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A NOTE ON IDEMPOTENT ELEMENTS IN $\mathbb{C}_3$

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Abstract This paper identifies twelve idempotent elements in $\mathbb{C}_3$, the space of tricomplex numbers.

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1 Introduction

We shall be using the following notations in this paper:

$\mathbb{C}_0 =$ Space of Real Numbers
$\mathbb{C}_1 =$ Space of Complex Numbers
$\mathbb{C}_2 =$ Space of Bicomplex Numbers
$\mathbb{C}_3 =$ Space of Tricomplex Numbers

Before defining tricomplex numbers, let us see the definition of *Bicomplex numbers*.

Corrado Segre [5] gave the concept of bicomplex numbers in 1892. The set of bicomplex numbers is defined as follows:

$$\mathbb{C}_2 = \{\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : {i_1}^2 = {i_1}^2 = -1, i_1 i_2 = i_2 i_1, x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

or equivalently as

$$\mathbb{C}_2 = \{\xi = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}.$$

In other words, we can say that a number $\mathbb{C}_2$, i.e., a bicomplex number can be written as the combination of four real numbers (or it is a four dimensional) or as the combination of two complex numbers, hence the term bicomplex.

Equipped with coordinate-wise addition and real scalar multiplication and term by term multiplication, $\mathbb{C}_2$ becomes a commutative algebra with identity $1 = 1 + i_1 \cdot 0 + i_2 \cdot 0 + i_1 \cdot i_2 \cdot 0$.

Basic definitions and properties of $\mathbb{C}_2$ can be found in the book by Price [2]. Multicomplex spaces are the generalizations of the space of complex numbers. The space of bicomplex numbers is the first in an infinite sequence of multicomplex spaces.

In recent years, substantial progress has been made in bicomplex analysis. Sager and Sağır [3] studied the completeness of certain bicomplex sequence spaces, while Wagh [7, 8] investigated functional analytic aspects of entire bicomplex sequence spaces and their subspaces. Ritu *et al*. [1] analyzed the boundary behavior of power series in bicomplex spaces together with gap theorems, and Urvashi *et al*. [4] focused on the bicomplex two-parameter Mittag-Leffler function, its properties, and its applications to the fractional time wave equation. A central tool in these developments has been the idempotent representation of bicomplex numbers, which plays a key role in advancing their theory and applications. Building on this foundation, the present work introduces tricomplex numbers and provides a detailed study of idempotent elements in tricomplex spaces.

2 Tricomplex Numbers

Definition 2.1. A tricomplex number is defined as

$$t = \zeta_1 + i_3\zeta_2,$$

where ζ_1 and ζ_2 are bicomplex numbers and $i_3^2 = -1$, $i_3 i_1 = i_1 i_3$, $i_3 i_2 = i_2 i_3$.

If $\zeta_1 = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4$ and $\zeta_2 = y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4$, then

$$t = (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) + i_3(y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4).$$

It is an 8-dimensional number. The set of tricomplex numbers is denoted by $\mathbb{C}_3$,

$$\mathbb{C}_3 = \{t : t = a_1 + i_1a_2 + i_2a_3 + i_3a_4 + i_1i_2a_5 + i_1i_3a_6 + i_2i_3a_7 + i_1i_2i_3a_8,$$
$$\alpha_j \in \mathbb{C}_0, 0 = j = 8, i_1^2 = i_2^2 = i_3^2 = -1, i_1i_2 = i_2i_1, i_1i_3 = i_3i_1, i_2i_3 = i_3i_1\}$$

is the set of tricomplex numbers. A tricomplex number is an eight-dimensional number. The basis elements are $1, i_1, i_2, i_3, i_1i_2, i_1i_3, i_2i_3, i_1i_2i_3$.

If $\kappa = \beta_1 + i_1\beta_2 + i_2\beta_3 + i_3\beta_4 + i_1i_2\beta_5 + i_1i_3\beta_6 + i_2i_3\beta_7 + i_1i_2i_3\beta_8$, then multiplication (term-by-term) of two tricomplex numbers t and κ is denoted by $t \circledast_{\mathbb{C}_3} \kappa$.

Wagh [9] has given binary compositions and idempotent representations in tricomplex space. It has also been shown there that $(\mathbb{C}_3, +, \circledast_{\mathbb{C}_3}, \|\ \|_3)$ forms a Banach Algebra.

3 Idempotent Elements in $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$

Definition 3.1. An idempotent element is one that remains unchanged when multiplying by itself.

Idempotent elements in $\mathbb{C}_0$: There are two idempotent elements, 0 and 1

Idempotent elements in $\mathbb{C}_1$: Like the space of real numbers, complex space too has two idempotent elements 0 and 1.

