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ON A WEAKLY SINGULAR NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION OF THE SECOND ORDER

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Abstract This work explores a new class of Volterra equations, specifically a nonlinear weakly singular integro-differential equation in which the unknown function and its first and second derivatives are implicitly involved in the kernel, alongside a nonlinear second term also depending on the unknown function. We investigate the existence and uniqueness of the solution to this equation, applying Krasnoselskii's fixed-point theorem under conditions that satisfy the requirements of this theorem. Additionally, we derive approximate equations for the equation and its derivatives using the Nyström and Product Integration methods, then we prove the existence and uniqueness of the solution to the resulting system. Finally, we present some illustrative examples using MATLAB software.

1 Introduction

Volterra integral equations arise in various scientific disciplines [11, 5, 16] and have been extensively studied from both theoretical and computational perspectives. In particular, the class of Volterra equations in which the solution's derivatives appear nonlinearly under the integral sign has attracted considerable attention from mathematicians due to their fundamental role in modeling diverse natural phenomena, including seismic activity [19], viscoelasticity [14], and heat conduction with memory effects [22, 8].

In recent decades, weakly singular Volterra integral equations have attracted significant attention because of their ability to model various hereditary and memory-dependent processes. These equations are often challenging to analyze and solve due to the presence of weakly singular kernels. To tackle such challenges, several analytical and numerical approaches have been proposed, including spectral methods [9, 15, 23], wavelet-based techniques [17, 18], collocation schemes [13], and other computational strategies [4, 1, 3].

An interesting study in this area is presented in [10], where the authors investigate a second-order weakly singular nonlinear Volterra integro-differential equation. Their analysis employs the Picard iterative method on suitable successive sequences, followed by a detailed numerical study using an adapted product integration scheme.

The present work is closely related to that study but differs in several important aspects. In our formulation, the kernel has an algebraic form similar to those arising in fractional models [20, 7, 12], while also depending explicitly on the unknown function and its first and second derivatives. Furthermore, the source term is nonlinear in the unknown function, introducing additional analytical difficulties that cannot be addressed using classical successive approximations. To handle these complexities, we employ Krasnoselskii's fixed-point theorem to establish the existence and uniqueness of the solution in the Banach space $C^2([0,1])$.

The problem is formulated as follows: given functions $f \in C^2([0,1] \times \mathbb{R})$ and $k \in C\left([0,1] \times \mathbb{R}^3\right)$, find $u \in C^2([0,1])$, such that

$$u(t) = f(t, u(t)) + \int_0^t (t-s)^\alpha k(s, u(s), u'(s), u''(s))\, ds, \quad \forall t \in [0,1], \tag{1.1}$$

where $1 < \alpha < 2$.

To completely describe the solution u, we derive two auxiliary equations by successive differentiation of the above relation. The first derivative yields

$$u'(t) = \frac{\partial f(t,u(t))}{\partial t} + u'(t)\frac{\partial f(t,u(t))}{\partial u} + \alpha \int_0^t (t-s)^{\alpha-1} k(s, u(s), u'(s), u''(s)) ds. \tag{1.2}$$

and a second differentiation gives

$$\begin{aligned} u''(t) =& \frac{\partial^2 f(t,u(t))}{\partial^2 t} + (u'(t))^2 \frac{\partial^2 f(t,u(t))}{\partial^2 u} + u''(t)\frac{\partial f(t,u(t))}{\partial u} + 2u'(t)\frac{\partial^2 f(t,u(t))}{\partial t \partial u} \\ &+ \alpha(\alpha-1) \int_0^t (t-s)^{\alpha-2} k(s, u(s), u'(s), u''(s)) ds. \end{aligned} \tag{1.3}$$

Our main analytical contribution lies in proving the existence and uniqueness of the solution to the coupled system (1.1)-(1.3). For the numerical study, we adopt a hybrid approach: the Nyström method is applied to equations (1.1) and (1.2), whose kernels are smooth, while the Product Integration method -a Sloan projection scheme based on a first-degree Lagrange basis- is used for equation (1.3) to efficiently handle the weak singularity in $(t-s)^{\alpha-2}$.

The MATLAB simulations presented in the final section confirm the accuracy, efficiency, and robustness of the proposed scheme.

2 Analytical study

In this section, we prove the existence and uniqueness of the solution to the system of equations (1.1)-(1.3). To simplify the expressions in equations (1.2) and (1.3), we introduce the following notation:

$$f_1 = \frac{\partial f}{\partial t}, \quad f_2 = \frac{\partial f}{\partial u}, \quad g_1 = \frac{\partial^2 f}{\partial^2 t}, \quad g_2 = \frac{\partial^2 f}{\partial^2 u}, \quad h = \frac{\partial^2 f}{\partial t \partial u}.$$

Thus, the equations (1.2) and (1.3) take the following form

$$u'(t) = f_1(t, u(t)) + u'(t) f_2(t, u(t)) + \alpha \int_0^t (t-s)^{\alpha-1} k(s, u(s), u'(s), u''(s))\, ds, \tag{2.1}$$

$$\begin{aligned} u''(t) =& g_1(t, u(t)) + (u'(t))^2 g_2(t, u(t)) + u''(t) f_2(t, u(t)) + 2u'(t) h(t, u(t)) \\ &+ \alpha(\alpha-1) \int_0^t (t-s)^{\alpha-2} k(s, u(s), u'(s), u''(s))\, ds. \end{aligned} \tag{2.2}$$

2.1 Existence

To demonstrate that equation (1.1) admits at least one solution in $C^2([0,1])$, we employ Krasnoselskii's fixed point theorem [6]. Before applying this theorem, we introduce the following assumptions:

