AUTOMEDIAN SETS OF SIGNED PERMUTATIONS IN HYPEROCTAHEDRAL GROUP OF TYPE $\mathcal{B}_n$

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Abstract. In this article, we investigate the structure of automedian sets within the hyperoctahedral group of type $\mathcal{B}_n$ of signed permutations under the generalized Kendall-τ distance metric. An automedian set in $\mathcal{B}_n$ is a term that refers to a subset of $\mathcal{B}_n$ that coincides with its own set of medians. We establish several characterizations and examples of such sets. We show that both the symmetric group $\mathcal{S}_n$ and the hyperoctahedral group of type $\mathcal{B}_n$ are automedian. Further, we characterize various automedian subsets, including singleton sets, specific two-element subsets consisting of pairs of permutations differing by first negation or transposition, and sets formed by cyclic shifts of a permutation. Explicit formulas for the generalized Kendall-τ distances within these sets are derived. Additionally, we classify larger families of automedian sets formed by concatenating signed permutations from $\mathcal{B}_k$ with unsigned permutations from $\mathcal{S}_{n-k}$ or with negatively signed permutations of the remaining elements, providing explicit formulas for their average distances. Finally, we explore the invariance properties of the generalized Kendall-τ distance under the natural group action of $\mathcal{B}_n$ on its power set, establishing foundational results that underpin the symmetry and structure of these automedian configurations.

1 Introduction

The problem of determining medians of a collection consisting of permutations with respect to the Kendall-τ distance [17, 24], which quantifies the number of pairwise disagreements in the relative ordering of two permutations, is a central challenge in rank aggregation and consensus-building across various disciplines. This problem, often referred to as the Kemeny Score Problem [16], has significant applications in social choice theory, decision-making, and data aggregation. Initially formulated in Kemeny's seminal work on ranking problems, the task involves determining a consensus order of n candidates based on rankings provided by m voters, such that the consensus minimizes the total Kendall-τ distance.

The Kemeny score problem is NP-complete when the number of input rankings m is an even number at least 4 [14, 6], and remains NP-hard for $m \geq 7$ when m is odd [4]. However, the complexity of the problem for smaller odd values such as $m = 3$ or $m = 5$ remains an open

question.

During the past two decades, significant advances have been made in the development of approximate algorithms, fixed-parameter approaches [9, 12], and a polynomial-time approximation scheme (PTAS) result [18]. Comparative studies of these methodologies can be found in [22, 2]. Moreover, theoretical frameworks have been proposed to reduce the search space for median-finding problems [8, 5], although no work has fully characterized polynomial-time solvable cases. Further refinements in data reduction with less restrictive conditions were proposed in [19], using the combinatorial properties of "almost adjacent" elements in median sets.

Moreover, modern efforts have explored novel paradigms, such as quantum optimization, to solve the Kemeny ranking aggregation problem. For instance, Combarro et al. [10] formulated the problem using multiple Quadratic Unconstrained Binary Optimization (QUBO) based encodings and evaluated their effectiveness using quantum approximate optimization algorithms and quantum annealing. These explorations, while still constrained by current hardware limitations, underscore the importance of efficient formulations in the preparation for scalable quantum solutions in the near future. Additionally, Rico et al. [21] proposed exact algorithms based on the necessary conditions for a ranking to be optimal under the Kemeny method, significantly reducing the computation time for instances with up to 14 alternatives. These advancements reinforce the centrality of the Kemeny problem in computational social choice and its evolving relevance in high-performance and hybrid computing contexts.

The notion of automedian sets, which are subsets of permutations that remain invariant under the median operation, offers a novel perspective on the median-finding problem in permutation spaces. These sets act as fixed points under the Kendall-τ distance, making them natural candidates for efficient solutions to combinatorial optimization problems that involve centrality. Recent research has explored the closure properties of automedian sets under operations such as the direct sum and shuffle product, providing constructive methods to build automedian sets in larger permutation groups [13]. Furthermore, new variants of automedian sets have been introduced that leverage direct sum constructions, along with parallel algorithms aimed at efficient median detection in separable permutation sets, enhancing the computational tractability of these problems. In particular, two simple, yet insightful cases of automedian sets have been identified in a recent study [15]. The first case considers sets consisting of a permutation and its cyclic shifts, while the second case analyzes sets composed of permutations sharing a S_k-kernel, potentially with some fixed or common elements. These cases shed light on the structural simplicity and symmetry contributing to the automedian property.

This problem has gained significant attention in computational biology, particularly in the study of genome rearrangement and gene expression patterns. In recent work, Cunha et al. [11] have conducted a comprehensive parameterized complexity analysis of the median and closest permutation problems under various genome rearrangement metrics. They have explored the computational complexity and structural aspects of permutation medians under various distance functions, such as swap, breakpoint, transposition, and block-interchange distances. They demonstrate that even when restricted to only three input permutations, most variants of the problem remain NP-hard. While their analysis focuses on unsigned permutations, it underscores the inherent intractability of consensus problems over permutation spaces and motivates the need to study the analogous median problem in the signed setting.

In our earlier work [23], we extended the idea of median sets to the hyperoctahedral group of type $\mathcal{B}_n$, which comprises signed permutations, which generalizes the symmetric group $\mathcal{S}_n$. The richer structure of $\mathcal{B}_n$ necessitates a refinement of classical notions like the Kendall-τ distance. We introduced a generalized Kendall-τ distance (see Definition 2.7) tailored to signed permutations by defining two types of order disagreements (Type I and Type II), along with a measure for sign differences. Our results demonstrated, for instance, that if a subset $\mathcal{A} \subseteq \mathcal{B}_n$ is closed under the total negation operation '$-$', then its median set equals the entire group: $\mathcal{M}(\mathcal{A}) = \mathcal{B}_n$. Likewise, if a subset consists solely of unsigned permutations and is closed under reversal, then its median set coincides with the symmetric group: $\mathcal{M}(\mathcal{A}) = \mathcal{S}_n$. In addition, we constructed a weighted distance graph $\mathcal{G}_n(V, E, \omega)$ over $\mathcal{B}_n$, where edges correspond to elementary opera-

tions such as adjacent transpositions and sign flips, allowing efficient computation of pairwise distances and medians.

Notably, the structure of $\mathcal{G}_n$ closely resembles that of a Cayley graph, as its edge set is induced by group-theoretic generators and captures the connectivity and symmetry inherent in $\mathcal{B}_n$. The graph-theoretic perspectives using Cayley graphs and signed structures have emerged as important tools. The work of Pranjali et al. [20] on canonically consistent Cayley signed graphs characterizes specific generating sets that produce consistent signed structures over finite groups, potentially providing symmetry-based insights relevant to automedian constructions.

An important motivation for extending the Kemeny ranking framework to signed permutations emerges from applications in computational biology. In gene regulatory network analysis, genes can be upregulated or downregulated under varying experimental conditions, resulting in signed orderings that encode both the rank and direction (activation or inhibition) of gene responses. Aggregating such data across multiple experiments necessitates identifying a consensus ranking that reflects not only the relative positioning of genes, but also their regulatory behavior. This leads to the problem of determining a signed consensus permutation that minimizes the generalized Kendall-τ distance from a collection of signed gene orderings. Solving this problem provides valuable insights into consistent regulatory patterns and supports a comprehensive understanding of gene expression dynamics. By incorporating directional information into rank aggregation, this extension enhances the applicability of median-based methods beyond the traditional symmetric group $\mathcal{S}_n$, offering new avenues for analysis in structured biological and computational systems.

It is also natural to ask whether the metric space formed by the hyperoctahedral group of type $\mathcal{B}_n$ under the generalized Kendall-τ distance exhibits $CAT(0)$ space-like behavior. If such a connection exists, it may offer a powerful geometric framework for studying the existence and uniqueness of medians, the convexity of automedian sets, and fixed point properties under group actions, potentially enriching both the combinatorial and geometric theory of signed permutations. For instance, in [3], Ali and Jubair establish fixed point and convergence results for generalized Suzuki-type contractions in $CAT(0)$ spaces, highlighting the analytical richness and geometric regularity of such structures. Algebraic methods have played a crucial role in understanding permutation-based structures, especially in [1].

