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INVOLUTION -W- CLEAN RINGS

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Abstract. In this paper, we introduce the notion of an involution weak idempotent clean rings(in short invo-w-clean rings), which is a generalization of involution-t-clean rings and a subclass of clean rings. We provide certain characterizations of invo-w-clean rings in terms of the Jacobson radical and the nil-radical. Furthermore, we show that every invo-w-clean ring is isomorphic to a product of three rings, R_1, R_2 and R_3, where R_1 is an invo-w-clean ring with $2 \in J(R_1)$, $R_2 = 0$ or is an invo-w-clean ring with $3 \in J(R_2)$ and $R_3 = 0$ or is an invo-w-clean ring with $5 \in J(R_3)$.

1 Introduction

Throughout this paper, R stands for an associative ring with unity. We denote the set of all idempotent elements, the set of weak idempotent elements, the set of tripotent elements, the set of involution elements, the set of nilpotent elements, the set of units, the Jacobson radical, and nil-radicals of a ring R by $Id(R)$, $wi(R)$, $Tri(R)$, $invo(R)$, $Nil(R)$, $U(R)$, $J(R)$ and $N(R)$ respectively. We recall the following definitions from $[1, 2, 5, 6, 7]$. A ring R is called:

1 Clean if every element can be written as a sum of a unit and an idempotent.

2. Invo-clean if every $r \in R$ can be written as $r = v + e$, where $v \in invo(R)$ and $e \in Id(R)$.

3. Weakly invo-clean if every $r \in R$ can be expressed as $r = v + e$ or $r = v - e$ where $v \in invo(R)$ and $e \in Id(R)$. If in addition, $ve = ev$, then R is called weakly invo-clean with the strong property.

4. Involution t-clean(for short invo-t-clean), if for all $a \in R$ can be expressed as $a = u + t$, where $u \in invo(R)$ and $t \in Tri(R)$.

5. Weak idempotent nil-clean (win-clean), if every $r \in R$ can be expressed as $r = n + w$, where $n \in Nil(R)$ and $w \in wi(R)$.

The following hold: invo-clean ⇒ weakly invo-clean ⇒ invo-t-clean ⇒ clean.
It is observed that every element can be represented as a sum of a certain element and an involution element in all the above-said rings. It is quite natural to ask whether the representation can be generalized by varying either of the components. To that extent, the answer is affirmative. In any ring R, if $a^4 = a^2$, then a is called weak idempotent element. Clearly, every idempotent is tripotent, and every tripotent is weak idempotent, but not conversely. For instance, consider the ring of integers modulo 4. Clearly, every element is a weak idempotent element but 2 is not a tripotent element and not idempotent. In view of these observations, we introduce the notion of involution-t-clean rings and invo-w-clean rings and establish that the later is a larger class. However, both are subclasses of the class of clean rings.

In this paper, we provide certain examples of involution -w-clean rings (for short invo-w-clean rings) and also obtain some basic results concerning invo-w-clean rings. The main result of this paper is that every invo-w-clean ring R is isomorphic to a direct product of invo-w-clean rings R_1, R_2, R_3, where $2 \in J(R_1), 3 \in J(R_2)$ and $5 \in J(R_3)$.

2 Involution weak idempotent clean rings

Definition 2.1. A ring R is said to be involution weak idempotent clean (in short invo-w-clean) if every $r \in R$ can be written as $r = v + w$, where $v \in invo(R)$ and $w \in wi(R)$. An invo-w-clean ring with $vw = wv$ is said to be strongly invo-w-clean ring.

