\ (2)\ {}^VZ\big(g^f({}^HX, {}^HY)\big) &= 2f'(r)g(Z, v)g(X, Y),\end{aligned}$$

for all vector fields X, Y and Z on M.

Let us derive the Levi-Civita connection ∇^f of the tangent bundle (TM, g^f). This connection is defined by the Koszul formula, which expresses the metric compatibility of the connection:

$$\begin{aligned}2g^f(\nabla^f_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}g^f(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}g^f(\widetilde{Z}, \widetilde{X}) - \widetilde{Z}g^f(\widetilde{X}, \widetilde{Y})\\ &\quad + g^f(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) + g^f(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) - g^f(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]), \qquad (3.1)\end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ on TM.

Theorem 3.3. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). The following results hold:*

$$\begin{aligned}1.\ \nabla^f_{{}^HX}{}^HY &= {}^H(\nabla_X Y) - f'(r)g(X, Y)^Vv - \frac{1}{2}{}^V(\mathrm{R}(X, Y)v),\\ 2.\ \nabla^f_{{}^HX}{}^VY &= {}^V(\nabla_X Y) + \frac{f'(r)}{f(r)}g(Y, v)^HX + \frac{1}{2f(r)}{}^H(\mathrm{R}(v, Y)X),\\ 3.\ \nabla^f_{{}^VX}{}^HY &= \frac{f'(r)}{f(r)}g(X, v)^HY + \frac{1}{2f(r)}{}^H(\mathrm{R}(v, X)Y),\\ 4.\ \nabla^f_{{}^VX}{}^VY &= 0,\end{aligned}$$

for any vector fields X and Y on M, where ∇ is the Levi-Civita connection of (M^m, g) and R *is its curvature tensor.*

Proof. In the proof, we make use of formulas (2.4), the Koszul formula (3.1), and Lemma 3.2.
1. By performing direct calculations, we obtain:

$$\begin{aligned}2g^f(\nabla^f_{{}^HX}{}^HY, {}^HZ) &= {}^HX\big(g^f({}^HY, {}^HZ)\big) + {}^HY\big(g^f({}^HZ, {}^HX)\big) - {}^HZ\big(g^f({}^HX, {}^HY)\big)\\ &\quad + g^f({}^HZ, [{}^HX, {}^HY]) + g^f({}^HY, [{}^HZ, {}^HX]) - g^f({}^HX, [{}^HY, {}^HZ])\\ &= g^f({}^H(\nabla_X Y), {}^HZ) + g^f({}^HX, {}^H(\nabla_X Z)) + g^f({}^H(\nabla_Y Z), {}^HX)\\ &\quad + g^f({}^HZ, {}^H(\nabla_Y X)) - g^f({}^H(\nabla_Z X), {}^HY) - g^f({}^HX, {}^H(\nabla_Z Y))\\ &\quad + g^f({}^HZ, {}^H[X, Y]) + g^f({}^HY, {}^H[Z, X]) - g^f({}^HX, {}^H[Y, Z])\\ &= 2g^f({}^H(\nabla_X Y), {}^HZ),\end{aligned}$$

and

$$\begin{aligned}2g^f(\nabla^f_{{}^HX}{}^HY, {}^VZ) &= {}^HX\big(g^f({}^HY, {}^VZ)\big) + {}^HY\big(g^f({}^VZ, {}^HX)\big) - {}^VZ\big(g^f({}^HX, {}^HY)\big)\\ &\quad + g^f({}^VZ, [{}^HX, {}^HY]) + g^f({}^HY, [{}^VZ, {}^HX]) - g^f({}^HX, [{}^HY, {}^VZ])\\ &= -2f'(r)g(Z, v)g(X, Y) - g^f({}^VZ, {}^V(\mathrm{R}(X, Y)v))\\ &= -g^f\big(2f'(r)g(X, Y)^Vv + {}^V(\mathrm{R}(X, Y)v), {}^VZ\big),\end{aligned}$$

from this, we derive

$$\nabla^f_{{}^HX}{}^HY = {}^H(\nabla_X Y) - f'(r)g(X, Y)^Vv - \frac{1}{2}{}^V(\mathrm{R}(X, Y)v).$$

2. Similar calculations as those provided above lead to:

$$\begin{aligned}2g^f(\nabla^f_{^HX}{}^VY,{}^HZ) &= {}^HX\big(g^f({}^VY,{}^HZ)\big)+{}^VY\big(g^f({}^HZ,{}^HX)\big)-{}^HZ\big(g^f({}^HX,{}^VY)\big)\\ &\quad +g^f({}^HZ,[{}^HX,{}^VY])+g^f({}^VY,[{}^HZ,{}^HX])-g^f({}^HX,[{}^VY,{}^HZ])\\ &= 2f'(r)g(Y,v)g(X,Z)-g^f({}^VY,{}^V(\mathrm{R}(Z,X)v))\\ &= \frac{2f'(r)}{f(r)}g(Y,v)g^f({}^HX,{}^HZ)+\frac{1}{f(r)}g^f({}^H(\mathrm{R}(v,Y)X),{}^HZ)\\ &= \frac{1}{f(r)}g^f\big(2f'(r)g(Y,v){}^HX+{}^H(\mathrm{R}(v,Y)X),{}^HZ\big).\end{aligned}$$

Also it follows that

$$\begin{aligned}2g^f(\nabla^f_{^HX}{}^VY,{}^VZ) &= {}^HX\big(g^f({}^VY,{}^VZ)\big)+{}^VY\big(g^f({}^VZ,{}^HX)\big)-{}^VZ\big(g^f({}^HX,{}^VY)\big)\\ &\quad +g^f({}^VZ,[{}^HX,{}^VY])+g^f({}^VY,[{}^VZ,{}^HX])-g^f({}^HX,[{}^VY,{}^VZ])\\ &= g^f({}^V(\nabla_XY),{}^VZ)+g^f({}^VY,{}^V(\nabla_XZ))+g^f({}^VZ,{}^V(\nabla_XY))\\ &\quad -g^f({}^VY,{}^V(\nabla_XZ))\\ &= 2g^f({}^V(\nabla_XY),{}^VZ).\end{aligned}$$

So, we see that

$$\nabla^f_{^HX}{}^VY={}^V(\nabla_XY)+\frac{f'(r)}{f(r)}g(Y,v){}^HX+\frac{1}{2f(r)}{}^H(\mathrm{R}(v,Y)X).$$

The remaining formulas can be derived through similar calculations. □

From Theorem 3.3, we obtain

Lemma 3.4. *Given a Riemannian manifold (M^m,g) and its tangent bundle (TM,g^f). The following results hold:*

$$\begin{aligned}(1)\ \nabla^f_{^HX}{}^Vv &= \frac{rf'(r)}{f(r)}{}^HX,\\ (2)\ \nabla^f_{^Vv}{}^HX &= \frac{rf'(r)}{f(r)}{}^HX,\\ (3)\ \nabla^f_{^VX}{}^Vv &= {}^VX,\\ (4)\ \nabla^f_{^Vv}{}^VX &= 0,\\ (5)\ \nabla^f_{^Vv}{}^Vv &= {}^Vv,\end{aligned}$$

for any vector field X on M.