Building upon these foundations, this article investigates the concept of automedian sets within the hyperoctahedral group of type $\mathcal{B}_n$. We propose novel constructions for automedian sets in this context and demonstrate their invariance under group actions. We begin by formally defining automedian sets and exploring fundamental examples and characterizations of automedian sets, including the symmetric group $\mathcal{S}_n$, the hyperoctahedral group of type $\mathcal{B}_n$ itself, singleton sets, and pairs of permutations differing only by first negation. Subsequently, we derive structural results for automedian sets formed through transpositions, cyclic shifts, and block-wise decompositions. The behavior of the generalized Kendall-τ distance under group actions is then examined, establishing invariance properties that are instrumental in characterizing the automedian nature of these sets.

Our main contributions include explicit characterizations of automedian sets in $\mathcal{B}_n$, along with insights into their combinatorial structure and geometric properties. These findings advance the theoretical understanding of automedian sets in signed permutation groups and suggest potential applications in ranking systems, clustering techniques, and data analysis involving signed or incomplete rankings.

This paper is structured as follows. Section 2 introduces essential definitions and results related to the generalized Kendall-τ distance in $\mathcal{B}_n$, along with a review of relevant findings from our previous work [23]. Section 3 presents core theoretical results on the structure and properties of automedian sets. In Section 4, we describe a natural action of the hyperoctahedral group of type $\mathcal{B}_n$ on the power set $\mathcal{P}(\mathcal{B}_n)$, and examine the behaviour of the generalized Kendall-τ distance with respect to this action. These results have important implications for the structure of automedian sets under the action of $\mathcal{B}_n$.

2 Preliminaries

In our previous work [23], we have investigated median sets in the hyperoctahedral group of type $\mathcal{B}_n$ with respect to a generalized Kendall-τ distance. Here, we recall the relevant definitions and results necessary for this article.

2.1 Definitions and Notations

Throughout this article, n is a positive integer. A **permutation** is a one-to-one and onto function from the set $[n] = \{1, 2, \ldots, n\}$ to itself. The collection of all such permutations, equipped with composition operation, forms a group known as the **symmetric group**, denoted by $\mathcal{S_n}$. The order of $\mathcal{S}_n$ is $n!$. By convention, we stick to the order of $\mathcal{S}_0$ as 1. We follow the one-line notation to write a permutation.

Definition 2.1 (Kendall-τ distance). [13] For π, $\sigma \in \mathcal{S}_n$, the Kendall-τ distance, denoted as d_{KT}, is defined as

$$d_{KT}(\pi, \sigma) = \left| \left\{ (i, j) \mid i < j \text{ and } \left[(\pi_i^{-1} < \pi_j^{-1} \text{ and } \sigma_i^{-1} > \sigma_j^{-1}) \right. \right. \right.$$
$$\left. \left. \left. \text{ or } (\pi_i^{-1} > \pi_j^{-1} \text{ and } \sigma_i^{-1} < \sigma_j^{-1}) \right] \right\} \right|.$$

That is, for any two permutations π and σ, this distance computes the number of disagreements in relative ordering between pairs of elements of π and σ.

Definition 2.2 (Hyperoctahedral group of type $\mathcal{B}_n$). [7] The hyperoctahedral group of type $\mathcal{B}_n$ is the group of signed permutations on n elements, representing the symmetry group of the n-dimensional hypercube. Formally, it consists of all bijections

$$\sigma : \{\pm 1, \pm 2, \ldots, \pm n\} \to \{\pm 1, \pm 2, \ldots, \pm n\}$$

such that $\sigma(-i) = -\sigma(i)$ for every integer $i \in \{1, 2, \ldots, n\}$.

Note that for any positive integer n, we have $|\mathcal{B}_n| = 2^n n!$ and $|\mathcal{B}_0| = 1$. For a signed permutation σ, the following window-like notation is used: $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$.

Definition 2.3 (Order disagreement of Type I). The order disagreement of Type I between pairs of elements of two signed permutations in $\mathcal{B}_n$ is defined as follows: For $\pi = \pi_1 \pi_2 \cdots \pi_n$, $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathcal{B}_n$,

$$O_1(\pi, \sigma) = \left\{ (i, j) \mid i < j \text{ and } \left[(|\pi_i^{-1}| < |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| > |\sigma_j^{-1}|) \right. \right.$$
$$\left. \left. \text{ or } (|\pi_i^{-1}| > |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| < |\sigma_j^{-1}|) \right] \right\}.$$

Definition 2.4 (Order disagreement of Type II). The order disagreement of Type II between pairs of elements of two signed permutations in $\mathcal{B}_n$ is defined as follows: For $\pi = \pi_1 \pi_2 \cdots \pi_n$, $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathcal{B}_n$,

$$O_2(\pi, \sigma) = \left\{ (i, j) \mid i < j \text{ and } \left[(|\pi_i^{-1}| < |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| < |\sigma_j^{-1}|) \right. \right.$$
$$\left. \text{ or } (|\pi_i^{-1}| > |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| > |\sigma_j^{-1}|) \right]$$
$$\text{ and } \left[(\pi_i^{-1} < \pi_j^{-1} \text{ and } \sigma_i^{-1} > \sigma_j^{-1}) \right.$$
$$\left. \left. \text{ or } (\pi_i^{-1} > \pi_j^{-1} \text{ and } \sigma_i^{-1} < \sigma_j^{-1}) \right] \right\}.$$

Remark 2.5. Each pair (i, j) contributes to at most one type of order disagreement (Type 1 or Type 2). That is, if a pair contributes to one type, it does not contribute to the other. It is also possible that a pair does not contribute to either type.

Definition 2.6 (Sign difference). The sign difference between the elements of signed permutations in $\mathcal{B}_n$ is defined as, for $\pi = \pi_1 \pi_2 \cdots \pi_n, \ \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathcal{B}_n$,

$$\text{sgn}(\pi, \sigma) = \{i \in \mathbb{N} \mid \text{sgn}(\pi_i^{-1}) \, \text{sgn}(\sigma_i^{-1}) < 0\}$$

where $\text{sgn}(\alpha)$ denotes the sign of $\alpha \in \mathbb{Z} \setminus \{0\}$, which is $+1$ for positive elements and -1 for negative elements.

Definition 2.7 (Generalized Kendall-τ distance). For $\pi, \ \sigma \in \mathcal{B}_n$, the generalized Kendall-τ distance, denoted as $\overline{d_{KT}}$, is defined as follows:

$$\overline{d_{KT}}(\pi, \sigma) = 2|O_1(\pi, \sigma)| + 4|O_2(\pi, \sigma)| + |\text{sgn}(\pi, \sigma)|.$$

Example 2.8. Consider $\pi = 1 - 3\ 2$, $\sigma = -1\ 2\ 3 \in \mathcal{B}_n$. To compute the generalized Kendall-τ distance between π and σ, consider $\pi^{-1} = 1\ 3 - 2$ and $\sigma^{-1} = -1\ 2\ 3$ to be the inverses of π and σ, respectively. Clearly, the pair $(2, 3)$ contributes to the order disagreement of Type I, and the pair $(1, 3)$ contributes to the order disagreement of Type II. In addition, the pair $(1, 2)$ does not contribute to any of the order disagreement types. It is easy to check $sgn(\pi, \sigma) = \{1, 3\}$. Thus we have

$$\overline{d_{KT}}(\pi, \sigma) = 2(1) + 4(1) + 2 = 8$$

The following describes the problem of determining the median of a set of signed permutations $\mathcal{A}$ in $\mathcal{B}_n$ under the generalized Kendall-τ distance.