Example 2.2. Let $R = T_2(\mathbb{Z}_4)$. Then

$$invo(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$$wi(R)=\left\{\begin{pmatrix}2&3\\0&2\end{pmatrix},\begin{pmatrix}2&1\\0&2\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&3\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&3\end{pmatrix},\begin{pmatrix}3&2\\0&3\end{pmatrix}\right.$$

$$\begin{pmatrix}0&1\\0&3\end{pmatrix},\begin{pmatrix}0&2\\0&1\end{pmatrix},\begin{pmatrix}0&2\\0&3\end{pmatrix},\begin{pmatrix}0&3\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&3\end{pmatrix},\begin{pmatrix}1&2\\0&2\end{pmatrix},\begin{pmatrix}3&0\\0&0\end{pmatrix},\begin{pmatrix}2&1\\0&1\end{pmatrix},\begin{pmatrix}2&2\\0&1\end{pmatrix},\begin{pmatrix}2&2\\0&0\end{pmatrix},$$

$$\begin{pmatrix}3&1\\0&0\end{pmatrix},\begin{pmatrix}3&2\\0&0\end{pmatrix},\begin{pmatrix}3&3\\0&0\end{pmatrix},\begin{pmatrix}3&0\\0&2\end{pmatrix},\begin{pmatrix}3&1\\0&2\end{pmatrix}\begin{pmatrix}3&2\\0&2\end{pmatrix},\begin{pmatrix}3&3\\0&2\end{pmatrix},\begin{pmatrix}0&2\\0&3\end{pmatrix},\begin{pmatrix}0&3\\0&3\end{pmatrix},\begin{pmatrix}2&0\\0&1\end{pmatrix},$$

$$\begin{pmatrix}2&3\\0&1\end{pmatrix},\begin{pmatrix}2&0\\0&3\end{pmatrix},\begin{pmatrix}2&1\\0&3\end{pmatrix},\begin{pmatrix}2&2\\0&3\end{pmatrix},\begin{pmatrix}2&3\\0&3\end{pmatrix}\begin{pmatrix}0&2\\0&0\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}2&2\\0&2\end{pmatrix},\begin{pmatrix}0&0\\0&2\end{pmatrix},\begin{pmatrix}0&2\\0&2\end{pmatrix},$$

$$\begin{pmatrix}1&1\\0&3\end{pmatrix},\begin{pmatrix}1&2\\0&3\end{pmatrix},\begin{pmatrix}1&3\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&1\end{pmatrix},\begin{pmatrix}3&1\\0&1\end{pmatrix}\begin{pmatrix}3&2\\0&1\end{pmatrix},\begin{pmatrix}3&3\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}3&0\\0&3\end{pmatrix},$$

$$\left.\begin{pmatrix}2&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&2\\0&0\end{pmatrix},\begin{pmatrix}1&3\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&2\end{pmatrix},\begin{pmatrix}1&1\\0&2\end{pmatrix}\right\}.$$

Hence, R is an invo-w-clean ring.

Example 2.3. Let $R = \mathbb{Z}_{10} \times \mathbb{Z}_2$. Then $invo(R) = \{(1,1),(9,1)\}$ and $wi(R) = \{(0,0),(0,1),(1,0),(4,1),(5,1),(6,1),(4,0),(5,0),(6,0),(1,1),(9,1)\}$. Clearly R is an invo-w-clean ring.

Example 2.4. Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then R is an invo-w-clean ring.

Remark 2.5. [3] If R is a ring and w is a weak idempotent element, then

1. $w^{2n} = w^2$ and $w^{2n+1} = w^3$.

2. $Id(R) \cup -Id(R) \subseteq wi(R)$.

It can be observed that a weakly invo -clean ring is an invo-w-clean ring. The converse fails. For instance $R = \mathbb{Z}_5 \times \mathbb{Z}_5$ is an invo-w-clean ring but not a weakly invo-clean. Observe that the involutions are $\{1,4\}$, idempotents are$\{0,4\}$, the nilpotent is $\{0\}$, and the units are $\{1,2,3,4\}$. Since $2 = 1+1$ and $3 = 4-1$, the units $(2,3)$ and $(3,2)$ of R cannot be expressed as $(1,4) \pm (1,1)$ and $(4,1) \mp (1,1)$ respectively$(see[7], Example 4.16)$.

Remark 2.6. Every invo-t-clean ring is an invo-w-clean ring.