Definition 3.5. Given a Riemannian manifold (M^m,g) and its tangent bundle (TM,g^f), let F be a $(1,1)$-tensor field on M. We define the horizontal and vertical vector fields ${}^HF$ and ${}^VF$ of F, on TM as follows:

$$\begin{aligned}{}^HF:TM &\to TTM\\ (x,v) &\mapsto {}^HF(x,v)={}^H(F_x(v)),\end{aligned}$$

$$\begin{aligned}{}^VF:TM &\to TTM\\ (x,v) &\mapsto {}^VF(x,v)={}^V(F_x(v)),\end{aligned}$$

locally we have

$$ {}^H(F(v)) = v^i\,{}^H(F(\partial_i)), \tag{3.2}$$

$$ {}^V(F(v)) = v^i\,{}^V(F(\partial_i)). \tag{3.3}$$

Proposition 3.6. *Given a Riemannian manifold (M^m, g), its tangent bundle (TM, g^f) and a $(1,1)$-tensor field F on M, the following holds:*

$$\begin{aligned}
i)\ \nabla^f_{{}^HX}{}^H(F(v)) &= {}^H((\nabla_X F)(v)) - f'(r)g(X, F(v))^V v - \frac{1}{2}{}^V(\mathrm{R}(X, F(v))v),\\
ii)\ \nabla^f_{{}^VX}{}^H(F(v)) &= {}^H(F(X)) + \frac{f'(r)}{f(r)}g(X, v)^H(F(v)) + \frac{1}{2f(r)}{}^H(\mathrm{R}(v, X)F(v)),\\
iii)\ \nabla^f_{{}^HX}{}^V(F(v)) &= {}^V((\nabla_X F)(v)) + \frac{f'(r)}{f(r)}g(F(v), v)^H X + \frac{1}{2f(r)}{}^H(\mathrm{R}(v, F(v))X),\\
iv)\ \nabla^f_{{}^VX}{}^V(F(v)) &= {}^V(F(X)),
\end{aligned}$$

for any vector field X on M.

Proof. The proof of Proposition 3.6 follows directly from Theorem 3.3, formulas (3.2) and (3.3). □

4 The Riemannian curvature tensors of the horizontally deformed Sasaki metric

We will compute the Riemannian curvature tensor R^f of (TM, g^f).

Theorem 4.1. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). The following formulas hold:*

$$\begin{aligned}
\mathrm{R}^f({}^HX, {}^HY)^H Z &= {}^H(\mathrm{R}(X,Y)Z) + \frac{1}{2f(r)}{}^H(\mathrm{R}(v, \mathrm{R}(X,Y)v)Z)\\
&\quad + \frac{1}{4f(r)}{}^H\big(\mathrm{R}(v, \mathrm{R}(X,Z)v)Y - \mathrm{R}(v, \mathrm{R}(Y,Z)v)X\big)\\
&\quad + \frac{r(f'(r))^2}{f(r)}\big(g(X,Z)^H Y - g(Y,Z)^H X\big)\\
&\quad + \frac{1}{2}{}^V\big((\nabla_Z \mathrm{R})(X,Y)v\big),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\mathrm{R}^f({}^HX, {}^VY)^H Z &= \frac{1}{2f(r)}{}^H((\nabla_X \mathrm{R})(v,Y)Z) - \frac{1}{4f(r)}{}^V(\mathrm{R}(X, \mathrm{R}(v,Y)Z)v)\\
&\quad + \frac{1}{2}{}^V(\mathrm{R}(X,Z)Y) - \frac{f'(r)}{2f(r)}g(Y,v)^V(\mathrm{R}(X,Z)v)\\
&\quad + f'(r)g(X,Z)^V Y - \frac{f'(r)}{2f(r)}g(\mathrm{R}(v,Y)Z, X)^V v\\
&\quad + \frac{2f''(r)f(r) - (f'(r))^2}{f(r)}g(Y,v)g(X,Z)^V v,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\mathrm{R}^f({}^HX, {}^HY)^V Z &= \frac{1}{2f(r)}{}^H\big((\nabla_X \mathrm{R})(v,Z)Y - (\nabla_Y \mathrm{R})(v,Z)X\big) + {}^V(\mathrm{R}(X,Y)Z)\\
&\quad + \frac{1}{4f(r)}{}^V\big(\mathrm{R}(Y, \mathrm{R}(v,Z)X)v - \mathrm{R}(X, \mathrm{R}(v,Z)Y)v\big)\\
&\quad - \frac{f'(r)}{f(r)}g(Z,v)^V(\mathrm{R}(X,Y)v) + \frac{f'(r)}{f(r)}g(\mathrm{R}(X,Y)v, Z)^V v,
\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathrm{R}^f({}^HX,{}^VY)^VZ &= \Big(\frac{(f'(r))^2-2f''(r)f(r)}{(f(r))^2}g(Y,v)g(Z,v)-\frac{f'(r)}{f(r)}g(Y,Z)\Big)^HX\\ &+\frac{f'(r)}{2(f(r))^2}\big(g(Y,v)^H(\mathrm{R}(v,Z)X)-g(Z,v)^H(\mathrm{R}(v,Y)X)\big)\\ &-\frac{1}{2f(r)}{}^H(\mathrm{R}(Y,Z)X)-\frac{1}{4(f(r))^2}{}^H(\mathrm{R}(v,Y)\mathrm{R}(v,Z)X), \end{aligned}\tag{4.4}$$

$$\begin{aligned}\mathrm{R}^f({}^VX,{}^VY)^HZ &= \frac{1}{4(f(r))^2}{}^H\big(\mathrm{R}(v,X)\mathrm{R}(v,Y)Z-\mathrm{R}(v,Y)\mathrm{R}(v,X)Z\big)\\ &+\frac{f'(r)}{(f(r))^2}\big(g(Y,v)^H(\mathrm{R}(v,X)Z)-g(X,v)^H(\mathrm{R}(v,Y)Z)\big)\\ &+\frac{1}{f(r)}{}^H(\mathrm{R}(X,Y)Z), \end{aligned}\tag{4.5}$$

$$\mathrm{R}^f({}^VX,{}^VY)^VZ = 0, \tag{4.6}$$

for all vector fields X,Y and Z on M.