Given any subset $\mathcal{A} \subseteq \mathcal{B}_n$ and a signed permutation $\pi \in \mathcal{B}_n$, we have

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \sum_{\sigma \in \mathcal{A}} \overline{d_{KT}}(\pi, \sigma).$$

Definition 2.9 (Medians). Given $\mathcal{A} \subseteq \mathcal{B}_n$, a median of $\mathcal{A}$ under the generalized Kendall-τ distance is a signed permutation $\pi^* \in \mathcal{B}_n$ satisfying $\overline{d_{KT}}(\pi^*, \mathcal{A}) \leq \overline{d_{KT}}(\pi, \mathcal{A}), \ \forall \pi \in \mathcal{B}_n$.

Define the set $\mathcal{M}(\mathcal{A})$ to be the set consisting of all medians of $\mathcal{A}$. i.e.,

$$\mathcal{M}(\mathcal{A}) = \{\sigma \in \mathcal{B}_n \mid \overline{d_{KT}}(\sigma, \mathcal{A}) \leq \overline{d_{KT}}(\pi, \mathcal{A}), \forall \pi \in \mathcal{B}_n\}$$

Definition 2.10 (Universal-median set). A set $\mathcal{A} \subseteq \mathcal{B}_n$ is said to be a Universal-median set if $\mathcal{M}(\mathcal{A}) = \mathcal{B}_n$.

Definition 2.11 ($\mathcal{S}_n$-median set). A set $\mathcal{A} \subseteq \mathcal{B}_n$ is said to be a $\mathcal{S}_n$-median set if $\mathcal{M}(\mathcal{A}) = \mathcal{S}_n$.

Definition 2.12 (Transposition operation). A unary operation on $\pi = \pi_1 \cdots \pi_n \in \mathcal{B}_n$, called "transposition operation" denoted by t_i, for any $1 \leq i \leq n - 1$, is given by

$$t_i : \mathcal{B}_n \to \mathcal{B}_n \text{ defined as}$$

$$t_i(\pi) = \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \pi_{i+2} \cdots \pi_n.$$

Definition 2.13 (Negation operation). A unary operation on $\pi = \pi_1 \cdots \pi_n \in \mathcal{B}_n$, called "negation operation" denoted by η_i, for any $1 \leq i \leq n$, is given by

$$\eta_i : \mathcal{B}_n \to \mathcal{B}_n \text{ defined as}$$

$$\eta_i(\pi) = \pi_1 \cdots \pi_{i-1} - \pi_i \ \pi_{i+1} \cdots \pi_n.$$

Definition 2.14 (Total negation operation). A unary operation '$-$' on the elements of $\mathcal{B}_n$ is defined to be the map

$$- : \mathcal{B}_n \to \mathcal{B}_n \text{ such that}$$

$$-(\pi_1 \pi_2 \cdots \pi_n) = -\pi_1 - \pi_2 \cdots - \pi_n.$$

The image of $\pi = \pi_1 \pi_2 \cdots \pi_n$ with respect to this operation '$-$' is denoted by π^- and we call the operation "$-$" a total negation operation.

2.2 Some results on median of signed permutation

Theorem 2.15. *The generalized Kendall-τ distance, $\overline{d_{KT}}$ forms a metric on $\mathcal{B}_n$, i.e., the function*

$$\overline{d_{KT}} : \mathcal{B}_n \times \mathcal{B}_n \to \mathbb{R}$$

that satisfies the following axioms for all signed permutations $\pi, \sigma, \alpha \in \mathcal{B}_n$:

(i) $\overline{d_{KT}}(\pi, \sigma) \geq 0$ *for $\pi \neq \sigma$ (Non-negativity)*

(ii) $\overline{d_{KT}}(\pi, \sigma) = 0$ *if and only if $\pi = \sigma$*

(iii) $\overline{d_{KT}}(\pi, \sigma) = \overline{d_{KT}}(\sigma, \pi)$ *(Symmetry)*

(iv) $\overline{d_{KT}}(\pi, \sigma) \leq \overline{d_{KT}}(\pi, \alpha) + \overline{d_{KT}}(\alpha, \sigma)$. *(Triangle inequality)*

Lemma 2.16. *Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be an element of $\mathcal{B}_n$ and consider $\widehat{\pi_r} = \eta_r(\pi) = \pi_1 \cdots \pi_{r-1} - \pi_r \, \pi_{r+1} \cdots \pi_n$. Then,*

$$\overline{d_{KT}}(\pi, \widehat{\pi_r}) = 1 + 4(r-1).$$

Theorem 2.17. *If $\mathcal{A}$ is a subset of signed permutations of $\mathcal{B}_n$ with cardinality m and is closed under operation '$-$', (i.e., for all $\pi \in \mathcal{A}$, we have $\pi^- \in \mathcal{A}$), then the set of medians is equal to $\mathcal{B}_n$. That is,*

$$\mathcal{M}(\mathcal{A}) = \mathcal{B}_n.$$

Moreover, for any $\pi \in \mathcal{A}$, the generalized Kendall-τ distance between π and $\mathcal{A}$ satisfies

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\pi^-, \mathcal{A}) = \frac{m}{2} n(2n-1).$$

Remark 2.18. *For any element $\pi \in \mathcal{B}_n$ and any non-empty subset $\mathcal{A} \subseteq \mathcal{B}_n$, we have*

$$\overline{d_{KT}}(\pi, \mathcal{A}) + \overline{d_{KT}}(\pi, \mathcal{A}^c) = \frac{|\mathcal{B}_n|}{2} n(2n-1).$$

Theorem 2.19. *If $\mathcal{A}$ is a subset of the symmetric group $\mathcal{S}_n \subset \mathcal{B}_n$ of cardinality m and $\mathcal{A}$ is closed under the unary operation 'rev' (i.e., whenever $\pi \in \mathcal{A}$, then $\pi^{rev} \in \mathcal{A}$), then $\mathcal{A}$ is a $\mathcal{S}_n$-median set. That is,*

$$\mathcal{M}(\mathcal{A}) = \mathcal{S}_n.$$

and also for any $\sigma \in \mathcal{S}_n$,

$$\overline{d_{KT}}(\sigma, \mathcal{A}) = \frac{m}{2} n(n-1).$$

2.3 Distance graph $\mathcal{G}_n$ of $\mathcal{B}_n$

We define the distance graph $\mathcal{G}_n(V, E, \omega)$ for $\mathcal{B}_n$, where

- the set of vertices V are the signed permutations in $\mathcal{B}_n$ written in one-line notation, i.e., $V = \mathcal{B}_n$,

- $e = (\pi, \sigma) \in E$ iff σ is obtained from π by using the 'first negation operation η_1' (refer Definition 2.13) or the 'transposition operation t_i' (refer Definition 2.12) and vice versa.

- the weight function $\omega : E \to \mathbb{N}$ defined as

$$\omega(e) = \omega((\pi, \sigma)) = \begin{cases} 1, & \text{if } \sigma = \eta_1(\pi) \\ 2, & \text{if } \sigma = t_i(\pi). \end{cases}$$

The graph $\mathcal{G}_n$ is a weighted, rooted, connected graph with identity-signed permutation as root. This graph clearly does not contain loops since $\pi \neq \eta_1(\pi)$ and $\pi \neq t_i(\pi)$ for any signed permutation $\pi \in \mathcal{B}_n$.

In the context of the distance graph $\mathcal{G}_n$, the generalized Kendall-τ distance (see Definition 2.7) between any two signed permutations $\pi, \sigma \in \mathcal{B}_n$ is defined as the minimum total weight of a path that connects π and σ in the graph $\mathcal{G}_n$.

The distance graph $\mathcal{G}_n$ can be constructed by following simple steps.

Step 1: Begin with the identity element $e = 12 \cdots n$ in $\mathcal{B}_n$ as the root of the graph.

Step 2: In each step, apply the operations η_1 and t_i to every vertex generated in the previous step. For a given node $\pi = \pi_1 \pi_2 \cdots \pi_n$:

- Connect π to the element $-\pi_1 \pi_2 \cdots \pi_n$ with an edge of weight 1.
- Connect π to the elements $\pi_1 \cdots \pi_{i+1} \pi_i \cdots \pi_n$ with an edge of weight 2, for every $1 \leq i \leq n - 1$.