Proof. Suppose R is an invo-t-clean ring and $r \in R$. Then $r = v + t$, where $v \in invo(R)$ and $t \in Tri(R)$. Since $t^3 = t$, it follows that $t^4 = t^2$. Thus, $t \in wi(R)$. Hence, R is an invo-w-clean ring. This establishes that the class of invo-w-clean rings is wider than the class of invo-t-clean rings.
The converse of Remark 2.6 is not true. Recall Example 2.2, which is an invo-w-clean ring, but it is not an invo-t-clean, since the elements

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

are not invo-t-clean in $T_2(\mathbb{Z}_4)$ $(see[1], Remark 2.3(1))$.

Theorem 2.7. *Every invo-w-clean ring is clean.*

Proof. Suppose R be an invo-w-clean ring and let $r \in R$. Then

$$r = v + w \ , forsome \ v \in invo(R) \ and \ w \in wi(R).$$

$$= (v + w - w^2) + w^2.$$

Now, we need to show $v + w - w^2$ is a unit element in R. Let $n = w - w^3$. Then, $n^2 = (w - w^3)^2 = w^2 - 2w^4 + w^6 = w^2 - 2w^2 + w^2 = 2w^2 - 2w^2 = 0$. Thus, $n = w - w^3$ is a nilpotent element and so, $v + n$ is a unit element. Let $u = v + w - w^2$. Consider the following product $(v+w-w^3)(1+(v+w-w^3)^{-1}(w^3-w^2)) = v+w-w^3+w^3-w^2 = v + w - w^2 = u$. Thus, we have shown that u can be factored into $(v + w - w^3)$ and $1 + (v + w - w^3)^{-1}(w^3 - w^2))$. For u, to be a unit both of these factors must be units. Let $z = w^3 - w^2$. Then, $z^2 = (w^3 - w^2)^2 = w^6 - 2w^5 + w^4 = w^2 - 2w^3 + w^2 = 2w^2 - 2w^3 = -2(w^3 - w^2) = -2z$. Thus, $z^2 = -2z$ implies $z^2 + 2z = 0$. Adding 1 to both sides, we get $z^2 + 2z + 1 = 1$ implies $(z + 1)^2 = 1$. So, $1 + z$ is a unit element. We know that z is a quasi-regular element if $1 + z$ is a unit. This is equivalent to $z \in J(R)$. Hence, $1 + (v + w - w^3)^{-1}(w^3 - w^2)$ is a unit element. Therefore, $u = v + w - w^2$ is a unit element because it is a product of two unit elements. Also, if we let $e = w^2$, then $e^2 = (w^2)^2 = w^4 = w^2 = e$. Thus, $e = w^2$ is an idempotent element. Thus, every element of R can be written as a sum of a unit element and an idempotent element. Hence, R is a clean ring.

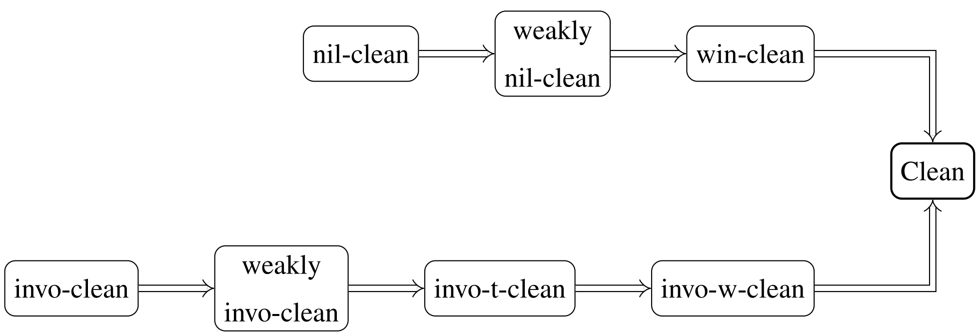
Remark 2.8. The converse of Theorem 2.7 is not true. Consider $\mathbb{Z}_7$. $invo(\mathbb{Z}_7) = \{1, 6\}$ and $wi(Z_7) = \{0, 1, 6\}$. Thus, all possible sums, $v + w$, where $v \in invo(\mathbb{Z}_7)$ and $w \in wi(\mathbb{Z}_7)$: $1 + 0 = 1$; $1 + 1 = 2$; $1 + 6 = 0$; $6 + 0 = 6$; $6 + 1 = 0$; $6 + 6 = 5$. So, 3 and 4 can not be expressed as a sum of $v + w$, for any $v \in invo(\mathbb{Z}_7)$ and $w \in wi(\mathbb{Z}_7)$. Hence, $\mathbb{Z}_7$ is not an invo-w-clean ring. But $\mathbb{Z}_7$ is clean ring.