Idempotent elements in $\mathbb{C}_2$: Beside 0, 1 there are two more idempotent elements namely $e(i_1i_2) = e_1$ and $e(-i_1i_2) = e_2$ in the space of bicomplex numbers, given by

$$e(i_1i_2) = e_1 = \frac{1 + i_1i_2}{2},$$
$$e(-i_1i_2) = e_2 = \frac{1 - i_1i_2}{2}.$$

$e_1 + e_2 = 1$ and $e_1 \cdot e_2 = 0$ can be easily verified. Every bicomplex number has a unique idempotent representation [1]. This idempotent representation plays a vital role in the development of the theory of bicomplex numbers

4 Main Work: Idempotent Elements in $\mathbb{C}_3$

Let us first see the product of two elements in $\mathbb{C}_3$.
If $t = a_1 + i_1a_2 + i_2a_3 + i_3a_4 + i_1i_2a_5 + i_1i_3a_6 + i_2i_3a_7 + i_1i_2i_3a_8$ and $\kappa = \beta_1 + i_1\beta_2 + i_2\beta_3 + i_3\beta_4 + i_1i_2\beta_5 + i_1i_3\beta_6 + i_2i_3\beta_7 + i_1i_2i_3\beta_8$ are two tricomplex numbers, following table gives their multiplication:

Table 1. $t \circledast_{\mathbb{C}_3} \kappa$

$\circledast_{\mathbb{C}_3}$	$\boldsymbol{\beta}_1$	$\boldsymbol{i}_1\boldsymbol{\beta}_2$	$\boldsymbol{i}_2\boldsymbol{\beta}_3$	$\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{\beta}_5$	$\boldsymbol{i}_3\boldsymbol{\beta}_4$	$\boldsymbol{i}_1\boldsymbol{i}_3\boldsymbol{\beta}_6$	$\boldsymbol{i}_2\boldsymbol{i}_3\boldsymbol{\beta}_7$	$\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3\boldsymbol{\beta}_8$
α_1	$\alpha_1\beta_1$	$i_1\alpha_1\beta_2$	$i_2a_1\beta_3$	$i_1i_2\alpha_1\beta_5$	$i_3a_1\beta_4$	$i_1i_3\alpha_1\beta_6$	$i_2i_3\alpha_1\beta_7$	$i_1i_2i_3\alpha_1\beta_8$
$\boldsymbol{i}_1\alpha_2$	$i_1\alpha_2\beta_1$	$-\alpha_2\beta_2$	$i_1i_2\alpha_2\beta_3$	$-i_2\alpha_2\beta_5$	$i_1i_3\alpha_2\beta_4$	$-i_3\alpha_2\beta_6$	$i_1i_2i_3\alpha_2\beta_7$	$-i_2i_3\alpha_2\beta_8$
$\boldsymbol{i}_2\alpha_3$	$i_2\alpha_3\beta_1$	$i_1i_2\alpha_3\beta_2$	$-\alpha_3\beta_3$	$-i_1\alpha_3\beta_5$	$i_2i_3\alpha_3\beta_4$	$i_1i_2i_3\alpha_3\beta_6$	$-i_3\alpha_3\beta_7$	$-i_1i_3\alpha_3\beta_8$
$\boldsymbol{i}_3\alpha_4$	$i_3\alpha_4\beta_1$	$i_1i_3\alpha_4\beta_2$	$i_2i_3\alpha_4\beta_3$	$i_1i_2i_3\alpha_4\beta_5$	$-\alpha_4\beta_4$	$-i_1\alpha_4\beta_6$	$-i_2\alpha_4\beta_7$	$-i_1i_2\alpha_4\beta_8$
$\boldsymbol{i}_1\boldsymbol{i}_2\alpha_5$	$i_1i_2\alpha_5\beta_1$	$-i_2\alpha_5\beta_2$	$-i_1\alpha_5\beta_3$	$\alpha_5\beta_5$	$i_1i_2i_3\alpha_5\beta_4$	$-i_2i_3\alpha_5\beta_6$	$-i_1i_3\alpha_5\beta_7$	$i_3\alpha_5\beta_8$
$\boldsymbol{i}_1\boldsymbol{i}_3\alpha_6$	$i_1i_3\alpha_6\beta_1$	$-i_3\alpha_6\beta_2$	$i_1i_2i_3\alpha_6\beta_3$	$-i_2i_3\alpha_6\beta_5$	$-i_1\alpha_6\beta_4$	$\alpha_6\beta_6$	$-i_1i_2\alpha_6\beta_7$	$-i_2\alpha_6\beta_8$
$\boldsymbol{i}_2\boldsymbol{i}_3\alpha_7$	$i_2i_3\alpha_7\beta_1$	$i_1i_2i_3\alpha_7\beta_2$	$-i_3\alpha_7\beta_3$	$-i_1i_3\alpha_7\beta_5$	$-i_2\alpha_7\beta_4$	$-i_1i_2\alpha_7\beta_6$	$\alpha_7\beta_7$	$i_1\alpha_7\beta_8$
$\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3\alpha_8$	$i_1i_2i_3\alpha_8\beta_1$	$-i_2i_3\alpha_8\beta_2$	$-i_1i_3\alpha_8\beta_3$	$i_3\alpha_8\beta_5$	$-i_1i_2\alpha_8\beta_4$	$i_2\alpha_8\beta_6$	$i_1\alpha_8\beta_7$	$-\alpha_8\beta_8$