Step 3: Repeat the process iteratively until all elements of $\mathcal{B}_n$ are included in the graph.

3 Automedian set of signed permutations

In the study of signed permutations in $\mathcal{B}_n$ under the generalized Kendall-τ distance, a natural question arises about subsets of the hyperoctahedral group of type $\mathcal{B}_n$ that remain unchanged under the median operation. Such subsets, which coincide with their own median set, possess some intrinsic symmetry and structure. To formalize this idea, we introduce the notion of an automedian set.

Definition 3.1 (Automedian set). An automedian set is the set $\mathcal{A}$ of all signed permutations in $\mathcal{B}_n$ such that $\mathcal{A}$ equals their collection of all medians, i.e., $\mathcal{A} = \mathcal{M}(\mathcal{A})$.
In other words,

- $\overline{d_{KT}}(\pi_1, \mathcal{A}) = \overline{d_{KT}}(\pi_2, \mathcal{A})$, $\forall \pi_1, \pi_2 \in \mathcal{A}$,
- $\overline{d_{KT}}(\pi, \mathcal{A}) < \overline{d_{KT}}(\sigma, \mathcal{A})$, $\forall \pi \in \mathcal{A}$, $\forall \sigma \in \mathcal{B}_n \setminus \mathcal{A}$.

Having introduced automedian sets, it is natural to ask whether the hyperoctahedral group of type $\mathcal{B}_n$ and the symmetric group $\mathcal{S}_n$ exhibit this property. The following theorem establishes that both of these are automedian sets.

Theorem 3.2. *The symmetric group $\mathcal{S}_n$ and the hyperoctahedral group of type $\mathcal{B}_n$ are automedian sets.*

Proof. Since $\mathcal{B}_n$ is closed under the operation '$-$' and $\mathcal{S}_n$ is closed under the unary operation 'rev ', the result easily follows from Theorems 2.17 and 2.19, respectively. □

To build intuition, we begin by considering the simplest possible cases of automedian sets in $\mathcal{B}_n$, namely, singleton sets and two-element subsets consisting of an element π and its 'first negation $\eta_1(\pi) = \hat{\pi}$'.

Theorem 3.3. *Let $\mathcal{A} = \{\pi\}$ be a singleton set, where $\mathcal{A} \subset \mathcal{B}_n$. Then $\mathcal{A}$ is an automedian set and $\overline{d_{KT}}(\pi, \mathcal{A}) = 0$, for $\pi \in \mathcal{A}$.*

Proof. By the metric properties of the generalized Kendall-τ distance $\overline{d_{KT}}$, we have the following:

$$\overline{d_{KT}}(\pi, \pi) = 0 \quad \text{and} \quad \overline{d_{KT}}(\pi, \sigma) > 0 \quad \text{for} \quad \pi \neq \sigma.$$

Now, consider the set $\mathcal{A} = \{\pi\}$. We have already defined the median set of $\mathcal{A}$ as

$$\mathcal{M}(\mathcal{A}) = \left\{\rho \in \mathcal{B}_n \mid \overline{d_{KT}}(\rho, \mathcal{A}) \leq \overline{d_{KT}}(\sigma, \mathcal{A}) \quad \forall \sigma \in \mathcal{B}_n\right\}.$$

Since $\overline{d_{KT}}(\pi, \mathcal{A}) = 0$ and $\overline{d_{KT}}(\sigma, \mathcal{A}) > 0$ for $\sigma \neq \pi$, it follows that

$$\mathcal{M}(\mathcal{A}) = \{\pi\}.$$

Thus, $\mathcal{A}$ is automedian, i.e., $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. Additionally, since $\overline{d_{KT}}(\pi, \mathcal{A}) = 0$, this completes the proof.

$\square$

Theorem 3.4. *Let* $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{B}_n$ *and let* $\hat{\pi} = -\pi_1 \pi_2 \cdots \pi_n$. *If* $\mathcal{A} = \{\pi, \hat{\pi}\}$, *then* $\mathcal{A}$ *is an automedian set and*

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\hat{\pi}, \mathcal{A}) = 1.$$

Proof. Let $\mathcal{A} = \{\pi, \hat{\pi}\}$ where $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{B}_n$ and $\hat{\pi} = -\pi_1 \pi_2 \cdots \pi_n$. We begin by observing that

$$\mathcal{M}(\mathcal{A}) = \{\rho \in \mathcal{B}_n \mid \overline{d_{KT}}(\rho, \mathcal{A}) \leq \overline{d_{KT}}(\sigma, \mathcal{A}) \text{ for all } \sigma \in \mathcal{B}_n\},$$

That is, the median set of $\mathcal{A}$ consists of the signed permutations that minimize the distance $\overline{d_{KT}}(\sigma, \mathcal{A})$ for all $\sigma \in \mathcal{B}_n$.

By the metric properties of the generalized Kendall-τ distance and the Lemma 2.16, we have the following distances:

$$\overline{d_{KT}}(\pi, \pi) = 0 \quad \text{and} \quad \overline{d_{KT}}(\hat{\pi}, \hat{\pi}) = 0,$$

$$\overline{d_{KT}}(\pi, \hat{\pi}) = 1, \quad \text{and} \quad \overline{d_{KT}}(\hat{\pi}, \pi) = 1,$$

the latter ones are easy to see since $\hat{\pi}$ differs from π only by the sign of the first element. This implies that

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\pi, \pi) + \overline{d_{KT}}(\pi, \hat{\pi}) = 1.$$

Similarly, for $\hat{\pi}$,

$$\overline{d_{KT}}(\hat{\pi}, \mathcal{A}) = \overline{d_{KT}}(\hat{\pi}, \pi) + \overline{d_{KT}}(\hat{\pi}, \hat{\pi}) = 1.$$

Now, for any $\sigma \in \mathcal{B}_n \setminus \mathcal{A}$, we compute the distance $\overline{d_{KT}}(\sigma, \mathcal{A})$. We have

$$\overline{d_{KT}}(\sigma, \mathcal{A}) = \overline{d_{KT}}(\sigma, \pi) + \overline{d_{KT}}(\sigma, \hat{\pi}) \geq 2,$$

since both $\overline{d_{KT}}(\sigma, \pi) > 0$ and $\overline{d_{KT}}(\sigma, \hat{\pi}) > 0$ as $\sigma \neq \pi, \hat{\pi}$.

Thus, no element outside of $\mathcal{A}$ achieves a smaller distance to $\mathcal{A}$ than π and $\hat{\pi}$. Therefore, $\mathcal{A} = \mathcal{M}(\mathcal{A})$, implying that $\mathcal{A}$ is its own median set.

Finally, we conclude that

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\hat{\pi}, \mathcal{A}) = 1,$$

as required.

$\square$

Remark 3.5. It is easy to see that if $\overline{d_{KT}}(\pi, \sigma) = 1$, then $\sigma = \hat{\pi} = -\pi_1 \pi_2 \cdots \pi_n$.

The next result identifies automedian sets formed by a signed permutation and one obtained by applying the transposition operation t_i, for any $i = 1, .., n$. The following lemma is helpful to prove this result.

Lemma 3.6. *Consider a signed permutation* $\pi = \pi_1 \cdots \pi_n \in \mathcal{B}_n$ *and for* $i < j$, *let us take* $\sigma = \pi_1 \cdots \pi_{i-1} \pi_j \pi_{i+1} \cdots \pi_{j-1} \pi_i \pi_{j+1} \cdots \pi_n$. *Then we have*

$$\overline{d_{KT}}(\pi, \sigma) = 2 + 4(j - i - 1).$$

Proof. Let $\pi_i = k_1$ and $\pi_j = k_2$.