$$
(\mathcal{H}1)\left\|\begin{array}{ll}
(\mathcal{H}1.a) & \exists K \in \mathbb{R}_+^*,\ \forall t \in [0,1],\ \forall x,y,z \in \mathbb{R}:\ |k(t,x,y,z)| \leq K. \\
(\mathcal{H}1.b) & \exists d_1, d_2 \in \mathbb{R}_+^*,\ \forall t \in [0,1],\ |u'(t)| \leq d_1,\ |u''(t)| \leq d_2. \\
(\mathcal{H}1.c) & \begin{array}{c}\exists F, F_1, F_2, G_1, G_2, H \in \mathbb{R}_+^*,\ \forall t \in [0,1],\ \forall x \in \mathbb{R}: \\ |f_1(t,x)| \leq F_1,\ |f_2(t,x)| \leq F_2, \\ |g_1(t,x)| \leq G_1,\ |g_2(t,x)| \leq G_2, \\ |h(t,x)| \leq H,\ |f(t,x)| \leq F. \end{array} \\
(\mathcal{H}1.d) & \begin{array}{c}\exists A, A_1, A_2, B_1, B_2, C \in \mathbb{R}_+^*,\ \forall t \in [0,1],\ \forall x,y \in \mathbb{R}: \\ |f_1(t,x)-f_1(t,y)| \leq A_1|x-y|,\ |f_2(t,x)-f_2(t,y)| \leq A_2|x-y|, \\ |g_1(t,x)-g_1(t,y)| \leq B_1|x-y|,\ |g_2(t,x)-g_2(t,y)| \leq B_2|x-y|, \\ |h(t,x)-h(t,y)| \leq C|x-y|,\ |f(t,x)-f(t,y)| \leq A|x-y|. \end{array} \\
(\mathcal{H}1.e) & \begin{array}{c}\exists M_1, M_2, M_3, \gamma \in \mathbb{R}_+^*: \\ M_1 = A + A_1 + d_1(A_2 + 2C) + B_1 + d_1^2 B_2 + d_2 A_2, \\ M_2 = F_2 + 2d_1 G_2 + 2H, \\ M_3 = F_2, \\ \gamma = \max(M_1, M_2, M_3) < 1. \end{array}
\end{array}\right.
$$

where $(\mathcal{H}1.a) - (\mathcal{H}1.c)$ ensure the boundedness of the kernel, the first and second derivatives of the unknown function u, and the nonlinear terms; $(\mathcal{H}1.d)$ imposes Lipschitz continuity, while $(\mathcal{H}1.e)$ provides a contraction-type condition required in the subsequent analysis.

Theorem 2.1. *Under the assumptions* $(\mathcal{H}1)$*, equation* (1.1) *admits at least one solution in* $C^2([0,1])$.

Proof. We consider the Banach space $C^2([0,1])$ equipped with the norm

$$
\begin{aligned}
\forall u \in C^2([0,1]),\ \|u\|_{C^2([0,1])} &= \|u\|_{C([0,1])} + \|u'\|_{C([0,1])} + \|u''\|_{C([0,1])}, \\
&= \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u'(t)| + \sup_{t\in[0,1]} |u''(t)|.
\end{aligned}
$$

Before proceeding with the proof, we introduce the following set:

$$
S_r := \left\{u \in C^2([0,1]);\ \|u\|_{C^2([0,1])} \leq r\right\},
$$

where

$$
r = K\left(\frac{1}{\alpha+1} + \alpha + 1\right) + d_1^2 G_2 + 2d_1 H + G_1 + (d_1 + d_2)F_2 + F_1 + F.
$$

It is evident that S_r is a nonempty, closed and convex subset of $C^2([0,1])$.
We also define the operator $T : C^2([0,1]) \to C^2([0,1])$ by

$$
Tu(t) = T_1 u(t) + T_2 u(t),
$$

where

$$
\begin{aligned}
T_1 u(t) &= \int_0^t (t-s)^\alpha k(s, u(s), u'(s), u''(s))ds, \\
T_2 u(t) &= f(t, u(t)).
\end{aligned}
$$

If the operator T admits a fixed point, then equation (1.1) admits at least one solution. To establish the existence of such a fixed point, we apply Krasnoselskii's fixed point theorem and

proceed through the following three steps.
• **First step:** We first show that $T_1u_1 + T_2u_2 \in S_r$ for every $u_1, u_2 \in S_r$.
For all $u_1, u_2 \in S_r$ and $t \in [0,1]$, we have

$$\begin{aligned}
|T_1u_1(t) + T_2u_2(t)| &\leq |T_1u_1(t)| + |T_2u_2(t)|, \\
&= \left|\int_0^t (t-s)^\alpha k(s, u_1(s), u_1'(s), u_1''(s))ds\right| + |f(t, u_2(t))|, \\
&\leq \int_0^t (t-s)^\alpha |k(s, u_1(s), u_1'(s), u_1''(s))| ds + |f(t, u_2(t))|.
\end{aligned}$$

Using assumptions $(\mathcal{H}1.a)$ and $(\mathcal{H}1.c)$, we obtain

$$|T_1u_1(t) + T_2u_2(t)| \leq K\frac{t^{\alpha+1}}{\alpha+1} + F \leq F + \frac{K}{\alpha+1}.$$

Hence,

$$\|T_1(u_1) + T_2(u_2)\|_{C([0,1])} \leq F + \frac{K}{\alpha+1}.$$

Next, we estimate the first derivative. Similarly, for all $t \in [0,1]$,

$$\begin{aligned}
|T_1'u_1(t) + T_2'u_2(t)| &\leq |T_1'u_1(t)| + |T_2'u_2(t)|, \\
&= \alpha\left|\int_0^t (t-s)^{\alpha-1} k(s, u_1(s), u_1'(s), u_1''(s))ds\right| \\
&\quad + |f_1(t, u_2(t)) + f_2(t, u_2(t))u_2'(t)|, \\
&\leq \alpha\int_0^t (t-s)^{\alpha-1} |k(s, u_1(s), u_1'(s), u_1''(s))| ds \\
&\quad + |f_1(t, u_2(t))| + |f_2(t, u_2(t))| |u_2'(t)|.
\end{aligned}$$

By applying assumptions $(\mathcal{H}1.a)$, $(\mathcal{H}1.b)$ and $(\mathcal{H}1.c)$, we deduce that

$$|T_1'u_1(t) + T_2'u_2(t)| \leq \alpha K\frac{t^\alpha}{\alpha} + F_1 + d_1F_2 \leq K + F_1 + d_1F_2.$$

Thus,

$$\|T_1'(u_1) + T_2'(u_2)\|_{C([0,1])} \leq K + F_1 + d_1F_2.$$

Now, let us consider the second derivative. For all $t \in [0,1]$, we have

$$\begin{aligned}
|T_1''u_1(t) + T_2''u_2(t)| &\leq |T_1''u_1(t)| + |T_2''u_2(t)|, \\
&\leq \alpha(\alpha-1)\int_0^t (t-s)^{\alpha-2} |k(s, u_1(s), u_1'(s), u_1''(s))| ds \\
&\quad + |g_1(t, u_2(t))| + |g_2(t, u_2(t))| \left|(u_2'(t))^2\right| \\
&\quad + |f_2(t, u_2(t))| |u_2''(t)| + 2|h(t, u_2(t))| |u_2'(t)|.
\end{aligned}$$