Proof. In the proof, we will use the formulas (2.2), (2.3), (2.4), Theorem 3.3, Lemma 3.4 and Proposition 3.6 we have:

(1) $\mathrm{R}^f({}^HX,{}^HY)^HZ=\nabla^f_{{}^HX}\nabla^f_{{}^HY}{}^HZ-\nabla^f_{{}^HY}\nabla^f_{{}^HX}{}^HZ-\nabla^f_{[{}^HX,{}^HY]}{}^HZ.$

Let $F:TM\to TM$ be bundle endomorphism given by $F(v)=\mathrm{R}(Y,Z)v$, then

$$\begin{aligned}\nabla^f_{{}^HX}\nabla^f_{{}^HY}{}^HZ &= \nabla^f_{{}^HX}\big({}^H(\nabla_YZ)-f'(r)g(Y,Z)^Vv-\frac{1}{2}{}^VF(v)\big)\\ &= {}^H(\nabla_X\nabla_YZ)-f'(r)g(X,\nabla_YZ)^Vv-\frac{1}{2}{}^V(\mathrm{R}(X,\nabla_YZ)v)\\ &\quad -f'(r)g(\nabla_XY,Z)^Vv-f'(r)g(Y,\nabla_XZ)^Vv-\frac{r(f'(r))^2}{f(r)}g(Y,Z)^HX\\ &\quad -\frac{1}{2}{}^V(\nabla_X(\mathrm{R}(Y,Z)v))+\frac{1}{2}{}^V(\mathrm{R}(Y,Z)\nabla_Xv)\\ &\quad -\frac{1}{4f(r)}{}^H(\mathrm{R}(v,\mathrm{R}(Y,Z)v)X). \end{aligned}\tag{4.7}$$

From which, with permutation of X by Y in the formula (4.7) we get

$$\begin{aligned}\nabla^f_{{}^HY}\nabla^f_{{}^HX}{}^HZ &= {}^H(\nabla_Y\nabla_XZ)-f'(r)g(Y,\nabla_XZ)^Vv-\frac{1}{2}{}^V(\mathrm{R}(Y,\nabla_XZ)v)\\ &\quad -f'(r)g(\nabla_YX,Z)^Vv-f'(r)g(X,\nabla_YZ)^Vv-\frac{r(f'(r))^2}{f(r)}g(X,Z)^HY\\ &\quad -\frac{1}{2}{}^V(\nabla_Y(\mathrm{R}(X,Z)v))+\frac{1}{2}{}^V(\mathrm{R}(X,Z)\nabla_Yv)\\ &\quad -\frac{1}{4f(r)}{}^H(\mathrm{R}(v,\mathrm{R}(X,Z)v)Y). \end{aligned}\tag{4.8}$$

Also, we find

$$\begin{aligned}\nabla^f_{[{}^HX,{}^HY]}{}^HZ &= \nabla^f_{{}^H[X,Y]}{}^HZ-\nabla^f_{{}^V(\mathrm{R}(X,Y)v)}{}^HZ\\ &= {}^H(\nabla_{[X,Y]}Z)-f'(r)g([X,Y],Z)^Vv-\frac{1}{2}{}^V(\mathrm{R}([X,Y],Z)v)\\ &\quad -\frac{1}{2f(r)}{}^H(\mathrm{R}(v,\mathrm{R}(X,Y)v)Z). \end{aligned}\tag{4.9}$$

From the formulas (4.7), (4.8) and (4.9) we get

$$\begin{aligned}\mathrm{R}^f({}^HX,{}^HY)^HZ &= {}^H(\mathrm{R}(X,Y)Z)+\frac{1}{2f(r)}{}^H(\mathrm{R}(v,\mathrm{R}(X,Y)v)Z)\\ &+\frac{1}{4f(r)}{}^H(\mathrm{R}(v,\mathrm{R}(X,Z)v)Y)-\frac{1}{4f(r)}{}^H(\mathrm{R}(v,\mathrm{R}(Y,Z)v)X)\\ &+\frac{r(f'(r))^2}{f(r)}g(X,Z)^HY-\frac{r(f'(r))^2}{f(r)}g(Y,Z)^HX\\ &-\frac{1}{2}{}^V\big((\nabla_X\mathrm{R})(Y,Z)v\big)+\frac{1}{2}{}^V\big((\nabla_Y\mathrm{R})(X,Z)v\big).\end{aligned}\tag{4.10}$$

Using the second Bianchi identity, we obtain the formula (4.1).

(2) $\mathrm{R}^f({}^HX,{}^VY)^HZ=\nabla^f_{^HX}\nabla^f_{^VY}{}^HZ-\nabla^f_{^VY}\nabla^f_{^HX}{}^HZ-\nabla^f_{[^HX,^VY]}{}^HZ.$

Let $F:TM\to TM$ be bundle endomorphism given by $F(v)=\mathrm{R}(v,Y)Z$, then

$$\begin{aligned}\nabla^f_{^HX}\nabla^f_{^VY}{}^HZ &= \nabla^f_{^HX}\Big(\frac{f'(r)}{f(r)}g(Y,v)^HZ+\frac{1}{2f(r)}{}^HF(v)\Big)\\ &= \frac{f'(r)}{f(r)}g(\nabla_XY,v)^HZ+\frac{f'(r)}{f(r)}g(Y,v)^H(\nabla_XZ)\\ &-\frac{(f'(r))^2}{f(r)}g(Y,v)g(X,Z)^Vv-\frac{f'(r)}{2f(r)}g(Y,v)^V(\mathrm{R}(X,Z)v)\\ &+\frac{1}{2f(r)}{}^H(\nabla_X(\mathrm{R}(v,Y)Z))-\frac{1}{2f(r)}{}^H(\mathrm{R}(\nabla_Xu,Y)Z)\\ &-\frac{f'(r)}{2f(r)}g(X,\mathrm{R}(v,Y)Z)^Vv-\frac{1}{4f(r)}{}^V(\mathrm{R}(X,\mathrm{R}(v,Y)Z)v).\end{aligned}\tag{4.11}$$