Consider the inverses of π and σ respectively as

$$\pi^{-1} = \pi_1^{-1} \cdots \pi_{|k_1|-1}^{-1} \ \epsilon(k_1)i \ \pi_{|k_1|+1}^{-1} \cdots \pi_{|k_2|-1}^{-1} \ \epsilon(k_2)j \ \pi_{|k_2|+1}^{-1} \cdots \pi_n^{-1},$$

and

$$\sigma^{-1} = \pi_1^{-1} \cdots \pi_{|k_1|-1}^{-1} \ \epsilon(k_1)j \ \pi_{|k_1|+1}^{-1} \cdots \pi_{|k_2|-1}^{-1} \ \epsilon(k_2)i \ \pi_{|k_2|+1}^{-1} \cdots \pi_n^{-1},$$

where $\epsilon(k_1)$ and $\epsilon(k_2)$ are sign of k_1 and k_2 respectively. i.e., $\pi^{-1} = \sigma^{-1}$ except at positions $|k_1|$ and $|k_2|$.

Clearly, $\text{sgn}(\pi, \sigma) = \emptyset$.

Let us examine the Type I and Type II order disagreements between π and σ.

For Type I, we look for the relative ordering between the absolute values of the elements of π^{-1} and σ^{-1}, i.e.,

$$|\pi^{-1}| = |\pi_1^{-1}| \cdots |\pi_{|k_1|-1}^{-1}| \ |\epsilon(k_1)i| \ |\pi_{|k_1|+1}^{-1}| \cdots |\pi_{|k_2|-1}^{-1}| \ |\epsilon(k_2)j| \ |\pi_{|k_2|+1}^{-1}| \cdots |\pi_n^{-1}|$$

$$|\sigma^{-1}| = |\pi_1^{-1}| \cdots |\pi_{|k_1|-1}^{-1}| \ |\epsilon(k_1)j| \ |\pi_{|k_1|+1}^{-1}| \cdots |\pi_{|k_2|-1}^{-1}| \ |\epsilon(k_2)i| \ |\pi_{|k_2|+1}^{-1}| \cdots |\pi_n^{-1}|.$$

The only positions where we get disagreements are the pairs containing $|\epsilon(k_1)i|$ (i.e., i) or $|\epsilon(k_2)j|$ (i.e., j) at positions $|k_1|$ and $|k_2|$ respectively. Then we have three possibilities to examine. They are:

(i) Pairs containing i and $|\pi_l^{-1}|$ in π^{-1} for $l \neq |k_1|$ and $|k_2|$,

(ii) Pairs containing j and $|\pi_l^{-1}|$ in π^{-1} for $l \neq |k_1|$ and $|k_2|$,

(iii) The pair (i, j) at positions $(|k_1|, |k_2|)$.

(i) Pairs containing $|\epsilon(\mathbf{k_1})\mathbf{i}| = \mathbf{i}$ and $|\pi_\mathbf{l}^{-1}|$ in π^{-1} for $\mathbf{l} \neq |\mathbf{k_1}|, |\mathbf{k_2}|$.

Without loss of generality, let us take $|\epsilon(k_1)i| = i < |\pi_l^{-1}|$ in π^{-1}.

Case 1. If $|\epsilon(k_1)j| > |\pi_l^{-1}|$ in $|\sigma^{-1}|$, for all $l \neq |k_1|, |k_2|$.

Since $i < |\pi_l^{-1}|$ and $j > |\pi_l^{-1}|$, these pairs contribute to Type I order disagreements, and because $i < |\pi_l^{-1}| < j$, we have $(j - i - 1)$ such pairs.

Case 2. If $|\epsilon(k_1)j| < |\pi_l^{-1}|$ in $|\sigma^{-1}|$ for all $l \neq |k_1|, |k_2|$.

Then we have $i < j < |\pi_l^{-1}|$ since $i < j$ and in this case there are no Type I order disagreements. So let us check for Type II.

Here for each l, π_l^{-1} could be positive or negative.

Subcase 1. $\pi_l^{-1} > 0$, for each $l \neq |k_1|, |k_2|$.

We must have $\epsilon(k_1)i < \pi_l^{-1}$ and $\epsilon(k_1)j < \pi_l^{-1}$. In this case, we clearly see that there are no Type II order disagreements for any l.

Subcase 2. $\pi_l^{-1} < 0$, for each $l \neq |k_1|, |k_2|$.

Then $i < j < -\pi_l^{-1}$ implies $\pi_l^{-1} < -j < -i < i < j$.

$$\epsilon(k_1)i > \pi_l^{-1} \text{ and } \epsilon(k_1)j > \pi_l^{-1} \text{ for any } l.$$

Therefore, these pairs contribute nothing to Type II order disagreements.

(ii) Pairs containing $|\epsilon(\mathbf{k_2})\mathbf{j}| = \mathbf{j}$ and $|\pi_1^{-1}|$ in π^{-1} for $l \neq |\mathbf{k_1}|, |\mathbf{k_2}|$.

The same argument follows for this, analogous to (i), there are $(j - i - 1)$ pairs contributing as Type I order disagreements.

(iii) The pair $(\mathbf{i}, \mathbf{j})$ at positions $(|\mathbf{k_1}|, |\mathbf{k_2}|)$.

Clearly, this pair contributes to $\overline{d_{KT}}(\pi, \sigma)$, since:

$$|\epsilon(k_1)i| < |\epsilon(k_2)j| \text{ in } |\pi^{-1}| \quad \text{and} \quad |\epsilon(k_1)j| > |\epsilon(k_2)i| \text{ in } |\sigma^{-1}|$$

Thus, this pair also contributes to Type I order disagreements.

By combining all the cases dealt in (i), (ii), and (iii), we get the total distance:

$$\overline{d_{KT}}(\pi, \sigma) = 2|O_1(\pi, \sigma)| = 2[1 + (j - i - 1) + (j - i - 1)] = 2 + 4 \cdot (j - i - 1)$$

where $\sigma = \pi$ except the swapping at positions i and j.   □

Theorem 3.7. *Let $\pi = \pi_1 \cdots \pi_n$ be any element in $\mathcal{B}_n$ and for any $1 \leq i \leq n-1$, let $\sigma = t_i(\pi) = \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i\pi_{i+2} \cdots \pi_n$. Consider the subset $\mathcal{A} = \{\pi, \sigma\}$ of $\mathcal{B}_n$. Then $\mathcal{A}$ is an automedian set. That is, $\mathcal{A} = \mathcal{M}(\mathcal{A})$.*

Proof. To show that $\mathcal{A}$ is automedian, we have to prove the following:

(i) for any $\pi, \sigma \in \mathcal{A}$, $\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\sigma, \mathcal{A})$,

(ii) for any $\rho \in \mathcal{B}_n \setminus A$, $\overline{d_{KT}}(\rho, \mathcal{A}) > \overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\sigma, \mathcal{A})$.

(i) By substituting $j = i + 1$ in the above lemma, we get $\overline{d_{KT}}(\pi, \sigma) = 2 + 4(i + 1 - i - 1) = 2$.

Now, $\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\pi, \pi) + \overline{d_{KT}}(\pi, \sigma) = 0 + 2 = 2$, and
$\overline{d_{KT}}(\sigma, \mathcal{A}) = \overline{d_{KT}}(\sigma, \pi) + \overline{d_{KT}}(\sigma, \sigma) = 2 + 0 = 2$.

This shows our claim.

(ii) To show that for any $\rho \in \mathcal{B}_n \setminus A$, $\overline{d_{KT}}(\rho, \mathcal{A}) > 2$, that is, $\overline{d_{KT}}(\rho, \mathcal{A}) = \overline{d_{KT}}(\rho, \pi) + \overline{d_{KT}}(\rho, \sigma) > 2$.

The above sum cannot be 0 as $\rho \neq \pi$ and $\rho \neq \sigma$. And this sum cannot be 1 as well because if it is the case, then we must have one of the two terms equal to 0, implying that $\rho = \pi$ or $\rho = \sigma$.