Using assumptions $(\mathcal{H}1.a)$, $(\mathcal{H}1.b)$, and $(\mathcal{H}1.c)$, we find

$$\begin{aligned}
|T_1''u_1(t) + T_2''u_2(t)| &\leq \alpha(\alpha-1)K\frac{t^{\alpha-1}}{\alpha-1} + d_1^2G_2 + 2d_1H + G_1 + F_2d_2, \\
&\leq K\alpha + d_1^2G_2 + 2d_1H + G_1 + F_2d_2.
\end{aligned}$$

Consequently,

$$\|T_1''(u_1) + T_2''(u_2)\|_{C([0,1])} \leq K\alpha + d_1^2G_2 + 2d_1H + G_1 + F_2d_2.$$

By combining the three estimates above, we obtain

$$\|T_1(u_1)+T_2(u_2)\|_{C^2([0,1])} \leq K\left(\frac{1}{\alpha+1} + \alpha + 1\right) + d_1^2G_2 + 2d_1H + G_1 + (d_1+d_2)F_2 + F_1 + F = r.$$

Therefore, $T_1u_1 + T_2u_2 \in S_r$ for every $u_1, u_2 \in S_r$. This completes the first step.
• **Second step:** Next, we show that the operator T_2 is a contraction on S_r.
For every $u_1, u_2 \in S_r$ and for all $t \in [0, 1]$, we have

$$|T_2u_1(t) - T_2u_2(t)| = |f(t, u_1(t)) - f(t, u_2(t))| .$$

Using assumption $(\mathcal{H}1.d)$, we obtain

$$\|T_2(u_1) - T_2(u_2)\|_{C([0,1])} \leq A\|u_1 - u_2\|_{C([0,1])}. \tag{2.3}$$

Moreover,

$$\begin{aligned} |T_2'u_1(t) - T_2'u_2(t)| &= |f_1(t, u_1(t)) + u_1'(t)f_2(t, u_1(t)) - f_1(t, u_2(t)) - u_2'(t)f_2(t, u_2(t))| , \\ &\leq |f_1(t, u_1(t)) - f_1(t, u_2(t))| + |u_1'(t)| \, |f_2(t, u_1(t)) - f_2(t, u_2(t))| \\ &\quad + |u_1'(t) - u_2'(t)| \, |f_2(t, u_2(t))| . \end{aligned}$$

By applying assumptions $(\mathcal{H}1.c)$ and $(\mathcal{H}1.d)$, we deduce that

$$\|T_2'(u_1) - T_2'(u_2)\|_{C([0,1])} \leq (A_1 + d_1A_2)\|u_1 - u_2\|_{C([0,1])} + F_2\|u_1' - u_2'\|_{C([0,1])}. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} |T_2''u_1(t) - T_2''u_2(t)| \leq & |g_1(t, u_1(t)) - g_1(t, u_2(t))| + \left|(u_1'(t))^2\right| |g_2(t, u_1(t)) - g_2(t, u_2(t))| \\ & + |g_2(t, u_2(t))| \left|(u_1'(t))^2 - (u_2'(t))^2\right| + |u_1''(t)| \, |f_2(t, u_1(t)) - f_2(t, u_2(t))| \\ & + |f_2(t, u_2(t))| \, |u_1''(t)) - u_2''(t)| + 2\,|u_1'(t)| \, |h(t, u_1(t)) - h(t, u_2(t))| \\ & + 2\,|h(t, u_2(t))| \, |u_1'(t) - u_2'(t)| . \end{aligned}$$

Using assumptions $(\mathcal{H}1.b)$, $(\mathcal{H}1.c)$ and $(\mathcal{H}1.d)$, it follows that

$$\begin{aligned} \|T_2''(u_1) - T_2''(u_2)\|_{C([0,1])} \leq & (B_1 + d_1^2B_2 + d_2A_2 + 2d_1C)\|u_1 - u_2\|_{C([0,1])} \\ & + (2d_1G_2 + 2H)\|u_1' - u_2'\|_{C([0,1])} + F_2\||u_1'' - u_2''\|_{C([0,1])}. \end{aligned} \tag{2.5}$$

By adding inequalities (2.3), (2.4), and (2.5) side by side, we obtain

$$\begin{aligned} \|T_2(u_1) - T_2(u_2)\|_{C^2([0,1])} &\leq (A + A_1 + (d_1 + d_2)A_2 + B_1 + d_1^2B_2 + 2d_1C)\|u_1 - u_2\|_{C([0,1])} \\ &\quad +(F_2 + 2d_1G_2 + 2H)\|u_1' - u_2'\|_{C([0,1])} + F_2\|u_1'' - u_2''\|_{C([0,1])} \\ &\leq \gamma\|u_1 - u_2\|_{C^2([0,1])}. \end{aligned}$$

Since $0 < \gamma < 1$, we conclude that T_2 is a contraction on S_r.
• **Third step:** Finally, we prove that $T_1(S_r)$ is relatively compact.
It is clear that T_1 is continuous. To show that the set $T_1(S_r)$ is relatively compact, it is sufficient, by the Arzelà-Ascoli theorem, to verify that it is uniformly bounded and equicontinuous.
From the results obtained in the first step, we have

$$\|T_1(u)\|_{C([0,1])} \leq \frac{K}{\alpha + 1}, \tag{2.6}$$

$$\|T_1'(u)\|_{C([0,1])} \leq K, \tag{2.7}$$

$$\|T_1''(u)\|_{C([0,1])} \leq K\alpha. \tag{2.8}$$

By adding inequalities (2.6) ,(2.7), and (2.8), we obtain

$$\|T_1(u)\|_{C^2([0,1])} \leq K\frac{(1 + \alpha)^2 + 1}{1 + \alpha} < r.$$

Hence, $T_1(S_r)$ is uniformly bounded and satisfies $T_1(S_r) \subseteq S_r$.
Next, we show that the set $T_1(S_r)$ is equicontinuous. For all $u \in S_r$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned} |T_1u(t_2) - T_1u(t_1)| &= \left| \int_0^{t_2} (t_2 - s)^{\alpha} k(s, u(s), u'(s), u''(s)) ds \right. \\ &\quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha} k(s, u(s), u'(s), u''(s)) ds \right|, \\ &\leq K \frac{|t_2^{\alpha+1} - t_1^{\alpha+1}|}{\alpha+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} |T_1'u(t_2) - T_1'u(t_1)| &= \alpha \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} k(s, u(s), u'(s), u''(s)) ds \right. \\ &\quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} k(s, u(s), u'(s), u''(s)) ds \right|, \\ &\leq K |t_2^{\alpha} - t_1^{\alpha}|, \end{aligned}$$

and

$$\begin{aligned} |T_1''u(t_2) - T_1''u(t_1)| &= \alpha(\alpha-1) \left| \int_0^{t_2} (t_2 - s)^{\alpha-2} k(s, u(s), u'(s), u''(s)) ds \right. \\ &\quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-2} k(s, u(s), u'(s), u''(s)) ds \right|, \\ &\leq \alpha K |t_2^{\alpha-1} - t_1^{\alpha-1}|. \end{aligned}$$