The following examples show that weak idempotent nil-clean rings and invo-w-clean rings are independent subclasses of clean rings.

Example 2.9. Let $R = \mathbb{Z}_9$. Then, $wi(R) = \{0, 1, 3, 6, 8\}$ and $Nil(R) = \{0, 3, 6\}$. Clerly $r = n + w$. Hence, R is win-clean ring but not an invo-w-clean ring, since 3 and 6 can not be expressed as a sum of involution element and a weak idempotent element.

Example 2.10. Let $R = \mathbb{Z}_5$. Then, $wi(R) = \{0, 1, 4\}$ and $invo(R) = \{1, 4\}$. Thus, if we take $v = 1 \in invo(R)$, then $1 + 0 = 1$; $1 + 1 = 2$; $1 + 4 = 0$, and also if $v = 4$, then,$4 + 0 = 4$; $4 + 1 = 0$; $4 + 4 = 3$. Thus for any $r \in R$, there exist $v \in invo(R)$ and $w \in wi(R)$ such that $r = v + w$. Hence, R is an invo-w-clean ring but not weak idempotent nil-clean[3].

The following is the hierarchy of the classes of rings.



Theorem 2.11. *The homorphic image of an ino-w-clean ring is an invo-w-clean.*

Proof. Let $f : R \to S$ be a homomorphic mappng from an invo-w-clean ring R into S. Let $r \in f(R)$. Then $r = f(x)$, for some $x \in R$. Since, R is an invo-w-clean, $x = v + w$, for some $v \in invo(R)$ and $w \in wi(R)$. Now, $r = f(v + w) = f(v) + f(w)$. Since $v \in invo(R)$, $(f(v))^2 = f(v^2) = f(1) = 1$ and $(f(w))^4 = f(w^4) = f(w^2) = (f(w))^2$. Hence, $f(R)$ is an invo-w-clean ring.

Corollary 2.12. *Let R be an invo-w-clean ring and I is an ideal of R. Then R/I is an invo-w-clean ring.*

Remark 2.13. The converse of Corollary 2.12 is not true. For instance, consider the canonical epimorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(3)$ given by $\alpha(n) = n + (3)$. Then $\mathbb{Z}_3 \cong \mathbb{Z}/(3)$ is invo-w-clean ring, but $\alpha^{-1}(\mathbb{Z}/(3)) = \mathbb{Z}$ is not invo-w-clean ring.

Proposition 2.14. *Let R_1, R_2, ..., R_m be rings for some natural number m. Then, direct product $R = R_1 \times R_2 \times ... \times R_m = \prod_{i=1}^{m} R_i$ is an invo-w-clean ring if and only if each R_i is an invo-w-clean ring for $i = 1, 2, ..., m$.*

Proof. Let $R_1, R_2, ..., R_m$ be rings.
($\Rightarrow$) Suppose $R = \prod_{i=1}^{m} R_i$ is an invo-w-clean ring. Since the quotient $R/(\prod_{i=1,i\neq k}^{m} R_i) \cong R_k$ is a homomorphic image of R, it follows that each R_k is also an invo-w-clean ring.
($\Leftarrow$) Suppose R_i is an invo-w-clean ring for each $i = 1, 2, ..., m$, and let $R = \prod_{i=1}^{m} R_i$. For any element $x \in R$, we can write $x = (x_1, x_2, ..., x_m)$, where $x_i \in R_i$ and hence $x_i = v_i + w_i$, where $v_i \in invo(R_i)$ and $w_i \in wi(R_i)$. Thus, $v = (v_1, v_2, ..., v_m) \in invo(R)$ and $w = (w_1, w_2, ..., w_m) \in wi(R)$ such that $v + w = (v_1, v_2, ..., v_m) + (w_1, w_2, ..., w_m) = (x_1, x_2, ..., x_m)$. Hence, R is an invo-w-clean ring.