Let $F:TM\to TM$ be bundle endomorphism given by $F(v)=\mathrm{R}(X,Z)v$, then

$$\begin{aligned}\nabla^f_{^VY}\nabla^f_{^HX}{}^HZ &= \nabla^f_{^VY}\Big({}^H(\nabla_XZ)-f'(r)g(X,Z)^Vv-\frac{1}{2}{}^VF(v)\Big)\\ &= \frac{f'(r)}{f(r)}g(Y,v)^H(\nabla_XZ)+\frac{1}{2f(r)}{}^H(\mathrm{R}(v,Y)\nabla_XZ)\\ &-2f''(r)g(Y,v)g(X,Z)^Vv-f'(r)g(X,Z)^VY\\ &-\frac{1}{2}{}^V(\mathrm{R}(X,Z)Y).\end{aligned}\tag{4.12}$$

Direct calculations give

$$\begin{aligned}\nabla^f_{[^HX,^VY]}{}^HZ &= \nabla^f_{^V(\nabla_XY)}{}^HZ\\ &= \frac{f'(r)}{f(r)}g(\nabla_XY,v)^HZ+\frac{1}{2f(r)}{}^H(\mathrm{R}(v,\nabla_XY)Z).\end{aligned}\tag{4.13}$$

From the formulas (4.11), (4.12) and (4.13), we obtain the formula (4.2).

(3) Applying the formula (4.2) and the first Bianchi identity,

$$\mathrm{R}^f({}^HX,{}^HY)^VZ=\mathrm{R}^f({}^HX,{}^VZ)^HY-\mathrm{R}^f({}^HY,{}^VZ)^HX,$$

we find the formula (4.3).
The other formulas are obtained by a similar calculation. □

Proposition 4.2. *Given a Riemannian manifold (M^m,g) and its tangent bundle (TM,g^f). The following assertions hold:*

(i) If M is flat and $f = const$ then, TM is flat.
(ii) If TM is flat then, M is flat.

Proof.
(i) It is a direct consequence of Theorem 4.1 that if $\mathrm{R}^f = 0$.
(ii) We assume that $\mathrm{R}^f = 0$, we calculate the Riemann curvature tensor R^f at $(x, 0)$, from the formula (4.3), we obtain, $\mathrm{R} = 0$. □

Corollary 4.3. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). If $f \neq const$, then TM cannot be flat.*

Let $\{e_i\}_{i=1,\ldots,m}$ be a local orthonormal frame, then

$$\{\frac{1}{\sqrt{f(r)}}{}^He_i, {}^Ve_j\}_{i,j=1,\ldots,m} \tag{4.14}$$

is a local orthonormal frame on (TM, g^f).

Theorem 4.4. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). If* Ric *(resp.* Ric^f*) denote the Ricci curvature of (M^m, g) (resp. (T^*M, g^f)), then we have*

$$\begin{aligned} \mathrm{Ric}^f({}^HX, {}^HY) &= \mathrm{Ric}(X,Y) - \frac{1}{2f(r)}\sum_{i=1}^m g(\mathrm{R}(e_i,X)v, \mathrm{R}(e_i,Y)v) \\ &\quad - \frac{2rf''(r)f(r) + (m-2)r(f'(r))^2 + mf'(r)f(r)}{f(r)} g(X,Y), \end{aligned} \tag{4.15}$$

$$\mathrm{Ric}^f({}^HX, {}^VY) = \frac{1}{2f(r)}\sum_{i=1}^m g((\nabla_{e_i}\mathrm{R})(v,Y)X, e_i), \tag{4.16}$$

$$\begin{aligned} \mathrm{Ric}^f({}^VX, {}^VY) &= \frac{1}{4(f(r))^2}\sum_{i=1}^m g(\mathrm{R}(v,X)e_i, \mathrm{R}(v,Y)e_i) - \frac{mf'(r)}{f(r)} g(X,Y) \\ &\quad + \frac{m((f'(r))^2 - 2f''(r)f(r))}{(f(r))^2} g(X,v)g(Y,v), \end{aligned} \tag{4.17}$$

for all vector fields X and Y on M.

Proof. Using the local orthonormal frame defined by the formula (4.14) on TM. From the formulas (4.1) and (4.2), we have

$$\begin{aligned} \mathrm{Ric}^f({}^HX, {}^HY) &= \sum_{i=1}^m g^f(\mathrm{R}^f(\frac{1}{\sqrt{f(r)}}{}^He_i, {}^HX)^HY, \frac{1}{\sqrt{f(r)}}{}^He_i) \\ &\quad + \sum_{j=1}^m g^f(\mathrm{R}^f({}^Ve_j, {}^HX)^HY, {}^Ve_j) \\ &= \frac{1}{f(r)}\sum_{i=1}^m g^f(\mathrm{R}^f({}^He_i, {}^HX)^HY, {}^He_i) - \sum_{j=1}^m g^f(\mathrm{R}^f({}^HX, {}^Ve_j)^HY, {}^Ve_j) \\ &= \mathrm{Ric}(X,Y) - \frac{3}{4f(r)}\sum_{i=1}^m g(\mathrm{R}(e_i,X)v, \mathrm{R}(e_i,Y)v) \\ &\quad + \frac{1}{4f(r)}\sum_{j=1}^m g(\mathrm{R}(e_j,v)X, \mathrm{R}(e_j,v)Y) \\ &\quad - \frac{2rf''(r)f(r) + (m-2)r(f'(r))^2 + mf'(r)f(r)}{f(r)} g(X,Y). \end{aligned} \tag{4.18}$$

On the other hand, we have

$$\sum_{i=1}^{m} g(\mathrm{R}(e_i,X)v,\mathrm{R}(e_i,Y)v) \;=\; \sum_{j=1}^{m} g(\mathrm{R}(e_j,v)X,\mathrm{R}(e_j,v)Y). \tag{4.19}$$

Using the formulas (4.18) and (4.19) , we obtain the formula (4.15).
The other formulas (4.16) and (4.17) are obtained by a similar calculation. □

We will now compute the sectional curvature on tangent bundle (TM, g^f), we know that

$$\mathrm{K}^f(\widetilde{X},\widetilde{Y}) = \frac{g^f(\mathrm{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{X})}{g^f(\widetilde{X},\widetilde{X})g^f(\widetilde{Y},\widetilde{Y}) - g^f(\widetilde{X},\widetilde{Y})^2}, \tag{4.20}$$

is the sectional curvature on (TM, g^f) for the plane P spanned by $\{\widetilde{X},\widetilde{Y}\}$, where $\widetilde{X}$ and $\widetilde{Y}$ are linearly independent tangent vectors on TM.