As $t_1 \to t_2$, the right-hand sides of these inequalities, which are independent of u, tend to zero. Consequently

$$|T_1u(t_2) - T_1u(t_1)| \to 0, \ \ \forall \ \ |t_2 - t_1| \to 0, \ u \in S_r.$$

$$|T_1'u(t_2) - T_1'u(t_1)| \to 0, \ \ \forall \ \ |t_2 - t_1| \to 0, \ u \in S_r.$$

$$|T_1''u(t_2) - T_1''u(t_1)| \to 0, \ \ \forall \ \ |t_2 - t_1| \to 0, \ u \in S_r.$$

Therefore, $T_1(S_r)$ is equicontinuous, implying that T_1 is equicontinuous on S_r. By the Arzelà-Ascoli Theorem, we conclude that T_1 is relatively compact on S_r; in other words, T_1 is compact. Finally, by applying Krasnoselskii's fixed point theorem, we conclude that T has at least one fixed point. Consequently, equation (1.1) admits at least one solution in $S_r \subseteq C^2([0,1])$. □

2.2 Uniqueness

To establish the uniqueness of the solution to equation (1.1), we impose the following assumptions.

$$(\mathcal{H}2) \left\| \begin{array}{ll} (\mathcal{H}2.a) & \exists K_1, K_2, K_3 \in \mathbb{R}_+^*, \ \forall s \in [0,1], \ \forall x, y, z, x', y', z' \in \mathbb{R} : \\ & |k(s,x,y,z) - k(s,x',y',z')| \leq K_1 |x - x'| + K_2 |y - y'| + K_3 |z - z'|, \\ (\mathcal{H}2.b) & \exists \tilde{\gamma}, \theta \in \mathbb{R}_+^*, \ \tilde{\gamma} = \left(\frac{1}{\alpha+1} + \alpha + 1\right) \max(K_1, K_2, K_3) \text{ and } \theta = \gamma + \tilde{\gamma} < 1. \end{array} \right.$$

where $(\mathcal{H}2.a)$ guarantees the Lipschitz continuity of the kernel, and $(\mathcal{H}2.b)$ provides a contraction-type condition.

Theorem 2.2. *Under Assumptions* $(\mathcal{H}1)$-$(\mathcal{H}2)$*, equation* (1.1) *admits a unique solution in* $C^2([0,1])$.

Proof. Suppose that $u(t), v(t) \in C^2([0,1])$ are two distinct solutions of equation (1.1). Then, for all $t \in [0,1]$, we have

$$\begin{aligned} |u(t)-v(t)| &\leq |f(t,u(t))-f(t,v(t))| \\ &+ \int_0^t (t-s)^{\alpha}\, |k(s,u(s),u'(s),u''(s))-k(s,v(s),v'(s),v''(s))|\, ds. \end{aligned}$$

By applying assumptions $(\mathcal{H}1.d)$ and $(\mathcal{H}2.a)$, we obtain

$$\begin{aligned} |u(t)-v(t)| &\leq A|u(t)-v(t)| \\ &+ \int_0^t (t-s)^{\alpha} \left(K_1|u(s)-v(s)| + K_2|u'(s)-v'(s)| + K_3|u''(s)-v''(s)|\right) ds. \end{aligned}$$

Thus,

$$\|u-v\|_{C([0,1])} \leq (A + \frac{K_1}{\alpha+1})\|u-v\|_{C([0,1])} + \frac{K_2}{\alpha+1}\|u'-v'\|_{C([0,1])} + \frac{K_3}{\alpha+1}\|u''-v''\|_{C([0,1])}. \tag{2.9}$$

Similarly, we have

$$\begin{aligned} |u'(t)-v'(t)| &\leq |f_1(t,u(t))-f_1(t,v(t))| \\ &+ |u'(t)|\,|f_2(t,u(t))-f_2(t,v(t))| + |f_2(t,v(t))|\,|u'(t)-v'(t)| \\ &+ \alpha\int_0^t (t-s)^{\alpha-1}\, |k(s,u(s),u'(s),u''(s))-k(s,v(s),v'(s),v''(s))|\, ds, \end{aligned}$$

and, using assumptions $(\mathcal{H}1.b)-(\mathcal{H}1.d)$ and $(\mathcal{H}2.a)$, we find

$$\|u'-v'\|_{C([0,1])} \leq (d_1A_2+A_1+K_1)\|u-v\|_{C([0,1])} + (F_2+K_2)\|u'-v'\|_{C([0,1])} + K_3\||u''-v''\|_{C([0,1])}. \tag{2.10}$$

In the same manner, for $|u''(t)-v''(t)|$, we obtain

$$\begin{aligned} |u''(t)-v''(t)| &\leq |g_1(t,u(t))-g_1(t,v(t))| \\ &+ \left|(u'(t))^2\right| |g_2(t,u(t))-g_2(t,v(t))| + 2\,|h(t,v(t))|\,|u'(t)-v'(t)| \\ &+ |g_2(t,v(t))| \left|(u'(t))^2-(v'(t))^2\right| + |u''(t)|\,|f_2(t,u(t))-f_2(t,v(t))| \\ &+ |f_2(t,v(t))|\,|u''(t)-v''(t)| + 2\,|u'(t)|\,|h(t,u(t))-h(t,v(t))| \\ &+ \alpha(\alpha-1)\int_0^t (t-s)^{\alpha-2}\, |k(s,u(s),u'(s),u''(s))-k(s,v(s),v'(s),v''(s))|\, ds. \end{aligned}$$

Using again assumptions $(\mathcal{H}1.b)-(\mathcal{H}1.d)$ and $(\mathcal{H}2.a)$, we derive

$$\begin{aligned} \|u''-v''\|_{C([0,1])} &\leq (B_1+d_1^2B_2+\alpha K_1+d_2A_2+2d_1C)\|u-v\|_{C([0,1])} \qquad (2.11) \\ &+ (2d_1G_2+2H+\alpha K_2)\|u'-v'\|_{C([0,1])} + (F_2+\alpha K_3)\||u''-v''\|_{C([0,1])}. \end{aligned}$$

By adding inequalities (2.9) , (2.10), and (2.11), we obtain

$$\|u-v\|_{C^2([0,1])} \leq \theta\|u-v\|_{C^2([0,1])}.$$

Hence, $\theta > 1$, which contradicts the assumption that $0 < \theta < 1$. Consequently $u(t) = v(t)$, $u'(t) = v'(t)$, $u''(t) = v''(t)$, for all $t \in [0,1]$. Therefore, equation (1.1) admits a unique solution in $C^2([0,1])$. □

3 Numerical approach

Having established the existence and uniqueness of the analytical solution, we now turn to its numerical approximation. The objective of this section is to construct a reliable numerical scheme capable of approximating the solution of equation (1.1). Such a discretization complements the theoretical analysis and provides a practical tool to illustrate the obtained results.