Theorem 2.15. *A direct product $R = \prod_{i\in I} R_i$ of rings $\{R_i\}_{i\in I}$ is an invo-w-clean if and only if each R_i is an invo-w-clean ring.*

Proof. ($\Rightarrow$) This follows from Theorem 2.11.
($\Leftarrow$) Suppose that each R_i is an invo-w-clean ring. Let $x = (x_i) \in \prod_{i\in I} R_i$. For each i, write $x_i = v_i + w_i$, where $v_i \in invo(R_i)$ and $w_i \in wi(R_i)$. Then $x = v + w$, where $v = (v_i) \in invo(\prod R_i)$ and $w = (w_i) \in wi(\prod R_i)$. So, $\prod_{i\in I} R_i$ is an invo-w-clean ring.

Lemma 2.16. *Let 3 be an invo-w-clean element in a ring R. Then 2,880=0. In particular* $30 \in Nil(R)$.

Proof. Given that $3 = v + w$, where $v \in invo(R)$ and $w \in wi(R)$, it follows that $3 - v = w$. This implies that $(3 - v)^4 = (3 - v)^2$, which leads to the identity $(9 - 6v + v^2)(9-6v+v^2) = 9-6v+v^2$. From this, we can derive that $114v = 126$. By squaring both sides, we obtain 2,880=0. In addition, $(30)^6 = 729,000,000 = 253,125 \times 2880 = 0$. Hence, $30 \in Nil(R)$.

Corollary 2.17. *Let R be an invo-w-clean. Then the following hold:*

i. $5 \in U(R) \Leftrightarrow 6 \in Nil(R)$.

ii. $6 \in U(R) \Leftrightarrow 5 \in Nil(R)$.

Proof. *Follow directly from the fact that $1 + Nil(R) \subseteq U(R)$ and that $30 = 5.6 \in Nil(R)$.*

Proposition 2.18. *Let R be a ring in which 2= 0. Then R is strongly invo-w-clean if and only if, all its elements are solutions of the equation $x^4 = x^2$.*

Proof. ($\Rightarrow$). Let $x \in R$. Then $x = v + w$, $for\ some\ v \in invo(R)\ and\ w \in wi(R)$ with $vw = wv$. This implies that $x^2 = (v+w)^2 = v^2+2vw+w^2 = 1+w^2 = (1+w^2)^2 = x^4$.
($\Leftarrow$).Let $r \in R$ with $r^4 = r^2$. Since, 2= 0, $r = (1+r+r^2)+(1+r^2)$. Then $(1+r+r^2)^2 = 1$ and $(1+r^2)^2 = 1+r^2$ implies that $1+r^2 \in Id(R)$. It follows that $1+r^2 \in wi(R)$. Thus, R is an invo-w-clean ring.

Theorem 2.19. *Let I be a nil ideal of a ring R and $2 \in U(R)$. Then, R is an invo-w-clean if and only if R/I is an invo-w-clean and weak idempotents of R can be lifted modulo I.*