Let $\mathrm{K}^f({}^HX,{}^HY)$, $\mathrm{K}^f({}^HX,{}^VY)$ and $\mathrm{K}^f({}^VX,{}^VY)$ denote the sectional curvature of the plane spanned by $\{{}^HX,{}^HY\}$, $\{{}^HX,{}^VY\}$ and $\{{}^VX,{}^VY\}$ on (TM, g^f) respectively, for all X,Y orthonormal vector fields on M.

Theorem 4.5. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). Then the sectional curvature K^f satisfies the following:*

$$\begin{aligned}
1.\ \mathrm{K}^f({}^HX,{}^HY) &= \frac{1}{f(r)}\mathrm{K}(X,Y) - \frac{3}{4(f(r))^2}|\mathrm{R}(X,Y)v|^2 - \frac{r(f'(r))^2}{(f(r))^2},\\
2.\ \mathrm{K}^f({}^HX,{}^VY) &= \frac{1}{4(f(r))^2}|\mathrm{R}(v,Y)X|^2 + \frac{(f'(r))^2 - 2f''(r)f(r)}{(f(r))^2}g(Y,v)^2 - \frac{f'(r)}{f(r)},\\
3.\ \mathrm{K}^f({}^VX,{}^VY) &= 0,
\end{aligned}$$

where K *denote the sectional curvature tensor of (M^m, g).*

Proof. Using Theorem 4.1, formula (4.20) and direct calculations, we have

$$\begin{aligned}
1.\ \mathrm{K}^f({}^HX,{}^HY) &= \frac{g^f(\mathrm{R}^f({}^HX,{}^HY){}^HY,{}^HX)}{g^f({}^HX,{}^HX)g^f({}^HY,{}^HY) - g^f({}^HX,{}^HY)^2}\\
&= \frac{1}{(f(r))^2}\Big(f(r)g(\mathrm{R}(X,Y)Y,X) + \frac{1}{2}g(\mathrm{R}(v,\mathrm{R}(X,Y)v)Y,X)\\
&\quad + \frac{1}{4}g(\mathrm{R}(v,\mathrm{R}(X,Y)v)Y,X) - r(f'(r))^2\Big)\\
&= \frac{1}{f(r)}\mathrm{K}(X,Y) - \frac{3}{4(f(r))^2}|\mathrm{R}(X,Y)v|^2 - \frac{r(f'(r))^2}{(f(r))^2}.
\end{aligned}$$

$$\begin{aligned}
2.\ \mathrm{K}^f({}^HX,{}^VY) &= \frac{g^f(\mathrm{R}^f({}^HX,{}^VY){}^VY,{}^HX)}{g^f({}^HX,{}^HX)g^f({}^VY,{}^VY) - g^f({}^HX,{}^VY)^2}\\
&= \frac{1}{f(r)}\Big(\frac{(f'(r))^2 - 2f''(r)f(r)}{f(r)}g(Y,v)^2 - f'(r)\\
&\quad + \frac{1}{4f(r)}|\mathrm{R}(v,Y)X|^2\Big)\\
&= \frac{1}{4(f(r))^2}|\mathrm{R}(v,Y)X|^2 + \frac{(f'(r))^2 - 2f''(r)f(r)}{(f(r))^2}g(Y,v)^2 - \frac{f'(r)}{f(r)}.
\end{aligned}$$

$$3.\ \mathrm{K}^f({}^VX,{}^VY) = \frac{g^f(\mathrm{R}^f({}^VX,{}^VY){}^VY,{}^VX)}{g^f({}^VX,{}^VX)g^f({}^VY,{}^VY) - g^f({}^VX,{}^VY)^2} = 0.$$

□

Proposition 4.6. *Given a Riemannian manifold (M^m, g) of constant sectional curvature κ and its tangent bundle (TM, g^f). Then the sectional curvature K^f satisfies the following:*

$$\begin{aligned}
\mathrm{K}^f({}^HX, {}^HY) &= \frac{\kappa}{f(r)} - \frac{3\kappa^2}{4(f(r))^2}\big(g(X,v)^2 + g(Y,v)^2\big) - \frac{r(f'(r))^2}{(f(r))^2}, \\
\mathrm{K}^f({}^HX, {}^VY) &= \frac{\kappa^2}{4(f(r))^2} g(X,v)^2 + \frac{(f'(r))^2 - 2f''(r)f(r)}{(f(r))^2} g(Y,v)^2 - \frac{f'(r)}{f(r)}, \\
\mathrm{K}^f({}^VX, {}^VY) &= 0.
\end{aligned}$$

Proof. M has constant curvature κ then, we have

$$\mathrm{R}(X,Y)Z = \kappa\big(g(Y,Z)X - g(X,Z)Y\big),$$

for all vector fields X, Y and Z on M, direct calculations we get

$$\begin{aligned}
|\mathrm{R}(X,Y)v|^2 &= \kappa^2\big(g(X,v)^2 + g(Y,v)^2\big), \\
|\mathrm{R}(v,Y)X|^2 &= \kappa^2 g(X,v)^2,
\end{aligned}$$

This completes the proof. □

Theorem 4.7. *Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). If σ (resp., σ^f) denote the scalar curvature of (M^m, g) (resp., (TM, g^f)), then we have*

$$\begin{aligned}
\sigma^f &= \frac{1}{f(r)}\sigma - \frac{1}{4(f(r))^2}\sum_{i,j=1}^{m} |\mathrm{R}(e_i,e_j)v|^2 \\
&\quad - \frac{4mrf''(r)f(r) + m(m-3)r(f'(r))^2 + 2m^2 f'(r)f(r)}{(f(r))^2}.
\end{aligned}$$