The product $t \circledast_{\mathbb{C}_3} \kappa$ will be given by the sum of 64 terms provided in Table 1.

Similarly, we can give multiplication table when the elements from $\mathbb{C}_3$ are written in bicomplex representation, i.e.,

$$\begin{aligned} t &= (a_1 + i_1a_2 + i_2a_3 + i_1i_2a_5) + i_3(a_4 + i_1a_6 + i_2a_7 + i_1i_2a_8) \\ &= \zeta_1 + i_3\zeta_2, \end{aligned}$$

where $\zeta_1 = a_1 + i_1a_2 + i_2a_3 + i_1i_2a_5$ and $\zeta_2 = a_4 + i_1a_6 + i_2a_7 + i_1i_2a_8$ are bicomplex numbers.

Or when they are written in complex representation, i.e.,

$$\begin{aligned} t &= (a_1 + i_1a_2) + i_2(a_3 + i_1a_5) + i_3(a_4 + i_1a_6) + i_2i_3(a_7 + i_1a_8) \\ &= w_1 + i_2w_2 + i_3w_3 + i_2i_3w_4, \end{aligned}$$

where $w_1 = a_1 + i_1a_2$, $w_2 = a_3 + i_1a_5$, $w_3 = a_4 + i_1a_6$, $w_4 = \alpha_7 + i_1a_8$.

Idempotent Elements in $\mathbb{C}_3$

Beside 0, 1 and $e(i_1i_2) = e_1 = (1 + i_1i_2)/2$, $e(-i_1i_2) = e_2 = (1 - i_1i_2)/2$ (these four are idempotent elements in $\mathbb{C}_2$), there are other idempotent elements in $\mathbb{C}_3$,

$$\begin{aligned} &e(i_1i_3) = e_3 = \frac{1 + i_1i_3}{2}, \quad (-i_1i_3) = e_4 = (1 - i_1i_3)/2, \\ &e(i_2i_3) = e_5 = (1 + i_2i_3)/2, \quad e(-i_2i_3) = e_6 = (1 - i_2i_3)/2 \end{aligned}$$

We know that $e_1e_2 = 0$. Similarly, we can check that $e_3e_4 = 0$, $e_5e_6 = 0$. We also know that the product of idempotent elements is again an idempotent element. Let us check different products of e_1, e_2, e_3, e_4, e_5 and e_6.

$$e_1e_3 = \frac{1 + i_1i_3 + i_1i_2 - i_2i_3}{4}, \tag{i}$$

$$e_1e_4 = \frac{1 - i_1i_3 + i_1i_2 + i_2i_3}{4}, \tag{ii}$$

$$e_1e_5 = \frac{1 - i_1i_3 + i_1i_2 + i_2i_3}{4}, \tag{iii}$$

$$e_1e_6 = \frac{1 + i_1i_3 + i_1i_2 - i_2i_3}{4}. \tag{iv}$$

From (i), (ii), (iii) and (iv), we can see that $e_1e_3 = e_1e_6$ and $e_1e_4 = e_1e_5$.

$$e_2e_3 = \frac{1 + i_1i_3 - i_1i_2 + i_2i_3}{4}, \tag{v}$$

$$e_2e_4 = \frac{1 - i_1i_3 - i_1i_2 - i_2i_3}{4}, \tag{vi}$$

$$e_2e_5 = \frac{1 + i_1i_3 - i_1i_2 + i_2i_3}{4}, \tag{vii}$$

$$e_2e_6 = \frac{1 - i_1i_3 - i_1i_2 - i_2i_3}{4}. \tag{viii}$$

From (v)-(viii), we see that $e_2e_4 = e_2e_6$ and $e_2e_3 = e_2e_5$.