Proof. Let I be a nil-ideal of a ring R.
($\Rightarrow$) Suppose R is an invo-w-clean ring. Then R/I is an invo-w-clean ring by corollary 2.12 and I lifts weak idempotents [3].
($\Leftarrow$) Suppose R/I is an invo-w-clean ring and the weak idempotents can can be lifted modulo I. Let $r \in R$. Then, $\overline{r} = r+I \in R/I$. Since, R/I is invo-w-clean ring, $\overline{r} = \overline{v}+\overline{w}$, for some $\overline{v} \in invo(R/I)$ and $\overline{w} \in wi(R/I)$. This implies that $r+I = (v+w)+I$. Since, $v+I \in invo(R/I)$ implies $(v+I)^2 = 1+I$. So, $v^2-1 \in I$. Since $2 \in U(R)$, we can define $x = (1-v)/2$. Then, $x^2 - x = ((1-v)/2)^2 - (1-v)/2$= $(1-2v+v^2)/4 - 2(1-v)/4$= $(1-2v+v^2-2+2v)/4 = (v^2-1)/4$. Since, $v^2-1 \in I$, then $(v^2-1)/4 \in I$. Thus, $x^2 - x \in I$. Since $x^2 - x \in I$, and x is an idempotent element which is weak idempotent, by hypothesis there exists $w \in wi(R)$ such that $w - x \in I$. Let $u = 1 - 2w^2$. Then $u^2 = (1-2w^2)^2 = 1 - 4w^2 + 4w^2 = 1$. Thus, u is involution in R. Let $k = w - x$, where $k \in I$. Then $u = 1 - 2w^2 = 1 - 2(k+x)^2 = 1 - 2((1-v)/2+k)^2 = 1 - 2((1-2v+v^2)/4 + 2((1-v)/2)k + k^2) = 1 - 2((2-2v)/4) + (1-v)k + 2k^2 = 1 - 4(1-v)/4 + (1-v)k + 2k^2 = v + (1-v)k + 2k^2$. Since $k \in I$ and I is an ideal, $(1-v)k \in I$ and $k^2 \in I$ implies $(1-v)k + 2k^2 \in I$. Therefore, $u = v + j$, where $j = (1-v)k + 2k^2 \in I$. Thus, $r = u + w$. We know that weak idempotents lift modulo any nil ideal[3] and this allows us to assume that w is a weak idempotent in R. Moreover, $r - v - w \in I$ and it follows that $r - w = v + j$, where $j \in I$. Since, $(v+j)^2 = 1$. So $v + j \in invo(R)$. Hence, R is an invo-w-clean , as desired.

Remark 2.20. In Theorem 2.19, the assumption that I is a nil ideal is necessary. Forexample, consider the ring $\mathbb{Z}_{75}$. Then $I = \{0, 3, 6, 9, 12, 15, ..., 72\}$ is an ideal of $\mathbb{Z}_{75}$, but not nil ideal and $\mathbb{Z}_{75}/I = \{I, 1+I, 2+I\}$ is an invo-w-clean ring. Now
$wi(\mathbb{Z}_{75}) = \{0, 1, 5, 10, 15, 20, 24, 25, 26, 30, 35, 40, 45, 49, 50, 51, 55, 60, 65, 70, 74\}$. Then, I lifts weak idempotents, but $\mathbb{Z}_{75}$ is not invo-w-clean ring.

Proposition 2.21. *If R is a ring with $J(R)$ is strongly invo-w-clean ring. Then $J(R)$ is nil with index of nilpotence at most 3 and $char(J(R)) = 4$.*

Proof. Let $j \in J(R)$. Then $j = v + w$, where $v \in invo(R)$ and $w \in wi(R)$. Since $J(R) + U(R) = U(R)$, we derive that $j - v = w \in U(R) \cap wi(R)$. Since $w^4 = w^2$ and $w \in U(R) \cap wi(R)$, we get $w^2 = 1$. Thus $(j)^2 = (v+w)^2 = v^2 + 2vw + w^2 = 2 + 2vw = 2v(v+w) = 2vj$. Multiplying both sides of the last equality by j, we get, $j^3 = 2vj^2 = 2v(2vj) = 4j$. Replacing j by $2j$, $8j^3 = 8j$. This implies $8j(1-j^2) = 0$.

Since $1 - j \in 1 + J(R) \subseteq U(R)$, it follows that $8j = 0$. For $j^2 = 2vj$ replacing j by $2j$, we get $4j^2 = 4vj$. By multiplying $j^2 = 2vj$ by 4, we get $4j^2 = 8vj$. Thus $8vj = 4vj$ implies $4vj = 0$. Hence, $4j = 0$. Therefore, $j^3 = 4j = 0$.