Since equations (1.1) and (2.1) involve regular kernels, they can be effectively treated using the Nyström method [2]. In contrast, equation (2.2) presents additional challenges because its kernel includes the term $(t-s)^{\alpha-2}$, which becomes singular as t approaches s for $1<\alpha<2$, resulting in a weak algebraic singularity. In this case, the classical Nyström approach fails because standard quadrature rules lose accuracy near the singularity. To overcome this limitation, we adopt the Product Integration method [21], specifically designed to handle such weakly singular kernels.

We first recall the classical Nyström method. Fix $N \in \mathbb{N}^*$ and set a uniform grid

$$t_j = jh,\ h = \frac{1}{N},\ j = 0, \cdots, N.$$

The main idea of this method is to approximate the integral operator by a quadrature rule. For any $f \in C^0([0,1])$,

$$\int_0^1 f(x)\,dx \approx \sum_{j=0}^{N} w_j f(x_j),$$

where w_j are given weights satisfying $\sup\limits_{0\le j\le N} w_j < \infty$.

In the Product Integration method, only the smooth component of the kernel k is approximated by piecewise linear interpolation over each subinterval $[t_j, t_{j+1}]$, $j=0,...N-1$. Specifically, for all $s \in [t_j, t_{j+1}]$, we approximate:

$$\begin{aligned} k(s,u(s),u'(s),u''(s)) \quad &\approx \quad \left(\frac{s-t_j}{h}\right) k(t_{j+1},u_{j+1},u'_{j+1},u''_{j+1}) \\ &\quad + \left(\frac{t_{j+1}-s}{h}\right) k(t_j,u_j,u'_j,u''_j). \end{aligned}$$

We then apply the Nyström method to equations (1.1)-(1.2), and the Product Integration method to equation (1.3), yielding the following discrete approximations

$$U_0 = f(0,U_0), \tag{3.1}$$

$$V_0 = f_1(0,U_0) + V_0 f_2(0,U_0), \tag{3.2}$$

$$W_0 = g_1(0,U_0) + (V_0)^2 g_2(0,U_0) + W_0 f_2(0,U_0) + 2V_0 h(0,U_0), \tag{3.3}$$

$$U_i = f(t_i,U_i) + \sum_{j=0}^{i} w_j (t_i - t_j)^{\alpha} k(t_j,U_j,V_j,W_j), \quad 1 \le i \le N, \tag{3.4}$$

$$V_i = f_1(t_i,U_i) + V_i f_2(t_i,U_i) + \alpha \sum_{j=0}^{i} w_j (t_i-t_j)^{\alpha-1} k(t_j,U_j,V_j,W_j), \quad 1 \le i \le N, \tag{3.5}$$

$$\begin{aligned} W_i \quad &= \quad g_1(t_i,Ui) + (V_i)^2 g_2(t_i,U_i) + W_i f_2(t_i,U_i) + 2V_i h(t_i,U_i) \\ &\quad + \alpha(\alpha-1)\sum_{j=0}^{i} \eta_j k(t_j,U_j,V_j,W_j), \quad 1 \le i \le N, \end{aligned} \tag{3.6}$$

where U_i , V_i and W_i denote the approximate values of $u(t_i)$, $u'(t_i)$ and $u''(t_i)$, respectively, and η_j are given by:

$$\eta_0 = \frac{1}{h}\int_0^{t_1} (t_1-s)(t_i-s)^{\alpha-2} ds,$$

$$\eta_j = \frac{1}{h}\left(\int_{t_{j-1}}^{t_j} (s-t_{j-1})(t_i-s)^{\alpha-2} ds + \int_{t_j}^{t_{j+1}} (t_{j+1}-s)(t_i-s)^{\alpha-2} ds\right),\ j = 1, ...i-1,$$

$$\eta_i = \frac{1}{h}\int_{t_{i-1}}^{t_i}(s-t_{i-1})(t_i-s)^{\alpha-2}ds.$$

The discrete relations (3.1)-(3.6) lead to a nonlinear algebraic system. To analyze the solvability of this system, we construct an associated nonlinear operator and apply Banach's fixed-point theorem to prove the existence and uniqueness of the approximate solution. The result is stated in the following theorem.

Theorem 3.1. *Suppose that assumptions* $(\mathcal{H}1)$-$(\mathcal{H}2)$ *hold. Then, for sufficiently small values of* h*, the discrete system* (3.1)-(3.6) *admits a unique solution.*

Proof. Assume that the space $\mathbb{R}^3$ is equipped with the following norm:

$$\forall \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3, \qquad \left\| \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \right\|_1 = |X|+|Y|+|Z|.$$

For all $i \geq 1$, we define the operator

$$\phi_i \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} f(t_i,X)+S_1 \\ f_1(t_i,X)+Yf_2(t_i,X)+S_2 \\ g_1(t_i,X)+Y^2g_2(t_i,X)+Zf_2(t_i,X)+2Yh(t_i,X)+\alpha(\alpha-1)\eta_i k(t_i,X,Y,Z)+S_3 \end{pmatrix}$$

where

$$S_1 = \sum_{j=0}^{i-1} w_j(t_i-t_j)^{\alpha}k(t_j,U_j,V_j,W_j),$$

$$S_2 = \alpha\sum_{j=0}^{i-1} w_j(t_i-t_j)^{\alpha-1}k(t_j,U_j,V_j,W_j),$$

$$S_3 = \alpha(\alpha-1)\sum_{j=0}^{i-1} \eta_j k(t_j,U_j,V_j,W_j).$$

Let

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} \in \mathbb{R}^3.$$

Then,

$$\left\| \phi_i\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \phi_i\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right\|_1,$$

where

$$\begin{aligned}
\beta_1 &= f(t_i,X)-f(t_i,X'),\\
\beta_2 &= f_1(t_i,X)-f_1(t_i,X')+Yf_2(t_i,X)-Y'f_2(t_i,X'),\\
\beta_3 &= g_1(t_i,X)-g_1(t_i,X')+Y^2g_2(t_i,X)-Y'^2g_2(t_i,X')+Zf_2(t_i,X)-Z'f_2(t_i,X'),\\
&\quad +2Yh(t_i,X)-2Y'h(t_i,X')+\alpha(\alpha-1)\eta_i(k(t_i,X,Y,Z)-k(t_i,X',Y',Z')).
\end{aligned}$$