Proof. Let $\{\frac{1}{\sqrt{f(r)}}{}^He_i, {}^Ve_i\}_{i=1,\dots,m}$ be a local orthonormal frame on (TM, g^f) defined by the formula (4.14). We will compute the scalar curvature of (TM, g^f), as follows:

$$\sigma^f = \frac{1}{f(r)}\sum_{j=1}^{m} \mathrm{Ric}^f({}^He_j, {}^He_j) + \sum_{j=1}^{m} \mathrm{Ric}^f({}^Ve_j, {}^Ve_j). \tag{4.21}$$

Using the formula (4.15), we have

$$\begin{aligned}
\frac{1}{f(r)}\sum_{j=1}^{m} \mathrm{Ric}^f({}^He_j, {}^He_j) &= \frac{1}{f(r)}\sigma - \frac{1}{2(f(r))^2}\sum_{i,j=1}^{m} |\mathrm{R}(e_i,e_j)v|^2 \\
&\quad - \frac{2mrf''(r)f(r) + m(m-2)r(f'(r))^2 + m^2 f'(r)f(r)}{(f(r))^2}.
\end{aligned} \tag{4.22}$$

Using the formula (4.17), we have

$$\begin{aligned}
\sum_{j=1}^{m} \mathrm{Ric}^f({}^Ve_j, {}^Ve_j) &= \frac{1}{4(f(r))^2}\sum_{i,j=1}^{m} |\mathrm{R}(v,e_j)e_i|^2 \\
&\quad - \frac{2mrf''(r)f(r) - mr(f'(r))^2 + m^2 f'(r)f(r)}{(f(r))^2}.
\end{aligned} \tag{4.23}$$

On the other hand, from the formula (4.19), we find

$$\sum_{i,j=1}^{m} |\mathrm{R}(v,e_j)e_i|^2 = \sum_{i,j=1}^{m} |\mathrm{R}(e_i,e_j)v|^2. \tag{4.24}$$

Substituting the formulas (4.22), (4.23) and (4.24) into the formula (4.21), we find

$$\begin{aligned}\sigma^f &= \frac{1}{f(r)}\sigma - \frac{1}{4(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)v|^2 \\ &\quad - \frac{4mrf''(r)f(r)+m(m-3)r(f'(r))^2+2m^2f'(r)f(r)}{(f(r))^2}.\end{aligned}$$

□

From Theorem 4.7, we obtain

Proposition 4.8. *Given a Riemannian manifold (M^m, g) of constant sectional curvature κ and its tangent bundle (TM, g^f). Then scalar curvature σ^f of (TM, g^f) is given by:*

$$\begin{aligned}\sigma^f &= \frac{m(m-1)\kappa}{f(r)} - \frac{\kappa^2(m-1)r}{2(f(r))^2} \\ &\quad - \frac{4mrf''(r)f(r)+m(m-3)r(f'(r))^2+2m^2f'(r)f(r)}{(f(r))^2}.\end{aligned}$$

5 Horizontally deformed Sasaki metric on hypersurface

Let us denote by TM_0 the tangent bundle of a manifold M, excluding the zero section. Consider a smooth positive function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(r) \neq 0$ and $f'(r) \neq 1$, where $r = g(v, v)$.

We cosider the hypersurface of the tangent bundle TM_0 defined by

$$T^{f(r)}M = \{(x, v) \in TM_0,\ f(r) = r\}.$$

We define the smooth function ϕ by

$$\begin{aligned}\phi : TM_0 &\to \mathbb{R} \\ (x, v) &\mapsto \phi(x, v) = f(r) - r,\end{aligned}$$

then the hypersurface $T^{f(r)}M$ is given by

$$T^{f(r)}M = \{(x, v) \in TM_0, \quad \phi(x, v) = 0\},$$

and the gradient gradϕ of ϕ with respect to g^f is a normal vector field to $T^{f(r)}M$. From the formula (2.3), we get

$$\begin{aligned}g^f({}^HZ, \mathrm{grad}\phi) &= {}^HZ(\phi) = {}^HZ(f(r) - r) = 0, \\ g^f({}^VZ, \mathrm{grad}\phi) &= {}^VZ(\phi) = {}^VZ(f(r) - r) = 2(f'(r) - 1)g(Z, v) = 2(f'(r) - 1)g^f({}^VZ, {}^Vv).\end{aligned}$$

for all vector fields Z on M, hence

$$\mathrm{grad}\phi = 2(f'(r) - 1)^Vv.$$

Then the unit normal vector field to $T^{f(r)}M$ is given by

$$\mathcal{N} = \frac{\mathrm{grad}\phi}{\sqrt{g^f(\mathrm{grad}\phi, \mathrm{grad}\phi)}} = \frac{{}^Vv}{\sqrt{g^f({}^Vv, {}^Vv)}} = \frac{1}{\sqrt{r}}{}^Vv.$$

The tangential lift ${}^TZ$ with respect to g^f of a vector $Z \in T_xM$ to $(x, v) \in T^{f(r)}M$ as the tangential projection of the vertical lift of Z to (x, v) with respect to $\mathcal{N}$, that is

$${}^TZ = {}^VZ - g^f_{(x,v)}({}^VZ, \mathcal{N}_{(x,v)})\mathcal{N}_{(x,v)} = {}^VZ - \frac{1}{r}g_x(Z, v)^Vv.$$

Using the formula (2.1), then for all $\widetilde{Z} \in T_{(x,v)}TM$, there exist $X, Y \in T_xM$ such that

$$\begin{aligned} \widetilde{Z} &= {}^HX + {}^VY \\ &= {}^HX + {}^TY + g^f_{(x,v)}({}^VY, \mathcal{N}_{(x,v)})\mathcal{N}_{(x,v)} \\ &= {}^HX + {}^TY + \frac{1}{r} g_x(X,v)^V v, \end{aligned}$$

where $(x,v) \in T^{f(r)}M$. From this, we get the direct sum decomposition

$$T_{(x,v)}TM = T_{(x,v)}T^{f(r)}M \oplus span\{\mathcal{N}_{(x,v)}\}, \tag{5.1}$$

From the above decomposition (5.1), it is evident that the tangent space $T_{(x,v)}T^{f(r)}M$ of $T^{f(r)}M$ at the point (x,v) is expressed as

$$T_{(x,v)}T^{f(r)}M = \{{}^HX + {}^TY \,/\, X, Y \in T_xM, Y \in v^{\perp}\}.$$

where $v^{\perp} = \{Y \in T_xM,\ g(Y,v) = 0\}$.