This sum cannot be 2 also; otherwise, the sum $\overline{d_{KT}}(\rho, \pi) + \overline{d_{KT}}(\rho, \sigma)$ can be either 2+0 or 0+2 or 1+1. The former two cannot happen, as we argued above. The latter also cannot be true; if it is, then by Theorem 3.4, ρ will be of the form $\rho = -\pi_1\pi_2 \cdots \pi_n$ and $\rho = -\sigma_1\sigma_2 \cdots \sigma_n$, according

to the first and second terms of the above sum, respectively. This further forces $\rho_j = \pi_j = \sigma_j$ for all $j \neq 1$, which is a contradiction to the fact that, at positions i and $i + 1$, we have π_i, π_{i+1} in π and π_{i+1}, π_i in σ, respectively. Therefore, $\overline{d_{KT}}(\rho, \mathcal{A}) = \overline{d_{KT}}(\rho, \pi) + \overline{d_{KT}}(\rho, \sigma)$ must be greater than 2, which proves the theorem. □

Another class of transformations with elegant distance properties is given by cyclic shifts of positions. We formally define these left-shift operations, which will be instrumental in constructing further examples of automedian sets.

Definition 3.8 (i^{th}-left shift operation). For $1 \leq i \leq n - 1$, define an unary operation 'l_i' on the elements of $\mathcal{B}_n$ as

$$l_i(\pi_1 \pi_2 \cdots \pi_n) = \pi_{i+1} \pi_{i+2} \cdots \pi_n \pi_1 \cdots \pi_i$$

for $\pi = \pi_1 \pi_2 \cdots \pi_n$. This operation is called the ith-left shift operation.

We now examine how the generalized Kendall-τ distance behaves between a signed permutation and its successive left shifts.

Lemma 3.9. *Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be an element of $\mathcal{B}_n$. Then we have,*

(i) $\overline{d_{KT}}(\pi, l_1(\pi)) = \overline{d_{KT}}(\pi_1 \cdots \pi_n, \ \pi_2 \cdots \pi_n \pi_1) = 2(n - 1)$

(ii) $\overline{d_{KT}}(\pi, l_2(\pi)) = \overline{d_{KT}}(\pi_1 \cdots \pi_n, \ \pi_3 \cdots \pi_n \pi_1 \pi_2) = 2 \cdot 2(n - 2)$

(iii) $\overline{d_{KT}}(\pi, l_i(\pi)) = \overline{d_{KT}}(\pi_1 \cdots \pi_n, \ \pi_{i+1} \cdots \pi_n \pi_1 \cdots \pi_i) = 2 \cdot i(n - i), \text{ for } 0 \leq i \leq n - 1.$

Proof. **(i)** To compute $\overline{d_{KT}}(\pi, l_1(\pi))$, we trace a path in the distance graph G_n connecting π to $l_1(\pi)$. The transformation proceeds as follows: Swap π_1 iteratively with $\pi_2, \pi_3, \ldots, \pi_n$, placing π_1 in position n. Thus, we have the following path in G_n:

$$\pi \to \pi_2 \pi_1 \pi_3 \cdots \pi_n \to \pi_2 \pi_3 \pi_1 \cdots \pi_n \to \cdots \to \pi_2 \pi_3 \cdots \pi_n \pi_1$$

This requires $n - 1$ swaps, contributing a total distance of $2(n - 1)$.

(ii) Now, to compute $\overline{d_{KT}}(\pi, l_2(\pi))$, we trace a path on the distance graph G_n connecting π to $l_2(\pi)$. The transformation proceeds as follows: First, swap π_2 iteratively with $\pi_3, \pi_4, \ldots, \pi_n$, placing π_2 in n^{th} position from second position. This requires $n - 2$ swaps contributing $2(n - 2)$. Repeat this process for π_1, moving it to the $(n - 1)^{th}$ position through $n - 2$ swaps, which adds $2(n - 2)$ to the distance. Thus, we have the following path in G_n:

$$\pi \to \pi_1 \pi_3 \pi_2 \pi_4 \cdots \pi_n \to \pi_1 \pi_3 \pi_4 \cdots \pi_n \pi_2 \to \pi_3 \pi_1 \pi_4 \cdots \pi_n \pi_2 \to \cdots \to \pi_3 \pi_4 \cdots \pi_n \pi_1 \pi_2$$

The total distance accumulated through these swaps is given by:

$$\begin{aligned} \overline{d_{KT}}(\pi, l_2(\pi)) &= 2(n - 2) + 2(n - 2) \\ &= 2 \cdot 2(n - 2). \end{aligned}$$

(iii) Now to compute $\overline{d_{KT}}(\pi, l_i(\pi))$, for $0 \leq i \leq n - 1$, we trace a path in the distance graph G_n connecting π to $l_i(\pi)$. The transformation proceeds as follows: Swap π_i iteratively with $\pi_{i+1}, \pi_{i+2}, \ldots, \pi_n$, placing π_i in position n. Thus, we have the following path in G_n:

$$\pi \to \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \pi_{i+2} \cdots \pi_n \to \cdots \to \pi_1 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_n \pi_i$$

This requires $n - i$ swaps, contributing a total distance of $2(n - i)$. Repeat this process for π_{i-1}, moving it to position $n - 1$ through $n - i$ swaps, contributing $2(n - i)$. In G_n, this is given by the path:

$$\pi_1 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_n \pi_i \to \pi_1 \cdots \pi_{i-2} \pi_{i+1} \pi_{i-1} \pi_{i+2} \cdots \pi_n \pi_i$$

$$\to \cdots \to \pi_1 \cdots \pi_{i-2} \pi_{i+1} \cdots \pi_n \pi_{i-1} \pi_i$$

Continue this process for all π_k until we reach π_1, which is moved to its required position with $n - i$ swaps, to obtain $\pi_{i+1} \cdots \pi_n \pi_1 \pi_2 \cdots \pi_i$.

The total distance accumulated through these swaps is given by:

$$\begin{aligned}
\overline{d_{KT}}(\pi, l_i(\pi)) &= 2(n - i) + 2(n - i) + \cdots + 2(n - i)(i \text{ times}) \\
&= 2i(n - i)
\end{aligned}$$

□

Extending this analysis, we compute the distance between two arbitrary left shifts of the same signed permutation in the following lemma.

Lemma 3.10. *Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be an element of $\mathcal{B}_n$. Then we have, for $0 \le i \le j \le n - 1$,*

$$\begin{aligned}
\overline{d_{KT}}(l_i(\pi), l_j(\pi)) &= \overline{d_{KT}}(\pi_{i+1} \cdots \pi_n \pi_1 \cdots \pi_i, \ \pi_{j+1} \cdots \pi_n \pi_1 \cdots \pi_j) \\
&= 2(j - i)(n - (j - i))
\end{aligned}$$

Proof. The proof is similar to the proof of the previous lemma. To compute $\overline{d_{KT}}(l_i(\pi), l_j(\pi))$, we trace a path in the distance graph G_n connecting $l_i(\pi)$ to $l_j(\pi)$. The transformation proceeds by iteratively swapping π_j with the elements next to its right in $l_i(\pi)$ and moving it to the last position. This requires $n - (j - i)$ swaps, contributing a total distance of $2(n - (j - i))$. Thus, we have the following path in G_n:

$$l_i(\pi) \to \pi_{i+1} \cdots \pi_{j-1} \pi_{j+1} \pi_j \pi_{j+2} \cdots \pi_n \pi_1 \cdots \pi_i \to \cdots \to \pi_{i+1} \cdots \pi_{j-1} \pi_{j+1} \cdots \pi_n \pi_1 \cdots \pi_i \pi_j$$

Next, we swap π_{j-1} iteratively with all the elements to its right and place π_{j-1} before the element π_j and next to π_i. This is done in $n - (j - i)$ swaps, adding $2(n - (j - i))$ to the total distance. For now we obtain $\pi_{i+1} \cdots \pi_{j-2} \pi_{j+1} \cdots \pi_n \pi_1 \cdots \pi_i \pi_{j-1} \pi_j$. Continue this process for all π_k until we swap π_{i+1} to its required position which is next (right) to π_i with $n - (j - i)$ swaps to obtain $\pi_{j+1} \cdots \pi_n \pi_1 \cdots \pi_i \pi_{i+1} \cdots \pi_{j-1} \pi_j$. The total distance accumulated through these swaps is given by:

$$\begin{aligned}
\overline{d_{KT}}(l_i(\pi), l_j(\pi)) &= 2(n - (j - i)) + \cdots + 2(n - (j - i)) \ [j - i \text{ times}] \\
&= 2(j - i)(n - (j - i)).
\end{aligned}$$

□

The cyclic nature of left shifts naturally leads us to consider the set of all such shifts of a given signed permutation. Remarkably, this set forms an automedian set, as established in the theorem below.