From assumptions $(\mathcal{H}1)$-$(\mathcal{H}2)$, we obtain

$$\begin{aligned}
|\beta_1| &\leq A|X-X'|,\\
|\beta_2| &\leq (A_1+d_1A_2)|X-X'|+F_2|Y-Y'|,\\
|\beta_3| &\leq (B_1+d_1^2B_2+d_2A_2+2d_1C+\alpha(\alpha-1)\eta_iK_1)|X-X'|\\
&\quad +(2d_1G_2+2H+\alpha(\alpha-1)\eta_iK_2)|Y-Y'|\\
&\quad +(F_2+\alpha(\alpha-1)\eta_iK_3)|Z-Z'|.
\end{aligned}$$

Consequently,

$$\left\|\phi_i\begin{pmatrix}X\\Y\\Z\end{pmatrix}-\phi_i\begin{pmatrix}X'\\Y'\\Z'\end{pmatrix}\right\|_1\leq(\gamma+\alpha(\alpha-1)\eta_i\max(K_1,K_2,K_3))\left\|\begin{pmatrix}X\\Y\\Z\end{pmatrix}-\begin{pmatrix}X'\\Y'\\Z'\end{pmatrix}\right\|_1.$$

Finally, note that

$$\alpha(\alpha-1)\eta_i=\frac{\alpha(\alpha-1)}{h}\int_{t_{i-1}}^{t_i}(s-t_{i-1})(t_i-s)^{\alpha-2}ds=h^{\alpha-1}.$$

Since $\alpha(\alpha-1)\eta_i\to 0$ as $h\to 0$ and $\gamma<1$, the operator ϕ_i is a contraction for all $i\geq 1$.

By Banach's fixed-point theorem, the result follows. □

4 Numerical Results

In this section, we present illustrative examples to demonstrate the practical performance of the proposed numerical method. All computations were performed using MATLAB software on a machine equipped with an Intel Core i5 CPU (1.9 GHz) and 8 GB RAM.

We begin by applying the Nyström method using the trapezoidal rule to discretize the integrals in the first and second equations while the third equation is handled using the Product Integration method. The resulting nonlinear algebraic system, derived from equations (3.1)-(3.6), is then solved numerically, by applying the successive approximation technique with a null initial vector and an iteration tolerance of order 10^{-9}. After computing the approximate solutions, we compare them with the exact solutions across various values of N, assessing the method's accuracy through the dedicated error functions:

$$E=\max_{0\leq i\leq N}(|u(t_i)-U_i|+|u'(t_i)-V_i|+|u''(t_i)-W_i|).$$

Example 1: Consider the following equation:

$$\begin{aligned}u(t)&=\frac{3t^2\sqrt{t}}{5}\cos\left(u(t)-t^2\sqrt{t}\right)\\&+\int_0^t(t-s)^{1.5}\sin\left(u(s)+u'(s)+u''(s)+\frac{\pi}{2}-\frac{15\sqrt{s}}{4}\left(1+\frac{2s}{3}+\frac{4s^2}{15}\right)\right)ds,\end{aligned}$$

where, the exact solution and its derivatives are:

$$u(t)=t^2\sqrt{t},\ \ u'(t)=2.5t\sqrt{t},\ \ u''(t)=3.75\sqrt{t},\ \ t\in[0,1].$$

We present the results in the following Table and Figures:

N	E	CPU time (sec)
5	3.03E-2	0.04
10	1.04E-2	0.10
50	9.06E-4	1.24
100	3.17E-4	4.59
250	7.98E-5	26.80

Table 1. The error function E of example 1 by varying of N.

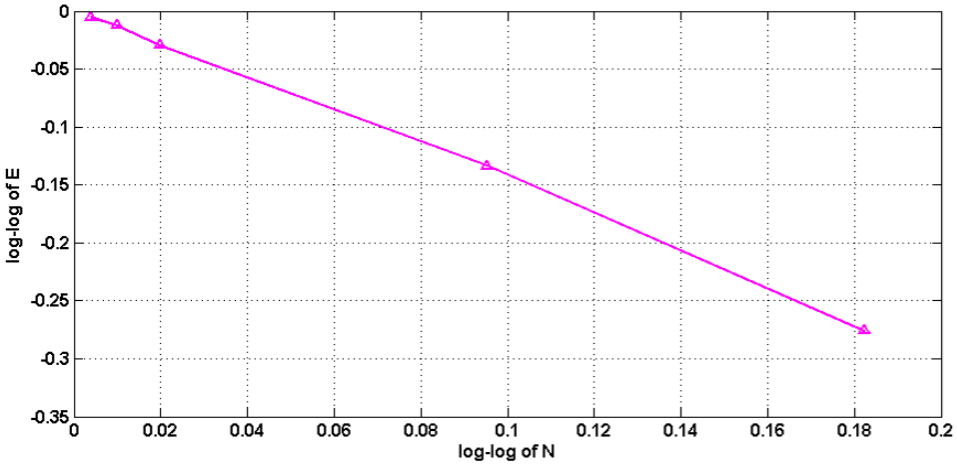

Figure 1. Computational rate of convergence in Example 1.

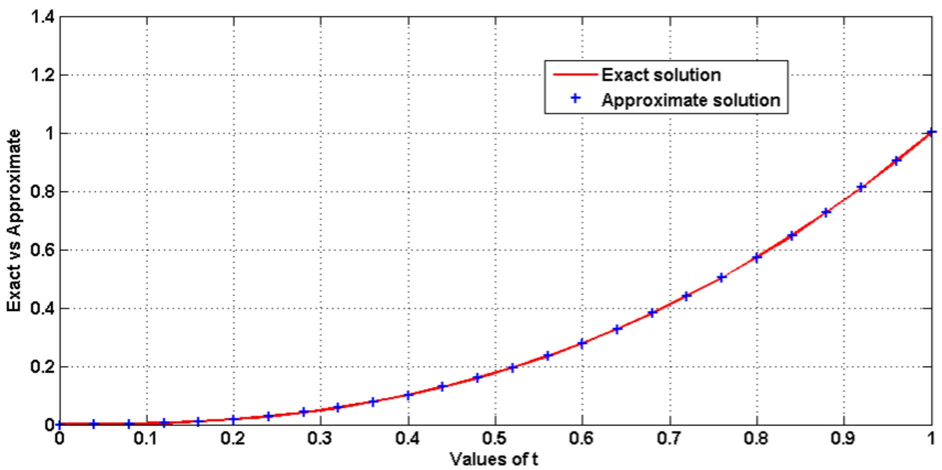

Figure 2. Comparison of the exact and approximate solutions of Example 1 for $N = 25$.