Given a vector field Z on M, the tangential lift ${}^TZ$ of Z is given by

$${}^TZ_{(x,v)} = \left({}^VZ - g^f({}^VZ, \mathcal{N})\mathcal{N}\right)_{(x,v)} = {}^VZ_{(x,v)} - \frac{1}{r} g_x(Z_x, v)^V v.$$

For any vector field Z on M, we have the followings

(1) $g^f({}^HZ, \mathcal{N}) = 0$,

(2) $g^f({}^TZ, \mathcal{N}) = 0$,

(3) ${}^TZ = {}^VZ$ if and only if $g(Z,v) = 0$,

(4) ${}^Tv = 0$,

(5) We put $\bar{Z} = Z - \frac{1}{r} g(Z,v)v$, then ${}^TZ = {}^V\bar{Z}$ and $g(\bar{Z}, v) = 0$.

Definition 5.1. Given a Riemannian manifold (M^m, g) and its tangent bundle (TM, g^f). The Riemannian metric $\hat{g}^f$ on $T^{f(r)}M$, induced by g^f, is completely determined by the identities

$$\begin{aligned} \hat{g}^f({}^HX, {}^HY) &= rg(X,Y), \\ \hat{g}^f({}^TX, {}^HY) &= \hat{g}^f({}^HX, {}^TY) = 0, \\ \hat{g}^f({}^TX, {}^TY) &= g(\bar{X}, \bar{Y}) = g(X,Y) - \frac{1}{r} g(X,v)g(Y,v), \end{aligned}$$

We will compute the Levi-Civita connection $\widehat{\nabla}^f$ of $(T^{f(r)}M, \hat{g}^f)$. This connection is described by Gauss's formula, which is given by

$$\widehat{\nabla}^f_{\widetilde{X}} \widetilde{Y} = \nabla^f_{\widetilde{X}} \widetilde{Y} - g^f(\nabla^f_{\widetilde{X}} \widetilde{Y}, \mathcal{N})\mathcal{N}. \tag{5.2}$$

for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ on $T^{f(r)}M$.

Theorem 5.2. *Given a Riemannian manifold (M^m, g) and its hypersurface $(T^{f(r)}M, \hat{g}^f)$, then we have the following*

$$\begin{aligned} 1.\ \widehat{\nabla}^f_{{}^HX} {}^HY &= {}^H(\nabla_X Y) - \frac{1}{2} {}^T(\mathrm{R}(X,Y)v), \\ 2.\ \widehat{\nabla}^f_{{}^HX} {}^TY &= {}^T(\nabla_X Y) + \frac{1}{2r} {}^H(\mathrm{R}(v,Y)X), \\ 3.\ \widehat{\nabla}^f_{{}^TX} {}^HY &= \frac{1}{2r} {}^H(\mathrm{R}(v,X)Y), \\ 4.\ \widehat{\nabla}^f_{{}^TX} {}^TY &= \frac{-1}{r} g(Y,v)^T X, \end{aligned}$$

for all vector fields X, Y on M.

Proof. In the proof, we will use the Theorem 3.3, Lemma 3.4 and the formula (5.2).
1. By direct calculation, we have

$$\begin{aligned}\widehat{\nabla}^f_{^HX}{}^HY &= \nabla^f_{^HX}{}^HY - g^f(\nabla^f_{^HX}{}^HY,\mathcal{N})\mathcal{N}\\ &= {}^H(\nabla_XY) - f'(r)g(X,Y)^Vv - \frac{1}{2}{}^V(\mathrm{R}(X,Y)v)\\ &\quad + f'(r)g(X,Y)g^f({}^Vv,\mathcal{N})\mathcal{N} + \frac{1}{2}g^f({}^V(\mathrm{R}(X,Y)v),\mathcal{N})\mathcal{N}\\ &= {}^H(\nabla_XY) - \frac{1}{2}{}^T(\mathrm{R}(X,Y)v).\end{aligned}$$

2. We have $\widehat{\nabla}^f_{^HX}{}^TY = \nabla^f_{^HX}{}^TY - g^f(\nabla^f_{^HX}{}^TY,\mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\nabla^f_{^HX}{}^TY = {}^T(\nabla_XY) + \frac{1}{2r}{}^H(\mathrm{R}(v,Y)X) \ \ and \ \ g^f(\nabla^f_{^HX}{}^TY,\mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}^f_{^HX}{}^TY = {}^T(\nabla_XY) + \frac{1}{2r}{}^H(\mathrm{R}(v,Y)X)\,.$$

The other formulas are obtained by a similar calculation. □

We will now compute the Riemannian curvature tensor of $(T^{f(r)}M,\hat{g}^f)$. Let $\widehat{\mathrm{R}}^f$ denote the Riemannian curvature tensor of $(T^{f(r)}M,\hat{g}^f)$. Using the Gauss equation for hypersurfaces, we deduce that $\widehat{\mathrm{R}}^f(\widetilde{X},\widetilde{Y})\widetilde{Z}$ satisfies the following

$$\widehat{\mathrm{R}}^f(\widetilde{X},\widetilde{Y})\widetilde{Z} = {}^t(\mathrm{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Z}) - B(\widetilde{X},\widetilde{Z}).A_{\mathcal{N}}\widetilde{Y} + B(\widetilde{Y},\widetilde{Z}).A_{\mathcal{N}}\widetilde{X}, \tag{5.3}$$

for all $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ vector fields on $T^{f(r)}M$. where ${}^t(\mathrm{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Z})$ is the tangential component of $\mathrm{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Z}$ with respect to the direct sum decomposition (5.1), $A_{\mathcal{N}}$ is the shape operator of $T^{f(r)}M$ in (TM,g^f) derived from $\mathcal{N}$, and B is the second fundamental form of $T^{f(r)}M$ (as a hypersurface immersed in TM), associated to $\mathcal{N}$ on $T^{f(r)}M$.
$A_{\mathcal{N}}\widetilde{X}$ is described by Weingarten's formula, which states that

$$\nabla^f_{\widetilde{X}}\mathcal{N} = -A_{\mathcal{N}}\widetilde{X} + {}^{\mathcal{N}}\nabla_{\widetilde{X}}\mathcal{N},$$

where ${}^{\mathcal{N}}\nabla$ is the normal connection of $T^{f(r)}M$. Thus, we have

$$A_{\mathcal{N}}\widetilde{X} = -{}^t(\nabla^f_{\widetilde{X}}\mathcal{N}). \tag{5.4}$$