Theorem 3.11. *Consider an element $\pi = \pi_1 \pi_2 \cdots \pi_n$ of $\mathcal{B}_n$ and let $l_i(\pi) = \pi_{i+1} \cdots \pi_n \pi_1 \cdots \pi_i$ be the ith-left shift of π. Let $\mathcal{A} = \{l_i(\pi) \mid 0 \le i < n\}$ be the set of all such shifts. Then $\mathcal{A}$ is an automedian set, and for each i with $0 \le i < n$, the generalized Kendall-τ distance satisfies*

$$\overline{d_{KT}}(l_i(\pi), \mathcal{A}) = \frac{n(n - 1)(n + 1)}{3}.$$

Proof. To show that $\mathcal{A}$ is automedian, we have to prove the following:

(i) $\overline{d_{KT}}(l_i(\pi), \mathcal{A}) = \dfrac{(n - 1)n(n + 1)}{3}$, for all $i \in \{0, \ldots, n - 1\}$,

(ii) $\overline{d_{KT}}(l_i(\pi), \mathcal{A}) < \overline{d_{KT}}(\rho, \mathcal{A})$ for all $\rho \in \mathcal{B}_n \setminus \mathcal{A}$.

Proof of (i): Using Lemma 3.10 we have,

$$
\overline{d_{KT}}(l_i(\pi), \mathcal{A}) = \left\{
\begin{array}{l}
\overline{d_{KT}}(l_i(\pi), l_0(\pi)) + \overline{d_{KT}}(l_i(\pi), l_1(\pi)) + \cdots + \overline{d_{KT}}(l_i(\pi), l_i(\pi)) \\
+ \cdots + \overline{d_{KT}}(l_i(\pi), l_{n-1}(\pi))
\end{array}
\right.
$$

$$
= \left\{
\begin{array}{l}
2[i(n-i) + (i-1)(n-(i-1)) + \cdots + \\
(i-(i-1))(n-(i-(i-1))) + 0+ \\
((i+1)-i)(n-((i+1)-i)) + \cdots + \\
((n-1)-i)(n-((n-1)-i))]
\end{array}
\right.
$$

$$
= \left\{
\begin{array}{l}
2[i(n-i) + (i-1)(n-(i-1)) + \cdots + 2(n-2) + (n-1) + 0+ \\
(n-1) + 2(n-2) \cdots + \\
(n-(i+2))(i+2) + (n-(i+1))(i+1)]
\end{array}
\right.
$$

$$
= 2\sum_{k=0}^{i} k(n-k) + 0 + 2\sum_{k=i+1}^{n-1} k(n-k)
$$

$$
= 2\sum_{k=0}^{n-1} k(n-k)
$$

$$
= 2\left[n\frac{n(n-1)}{2} - \frac{n(n-1)(2n-1)}{6} \right]
$$

$$
= n(n-1)\left[n - \frac{2n-1}{3} \right]
$$

$$
= n(n-1)\left[\frac{3n-2n+1}{3} \right]
$$

$$
= \frac{n(n-1)(n+1)}{3}.
$$

Thus we have,

$$
\overline{d_{KT}}(l_i(\pi), \mathcal{A}) = \frac{n(n-1)(n+1)}{3}, \ \ \text{for all } i \in \{0, \ldots, n-1\}
$$

Proof of (ii): We start with the triangle inequalities (refer to Theorem 2.15):

$$
\begin{aligned}
\overline{d_{KT}}(\rho, l_0) + \overline{d_{KT}}(l_0, l_i) &\geq \overline{d_{KT}}(\rho, l_i) \\
\overline{d_{KT}}(\rho, l_1) + \overline{d_{KT}}(l_1, l_i) &\geq \overline{d_{KT}}(\rho, l_i) \\
&\vdots \\
\overline{d_{KT}}(\rho, l_{n-1}) + \overline{d_{KT}}(l_{n-1}, l_i) &\geq \overline{d_{KT}}(\rho, l_i)
\end{aligned}
$$

Summing all the above n inequalities, we get:

$$
\overline{d_{KT}}(\rho, \mathcal{A}) + \overline{d_{KT}}(l_i, \mathcal{A}) \geq n \cdot \overline{d_{KT}}(\rho, l_i)
$$

$$
\Rightarrow \overline{d_{KT}}(\rho, \mathcal{A}) \geq n \cdot \overline{d_{KT}}(\rho, l_i) - \overline{d_{KT}}(l_i, \mathcal{A}).
$$

To prove

$$
\overline{d_{KT}}(\rho, \mathcal{A}) > \overline{d_{KT}}(l_i, \mathcal{A}),
$$

it is equivalent to proving:

$$
n \cdot \overline{d_{KT}}(\rho, l_i) - \overline{d_{KT}}(l_i, \mathcal{A}) > \overline{d_{KT}}(l_i, \mathcal{A}).
$$

$$
\text{i.e.} \ \ n \cdot \overline{d_{KT}}(\rho, l_i) > 2 \cdot \overline{d_{KT}}(l_i, \mathcal{A}).
$$

Substituting the value of $\overline{d_{KT}}(l_i, \mathcal{A})$ from part(i), it is enough to prove:

$$\overline{d_{KT}}(\rho, l_i) > \frac{2}{3}(n^2 - 1)$$

Suppose for contradiction, that

$$\overline{d_{KT}}(\rho, l_i) \le \frac{2}{3}(n^2 - 1).$$

Taking the summation on both sides as ρ varies over $\mathcal{A}^c$, we get:

$$\overline{d_{KT}}(l_i, \mathcal{A}^c) \le \frac{2}{3}(n^2 - 1)(|\mathcal{A}^c|).$$

Substituting the value of $\overline{d_{KT}}(l_i, \mathcal{A}^c)$ into the equation, we get:

$$|\mathcal{B}_n|\frac{n(2n-1)}{2} - n\frac{n^2-1}{3} \le \frac{2}{3}(n^2-1)(|\mathcal{B}_n|-n)$$
$$\Rightarrow 3|\mathcal{B}_n| \cdot n(2n-1) - 2n \cdot (n^2-1) \le 4(n^2-1)(|\mathcal{B}_n|-n)$$
$$\Rightarrow 6n^2|\mathcal{B}_n| - 3n|\mathcal{B}_n| - 2n^3 + 2n \le 4n^2|\mathcal{B}_n| - 4|\mathcal{B}_n| - 4n^3 + 4n.$$

Grouping similar terms, we get:

$$(2n^2 - 3n + 4)|\mathcal{B}_n| + 2n^3 - 2n \le 0.$$

For sufficiently large n, the dominant term is $2n^2|\mathcal{B}_n|$, which remains strictly positive. Thus, the left-hand side is always positive, contradicting our assumption. Therefore, we conclude:

$$\overline{d_{KT}}(\rho, l_i) > \frac{2}{3}(n^2 - 1).$$

This completes the proof for

$$\overline{d_{KT}}(\rho, \mathcal{A}) > \overline{d_{KT}}(l_i, \mathcal{A}), \quad \text{for all } \rho \in \mathcal{B}_n \setminus \mathcal{A}.$$

$$\square$$

Another rich source of automedian sets emerges from the construction of subsets of $\mathcal{B}_n$ by combining elements from a subgroup $\mathcal{B}_k$ with permutations of the remaining entries. The following two theorems show that these structured subsets are automedian.