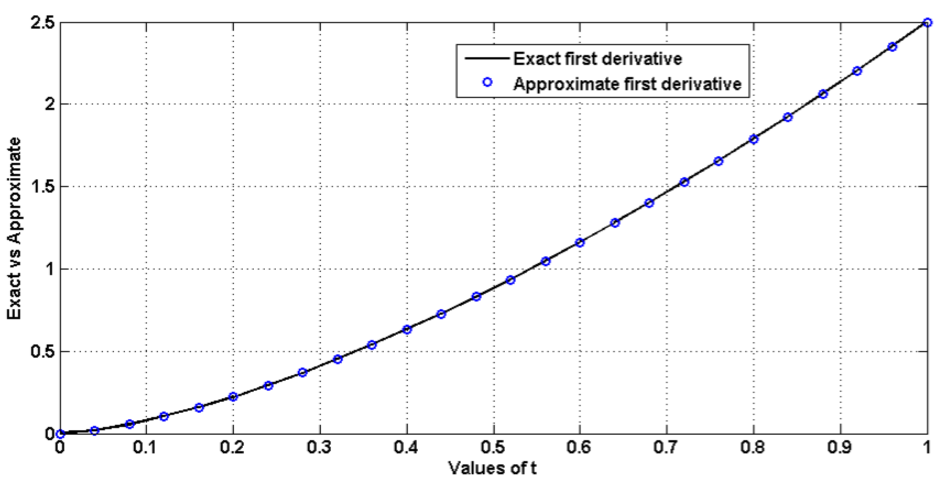

Figure 3. Comparison of the exact and approximate derivatives of the solution of Example 1 for $N = 25$.

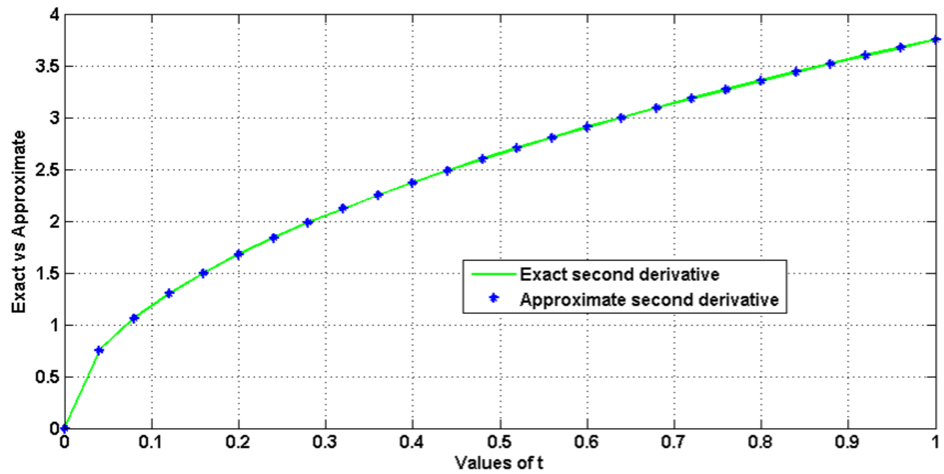

Figure 4. Comparison of the exact and approximate second derivatives of the solution of Example 1 for $N = 25$.

Table 1 and Figure 1 clearly show that the error function E decreases as the discretization parameter N increases, confirming the convergence of the approximate solution and its first and second derivatives to the exact ones. Moreover, Figures 2-4 illustrate that even for a relatively small value of $N = 25$, the approximate and exact solutions, along with their derivatives, are almost identical, highlighting the effectiveness and accuracy of the proposed numerical method.
Example 2: Consider the following equation:

$$u(t) = \frac{(\alpha+1)\cos(t) - t^{\alpha+1}}{u(t)^2 + \alpha + \sin(t)^2} + \int_0^t \frac{3(t-s)^\alpha}{u(s)^2 + u'(s)^2 + u''(s)^2 + 1 + \sin(s)^2} ds, \ \alpha \in]1,2[,$$

the exact solution and its derivatives are:

$$u(t) = \cos(t), \ \ u'(t) = -\sin(t), \ \ u''(t) = -\cos(t), \ \ t \in [0,1].$$

In this case we determine the error function E according of the values of:

$$\alpha \in \{1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9\}, \ \ N \in \{5, 10, 50, 100, 250\}.$$

$\alpha \backslash N$	5	10	50	100	250
1.1	8.43E-1	5.32E-1	2.30E-1	1.95E-1	1.73E-1
1.2	5.27E-1	2.58E-1	4.28E-2	2.05E-2	8.88E-3
1.3	3.59E-1	1.60E-1	2.12E-2	8.72E-3	2.70E-3
1.4	2.41E-1	1.00E-1	1.13E-2	4.35E-3	1.21E-3
1.5	1.58E-1	6.16E-2	6.06E-3	2.18E-3	5.62E-4
1.6	1.03E-1	3.72E-2	3.17E-3	1.08E-3	2.56E-4
1.7	6.41E-2	2.17E-2	1.60E-3	5.11E-4	1.12E-4
1.8	3.54E-2	1.14E-2	7.53E-4	2.28E-4	4.63E-5
1.9	1.48E-2	4.47E-3	2.68E-4	7.78E-5	1.49E-5

Table 2. The error function E of example 2 by varying the values of α and N.