$B(\widetilde{X},\widetilde{Y})$ is defined by Gauss's formula as

$$\nabla^f_{\widetilde{X}}\widetilde{Y} = \widehat{\nabla}^f_{\widetilde{X}}\widetilde{Y} + B(\widetilde{X},\widetilde{Y})\mathcal{N}.$$

Therefore, we can express $B(\widetilde{X},\widetilde{Y})$ as

$$B(\widetilde{X},\widetilde{Y}) = g^f(\nabla^f_{\widetilde{X}}\widetilde{Y},\mathcal{N}). \tag{5.5}$$

Lemma 5.3. *Given a Riemannian manifold (M^m,g) and its hypersurface $(T^{f(r)}M,\hat{g}^f)$, then we have the following*

$$\begin{aligned}A_{\mathcal{N}}{}^HX &= \frac{-f'(r)}{\sqrt{r}}{}^HX,\\ A_{\mathcal{N}}{}^TX &= \frac{-1}{\sqrt{r}}{}^TX,\\ B({}^HX,{}^HY) &= -\sqrt{r}f'(r)g(X,Y),\\ B({}^HX,{}^TY) &= B({}^TX,{}^HY) = 0,\\ B({}^TX,{}^TY) &= \frac{-1}{\sqrt{r}}g(\bar{X},\bar{Y}),\end{aligned}$$

for all vector fields X,Y on M.

Proof. Using Theorem 3.3, Lemma 3.4, the formulas (5.4) and (5.5) we get the following:

$$\nabla^f_{^HX}\mathcal{N} = \nabla^f_{^HX}(\frac{1}{\sqrt{r}}{}^Vv) = {}^HX(\frac{1}{\sqrt{r}})^Vv + \frac{1}{\sqrt{r}}\nabla^f_{^HX}{}^Vv = \frac{f'(r)}{\sqrt{r}}{}^HX.$$

then

$$A_{\mathcal{N}}{}^HX = -{}^t(\nabla^f_{^HX}\mathcal{N}) = \frac{-f'(r)}{\sqrt{r}}{}^HX,$$

$$\begin{aligned} B({}^HX,{}^HY) &= g^f(\nabla^f_{^HX}{}^HY,\mathcal{N}) \\ &= g^f({}^H(\nabla_XY),\mathcal{N}) - f'(r)g(X,Y)g^f({}^Vv,\frac{1}{\sqrt{r}}{}^Vv) \\ &\quad -\frac{1}{2}g^f({}^V(\mathrm{R}(X,Y)v),\frac{1}{\sqrt{r}}{}^Vv) \\ &= -\sqrt{r}f'(r)g(X,Y), \end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 5.4. *Let* (M^m,g) *be a Riemannian manifold and its hypersurface* $(T^{f(r)}M,\hat{g}^f)$*, then we have the following formulas*

$$\begin{aligned}
1.\ \widehat{\mathrm{R}}^f({}^HX,{}^HY)^HZ &= {}^H(\mathrm{R}(X,Y)Z) + \frac{1}{2r}{}^H(\mathrm{R}(v,\mathrm{R}(X,Y)v)Z) \\
&\quad +\frac{1}{4r}{}^H(\mathrm{R}(v,\mathrm{R}(X,Z)v)Y) - \frac{1}{4r}{}^H(\mathrm{R}(v,\mathrm{R}(Y,Z)v)X) \\
&\quad +\frac{1}{2}{}^T((\nabla_ZR)(X,Y)v), \\
2.\ \widehat{\mathrm{R}}^f({}^HX,{}^TY)^HZ &= \frac{1}{2r}{}^H((\nabla_XR)(v,Y)Z) - \frac{1}{4r}{}^T(\mathrm{R}(X,\mathrm{R}(v,Y)Z)v) \\
&\quad +\frac{1}{2}{}^T(\mathrm{R}(X,Z)\bar{Y}), \\
3.\ \widehat{\mathrm{R}}^f({}^HX,{}^HY)^TZ &= \frac{1}{2r}{}^H((\nabla_XR)(v,Z)Y) - \frac{1}{2r}{}^H((\nabla_YR)(v,Z)X) \\
&\quad +\frac{1}{4r}{}^T(\mathrm{R}(Y,\mathrm{R}(v,Z)X)v) - \frac{1}{4r}{}^T(\mathrm{R}(X,\mathrm{R}(v,Z)Y)v) \\
&\quad +{}^T(\mathrm{R}(X,Y)\bar{Z}), \\
4.\ \widehat{\mathrm{R}}^f({}^HX,{}^TY)^TZ &= -\frac{1}{2r}{}^H(\mathrm{R}(\bar{Y},\bar{Z})X) - \frac{1}{4r^2}{}^H(\mathrm{R}(v,Y)\mathrm{R}(v,Z)X), \\
5.\ \widehat{\mathrm{R}}^f({}^TX,{}^TY)^HZ &= \frac{1}{4r^2}{}^H(\mathrm{R}(v,X)\mathrm{R}(v,Y)Z) - \frac{1}{4r^2}{}^H(\mathrm{R}(v,Y)\mathrm{R}(v,X)Z) \\
&\quad +\frac{1}{r}{}^H(\mathrm{R}(\bar{X},\bar{Y})Z), \\
6.\ \widehat{\mathrm{R}}^f({}^VX,{}^VY)^VZ &= \frac{1}{r}g(\bar{Y},\bar{Z})^TX - \frac{1}{r}g(\bar{X},\bar{Z})^TY,
\end{aligned}$$

for all vector fields X,Y *and* Z *on* M*, where* $\bar{X} = X - \frac{1}{r}g(X,v)v$.

Proof. The proof follows directly from Theorem 4.1, Gauss's equation (5.3) and Lemma 5.3. □

6 Conclusion remarks

In this paper, we have introduced and studied the horizontally deformed Sasaki metric on the tangent bundle of a Riemannian manifold, expanding the class of invariant metrics in this context. Through a detailed geometric analysis, we derived the Levi-Civita connection and examined curvature properties including the Riemann, Ricci, and scalar curvatures, as well as sectional curvature revealing how horizontal deformations distinguish this metric from classical Sasaki-type structures. Furthermore, we investigated the geometry of hypersurfaces in the tangent bundle, providing explicit descriptions of their connections and curvature tensors.

Our results offer new insights into the interplay between base manifolds and their tangent bundles under deformed metrics.The findings advance the understanding of these structures and have potential implications for future research in differential geometry, where tangent bundles play a key role.

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