Proposition 2.22. *Let a is a strongly invo-w-clean element in R. Then*

1. *$-a$ is strongly invo-w-clean element.*
2. *av is strongly invo-w-clean element.*
3. *aw is a sum of idempotent and weak idempotent element.*
4. *If char(R)= 4 and $3 \in wi(R)$. Then $a = (v - 2w) + 3w, v - 2w \in invo(R), 3w \in wi(R)$.*

Proof. (1), (2) and (3) is clearly.
(4) Since $a = v + w = (v - 2w) + 3w$, $(v - 2w)^2 = v^2 - 4vw + 4w^2 = v^2 = 1$, $v - 2w \in invo(R)$, $(3w)^4 = 81w^4 = 3^4w^4 = 3^2w^2 = (3w)^2$, $3w \in wi(R)$.

Remark 2.23. It is worth noticing that $\mathbb{Z}_8 = \mathbb{Z}_{2^3}$ is both invo-w-clean and nil-clean ring containing the element 2 of nilpotence index 3.
Likewise $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$ is a nil-clean ring which is not necessarily invo-w-clean. In fact, $\mathbb{Z}_{16}$ is indecomposable, that is $invo(\mathbb{Z}_{16}) = \{1, 7, 9, 15\}$ as well as $wi(\mathbb{Z}_{16}) = \{0, 1, 4, 7, 8, 9, 12, 15\}$. So 4 and 12 cannot be represented as a sum of an involution and weak idempotent element.

Theorem 2.24. *A ring R is an invo-w-clean if and only if $R \cong R_1 \times R_2 \times R_3$, where R_1 is an invo-w-clean ring with $2 \in J(R_1)$ and $R_2 = 0$ or is an invo-w-clean ring with $3 \in J(R_2)$ and $R_3 = 0$ or is an invo-w-clean ring with $5 \in J(R_3)$.*

Proof.($\Rightarrow$) Suppose that R is an invo-w-clean ring. By Lemma 2.16, $30^n = 0$, for some $n \in \mathbb{N}$. Since, $(2^n, 3^n, 5^n) = 1$, i.e. there exist nozero integers k, l, m such that $2^nk + 3^nl + 5^nm = 1$, it follows that $R = 2^nR + 3^nR + 5^nR$, because $2^nR \cap 3^nR \cap 5^nR = \{0\}$. Thus, $R \cong (R/2^nR) \times (R/3^nR) \times (R/5^nR)$ by Chines Remainder Theorem. By Theorem 2.11, $R_1 = R/2^nR$, $R_2 = R/3^nR$ and $R_3 = R/5^nR$ are invo-w-clean rings. Thus, 2 is centeral nilpotent element in R_1 and hence $2 \in J(R_1)$. Assume that $R_2 \neq 0$, then 3 is centeral nilpotent element in R_2 and hence $3 \in J(R_2)$. If $R_3 \neq 0$, then 5 is centeral nilpotent element and hence $5 \in J(R_3)$.
($\Leftarrow$) Suppose $R \cong R_1 \times R_2 \times R_3$, where R_1 is an invo-w-clean ring with $2 \in J(R_1)$, $R_2 = 0$ or is an invo-w-clean ring with $3 \in J(R_2)$ and $R_3 = 0$ or is an invo-w-clean ring with $5 \in J(R_3)$. Since R_1, R_2 and R_3 are invo-w-clean rings, by Proposition 2.14, $R_1 \times R_2 \times R_3$ is an invo-w-clean ring. So, $R \cong R_1 \times R_2 \times R_3$ implies that R is an

invo-w-clean ring.
The following example illustrate that indecomposability is not mandatory, as it is in weakly nil-clean rings.

Example 2.25. Consider the matrix ring $R = M_2(\mathbb{Z}_2)$. Then

$$invo(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

$$wi(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Thus, R is an invo-w-clean ring and $J(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$.
Also $2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in J(R)$, $3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Thus, $R_1 = R/2R = R$, $R_2 = R/3R = 0$, $R_3 = R/5R = 0$, $R = R_1 \cap R_2 \cap R_3 = 0$ and $R_1 + R_2 + R_3 = R$. Hence, $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2R$ with $2 \in J(R_1)$, $R_2 = 0$ and $R_3 = 0$.

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