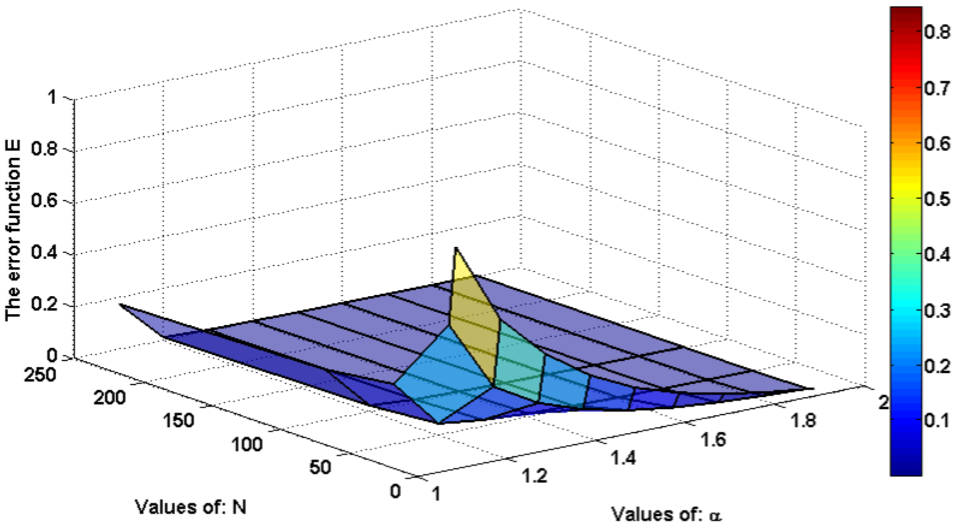

Figure 5. Graph illustrating the dependence of the error function E in Example 2 on α and N.

Table 2 and Figure 5 illustrate that for any value of α, the error function E decrease as N increases, which affirms the effectiveness and accuracy of the numerical method. Additionally, the results show that the performance of the method is significantly influenced by the kernel's regularity, which is tied to the value of α. Specifically, when α is close to 1, indicating lower regularity, the method converges more slowly. On the other hand, as α approaches 2, indicating higher regularity, the convergence becomes much faster.

5 Conclusion

In this study, we have rigorously established the existence and uniqueness of the solution to a second-order nonlinear Volterra integro-differential equation involving weakly singular kernels. Moreover, a hybrid Nyström-Product Integration scheme has been developed to obtain accurate numerical approximations of the solution. The numerical experiments carried out in MATLAB confirm the proposed method's accuracy, efficiency, and robustness.

References

[1] A. Altürk, *Application of the homotopy perturbation method for weakly singular Volterra integral equations*, J. New Results Sci., **13(3)**, 201-213, (2024).

[2] K. E. Atkinson, W. Han, *Theoretical Numerical Analysis: A Functional Analysis Approach*, Springer, New York, (2009).

[3] I. Bazine, S. Lemita, *An investigation for a second order Volterra-Fredholm integro-differential equation with two algebraic weakly singular kernels*, FILOMAT, **39(30)**, 10919 - 10934, (2025).

[4] I. A. Bhata, L. Narayan Mishra, V. Narayan Mishra, M. Abdel-Atyc, M. Qasymeh, *Acomprehensive analysis for weakly singular nonlinear functional Volterra integral equations using discretization techniques*, Alex. Eng. J., **104**, 564-575, (2024).

[5] H. Brunner, *Volterra Integral Equations: An Introduction to the Theory and Applications*, Cambridge Univ, Press, Cambridge, (2017).

[6] T. A. Burton, *A Fixed-Point Theorem of Krasnoselskii*, Appl. Math. Lett., **11(1)**, 85-88, (1998).

[7] P. Bwire, S. Kumar, M. Mursaleen, et al. *Numerical solution of fractional Volterra integral equations using Szász-Mirakyan approximation operators*, J. Inequal. Appl. **2025(108)**, (2025).

[8] S. Carillo, *Singular kernel problems in materials with memory*, Meccanica. **50(3)**, 603-615, (2015).

[9] Y. Chen, T. Tang, *Spectral methods for weakly singular Volterra integral equations with smooth solutions*, J. Comput. Appl. Math., **233(4)**, 938-950, (2009).

[10] M. Ghiat, H. Guebbai, M. Kurulay, S. Segni, *On the weakly singular integro-differential nonlinear Volterra equation depending in acceleration term*, Comp. App. Mat., **39(206)**, 1-13, (2020).

[11] J. M. Gushing, *Volterra Integrodifferential Equations in Population Dynamics*, Springer, Berlin, Heidelberg, (2010).

[12] P. Karthikeyan, A. Manikandan, D. Vijay, *Existence results for fractional integro differential equations involving two boundary conditions with ψ-Caputo fractional derivative*, Palest. J. Math, **14(4)**, 352-366, (2025).

[13] K. Kherchouche, A. Bellour, P. Lima, *Numerical solution of nonlinear third-kind Volterra integral equations using an iterative collocation method*, Int. J. Comput. Math., **100(11)**, 2063-2076, (2023).

[14] V. Lakshmikantham, M. Rama Mohana Rao, *Theory of Integro-Differential Equations*, CRC Press, (1995).

[15] Z. Liua, D. Taoband, C. Zhang, *An Efficient Spectral Method for Nonlinear Volterra Integro-Differential Equations withWeakly Singular Kernels*, Math. Model. Anal., **29(3)**, 387-405, (2024).

[16] W. Okrasinski, *Nonlinear Volterra equations and physical applications*, Extr. Math., **4(2)**, 51-80, (1989).

[17] Y. Rostami, *A new wavelet method for solving a class of nonlinear partial integro-differential equations with weakly singular kernels*, Math. Sci., **16**, 225-235, (2022).

[18] Y. Rostami, *An effective computational approach based on Hermite wavelet Galerkin for solving parabolic Volterra partial integro differential equations and its convergence analysis*, Math. Model. Anal., **28(1)**, 163-179, (2023).

[19] S. Salah, H. Guebbai, S. Lemita, M. Z. Aissaoui, *Solution of an integro-differential nonlinear equation of Volterra arising of earthquake model*, Bol. Soc. Paran. Mat. **39**, 82-96, (2021).

[20] S. Santra, J. Mohapatra, *Numerical Analysis of Volterra Integro-differential Equations with Caputo Fractional Derivative*, Iran. J. Sci. Technol. Trans. Sci., **45**, 1815-1824, (2021).

[21] C. Schneider, *Product integration for weakly singular integral equations*, Math. Comput., **36(153)**, 207-213, (1981).

[22] G. Seifert, *A* temperature equation for a rigid heat conductor with memory, Q. Appl. Math. **38(2)**, 246-252, (1980).

[23] M. A. Taema, M. A. Dagher, Y. H. Youssri, *Spectral Collocation Method via Fermat Polynomials for Fredholm-Volterra Integral Equations with Singular Kernels and Fractional Differential Equations*, Palest. J. Math, **14(2)**, 481-492, (2025).

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