Now,

$$e_3e_5 = \frac{1 + i_1i_3 - i_1i_2 + i_2i_3}{4}. \tag{ix}$$

From (v), (vii) and (ix), we see that

$$e_2e_3 = e_2e_5 = e_3e_5. \tag{x}$$

Eq. (x) gives us

$$e_7 = \frac{1 - i_1i_2 + i_2i_3 + i_1i_3}{4} .$$

Similarly, we can easily check that

$$e_1e_3 = e_1e_6 = e_3e_6. \tag{xi}$$

Eq. (xi) gives us

$$e_8 = \frac{1 + i_1i_3 + i_1i_2 - i_2i_3}{4},$$

$$e_4e_5 = e_1e_4 = e_1e_5., \tag{xii}$$

Eq. (xii) gives us

$$e_9 = \frac{1 - i_1i_3 + i_1i_2 + i_2i_3}{4},$$

$$e_2e_4 = e_2e_6 = e_4e_6. \tag{xiii}$$

Eq. (xii) gives us

$$e_{10} = \frac{1 - i_1i_3 - i_1i_2 - i_2i_3}{4}.$$

Also, we can verify that $e_2e_4e_6 = e_2e_4$ and $e_1e_3e_5 = 0$.

Hence, we have the following results:

Theorem 4.1. *There are ten nontrivial idempotent elements in* $\mathbb{C}_3$.

$$e_1 = (1 + i_1 i_2)/2,$$
$$e_2 = (1 - i_1 i_2)/2,$$
$$e_3 = \frac{1 + i_1 i_3}{2},$$
$$e_4 = (1 - i_1 i_3)/2,$$
$$e_5 = (1 + i_2 i_3)/2,$$
$$e_6 = (1 - i_2 i_3)/2,$$
$$e_7 = \frac{1 - i_1 i_2 + i_2 i_3 + i_1 i_3}{4},$$
$$e_8 = \frac{1 + i_1 i_3 + i_1 i_2 - i_2 i_3}{4},$$
$$e_9 = \frac{1 - i_1 i_3 + i_1 i_2 + i_2 i_3}{4},$$
$$e_{10} = \frac{1 - i_1 i_3 - i_1 i_2 - i_2 i_3}{4}.$$

Note 1. Including two trivial idempotent elements 0 and 1, there are twelve idempotent elements in $\mathbb{C}_3$.

Note 2. If we multiply two or more idempotent elements we are getting the result from these twelve elements only. Multiplication table of these twelve idempotent elements is given below:

Table 2. $ID_{\mathbb{C}_3} \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3}$

$\circledast_{\mathbb{C}_3}$	0	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	0	e_1	e_1	0	e_8	e_9	e_9	e_8	0	e_8	e_9	0
e_2	0	e_2	0	e_2	e_7	e_{10}	e_7	e_{10}	e_7	0	0	e_{10}
e_3	0	e_3	e_8	e_7	e_3	0	e_7	e_8	e_7	e_8	0	0
e_4	0	e_4	e_9	e_{10}	0	e_4	e_9	e_{10}	0	0	e_9	e_{10}
e_5	0	e_5	e_9	e_7	e_7	e_9	e_5	0	e_7	0	e_9	0
e_6	0	e_6	e_8	e_{10}	e_8	e_{10}	0	e_6	0	e_8	0	e_{10}
e_7	0	e_7	0	e_7	e_7	0	e_7	0	e_7	0	0	0
e_8	0	e_8	e_8	0	e_8	0	0	e_8	0	e_8	0	0
e_9	0	e_9	e_9	0	0	e_9	e_9	0	0	0	e_9	0
e_{10}	0	e_{10}	0	e_{10}	0	0	0	e_{10}	0	0	0	e_{10}

Observations. Let $ID_{\mathbb{C}_3} = \{0, 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$,

$0 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0\}$,

$1 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = ID_{\mathbb{C}_3}$,

$e_1 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_1, e_8, e_9\}$,

$e_2 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_2, e_7, e_{10}\}$,

$e_3 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_3, e_7, e_8\}$,

$e_4 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_4, e_9, e_{10}\}$,

$e_5 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_5, e_7, e_9\}$,

$e_6 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_6, e_8, e_{10}\}$,

$$e_7 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_7\},$$
$$e_8 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_8\},$$
$$e_9 \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_9\},$$
$$e_{10} \circledast_{\mathbb{C}_3} ID_{\mathbb{C}_3} = \{0, e_{10}\}.$$

5 Conclusion

In this study, we identified twelve idempotent elements in $\mathbb{C}_3$, which includes two trivial idempotent elements, 0 and 1, as well as two non-trivial idempotent elements from $\mathbb{C}_2$, namely e_1 and e_2. Table 1 presents the product of two tricomplex elements, while Table 2 provides the multiplication of idempotent elements, confirming that the set $ID_{\mathbb{C}_3}$ is closed under the operation $\circledast_{\mathbb{C}_3}$. Ongoing work focuses on a deeper investigation of tricomplex idempotent elements and their properties, which will be further developed in future research.

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