Theorem 3.12. *Let* $\mathcal{A} = \{\pi = \alpha\beta \mid \alpha \in \mathcal{B}_k \text{ and } \beta \text{ is a permutation of } \{k+1, \ldots, n\}\}$ *for any* $0 \le k \le n$, *then* $\mathcal{A}$ *is automedian, i.e.* $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. *Also, for any* $\sigma \in \mathcal{A}$, *we have*

$$\overline{d_{KT}}(\sigma, \mathcal{A}) = \frac{|S_{n-k}||B_k|}{2} \left[k(2k-1) + (n-k)(n-k-1) \right].$$

Proof. Observe that $\mathcal{A} = \mathcal{S}_n$ is the particular case where $k = 0$, and $\mathcal{A} = \mathcal{B}_n$ is the particular case where $k = n$. These two cases have already been discussed and shown to be automedian sets by Theorem 3.2.

For $k = n - 1$, we have $\mathcal{A} = \{\pi = \alpha\beta \mid \alpha \in \mathcal{B}_{n-1} \text{ and } \beta \text{ is a permutation of } \{n\}\}$. Each element in $\mathcal{A}$ is of the form

$$\pi = \alpha_1\alpha_2 \cdots \alpha_{n-1}\, n,$$

where $\alpha_1\alpha_2 \cdots \alpha_{n-1} \in \mathcal{B}_{n-1}$. Thus, any signed permutation $\pi \in \mathcal{A}$ is obtained by taking an element of $\mathcal{B}_{n-1}$ and appending n as the last entry. So elements of $\mathcal{A}$ forms a subgroup isomorphic to $\mathcal{B}_{n-1}$. By Theorem 3.2, $\mathcal{B}_{n-1}$ is known to be automedian in itself and from Theorem 2.17, we have for any $\pi \in \mathcal{A}$,

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \overline{d_{KT}}(\pi, \mathcal{B}_{n-1}) = \frac{|\mathcal{B}_{n-1}|}{2}(n-1)(2(n-1) - 1).$$

This proves our claim for this case.

Now for any $0 < k < n - 1$, we want to prove that $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. To accomplish this, we must show the following:

(i) $\overline{d_{KT}}(\sigma, \mathcal{A}) = \dfrac{|S_{n-k}||B_k|}{2}[k(2k-1) + (n-k)(n-k-1)]$, for every $\sigma \in \mathcal{A}$,

(ii) $\overline{d_{KT}}(\sigma, \mathcal{A}) < \overline{d_{KT}}(\rho, \mathcal{A})$ for every $\sigma \in \mathcal{A}, \rho \in \mathcal{B}_n \setminus \mathcal{A}$.

Let $\mathcal{B}_k = \{\alpha_1, \alpha_2, \ldots, \alpha_{|\mathcal{B}_k|}\}$, and write $\mathcal{A} = \bigcup\limits_{i=1}^{|\mathcal{B}_k|} \mathcal{A}_i$, where $\mathcal{A}_i = \{\alpha_i \beta \mid \beta \in \mathcal{S}_{n-k}\}$.

Let $\sigma = \alpha' \beta' \in \mathcal{A}$, for some $\alpha' \in \mathcal{B}_k$ and $\beta' \in \mathcal{S}_{n-k}$. We calculate $\overline{d_{KT}}(\sigma, \mathcal{A})$ by looking at each pair of positions (i, j), $1 \leq i < j \leq n$. That is, we check the contribution of each pair (i, j) to the total distance $\overline{d_{KT}}(\sigma, \mathcal{A})$. This can be done by splitting it into 3 cases, namely,

Case 1: the pairs (i, j), where $k + 1 \leq i < j \leq n$,

Case 2: the pairs (i, j), where $1 \leq i \leq k, k + 1 \leq j \leq n$, and

Case 3: the pairs (i, j), where $1 \leq i < j \leq k$.

Let $\pi = \alpha_i \beta$ be an element of $\mathcal{A}$. Clearly, for case 1, a pair (i, j) contributes to $O_1(\sigma, \pi)$ and $O_2(\sigma, \pi)$ while calculating $\overline{d_{KT}}(\sigma, \pi)$ iff its corresponding pair of β' and β in $\mathcal{S}_{n-k}$ contributes to $O_1(\beta', \beta)$ and $O_2(\beta', \beta)$ while calculating $\overline{d_{KT}}(\beta', \beta)$.

It can be easily seen that for case 2, pair (i, j) does not contribute to $O_1(\sigma, \pi)$ and $O_2(\sigma, \pi)$ while calculating $\overline{d_{KT}}(\sigma, \pi)$.

Also for case 3, pair (i, j) contributes to $O_1(\sigma, \pi)$ and $O_2(\sigma, \pi)$ while calculating $\overline{d_{KT}}(\sigma, \pi)$ iff its corresponding pair of α' and α in $\mathcal{B}_k$ contributes to $O_1(\alpha', \alpha)$ and $O_2(\alpha', \alpha)$ while calculating $\overline{d_{KT}}(\alpha', \alpha)$.

Also, it is clear that, for $i > k$, $i \notin \mathrm{sgn}(\sigma, \pi)$, and for $i \leq k$, $i \in \mathrm{sgn}(\sigma, \pi)$ iff the corresponding i for α' and α in $\mathcal{B}_k$ contributes to $\mathrm{sgn}(\alpha', \alpha)$.
Then we have,

$$
\begin{aligned}
\overline{d_{KT}}(\sigma, \mathcal{A}) &= \sum_{i=1}^{|\mathcal{B}_k|} \overline{d_{KT}}(\sigma, \mathcal{A}_i) \\
&= \sum_{i=1}^{|\mathcal{B}_k|} \left[\overline{d_{KT}}(\alpha, \alpha_i)|S_{n-k}| + \frac{|S_{n-k}|(n-k)(n-k-1)}{2} \right] \\
&= |S_{n-k}| \sum_{i=1}^{|\mathcal{B}_k|} \overline{d_{KT}}(\alpha, \alpha_i) + \frac{|S_{n-k}||B_k|}{2}(n-k)(n-k-1) \\
&= |S_{n-k}|\overline{d_{KT}}(\alpha, B_k) + \frac{|S_{n-k}||B_k|}{2}(n-k)(n-k-1) \\
&= \frac{|S_{n-k}||B_k|}{2}k(2k-1) + \frac{|S_{n-k}||B_k|}{2}(n-k)(n-k-1) \\
&= \frac{|S_{n-k}||B_k|}{2}[k(2k-1) + (n-k)(n-k-1)]
\end{aligned}
$$

Now for $\rho \in \mathcal{B}_n \setminus \mathcal{A}$, using the triangle inequality, we have:

$$
\overline{d_{KT}}(\rho, \pi_i) + \overline{d_{KT}}(\pi_i, \sigma) > \overline{d_{KT}}(\rho, \sigma) \quad \text{for } \sigma, \pi_i \in \mathcal{A}.
$$

Taking summation of the above inequalities for all $\pi_i \in \mathcal{A}$, we obtain:

$$\overline{d_{KT}}(\rho, \mathcal{A}) + \overline{d_{KT}}(\sigma, \mathcal{A}) > |\mathcal{A}|\overline{d_{KT}}(\rho, \sigma)$$
$$\implies \overline{d_{KT}}(\rho, \mathcal{A}) > |\mathcal{A}|\overline{d_{KT}}(\rho, \sigma) - \overline{d_{KT}}(\sigma, \mathcal{A})$$

In order to prove:

$$\overline{d_{KT}}(\rho, \mathcal{A}) > \overline{d_{KT}}(\sigma, \mathcal{A}),$$

it is equivalent to proving

$$|\mathcal{A}|\overline{d_{KT}}(\rho, \sigma) - \overline{d_{KT}}(\sigma, \mathcal{A}) > \overline{d_{KT}}(\sigma, \mathcal{A})$$

$$\text{i.e. } |\mathcal{A}|\overline{d_{KT}}(\rho, \sigma) > 2\overline{d_{KT}}(\sigma, \mathcal{A})$$

Suppose for the contradiction that:

$$|\mathcal{A}|\overline{d_{KT}}(\rho, \sigma) \leq 2\overline{d_{KT}}(\sigma, \mathcal{A})$$

Taking summation on both sides as ρ varies over $\mathcal{A}^c$, we get:

$$|\mathcal{A}|\overline{d_{KT}}(\sigma, \mathcal{A}^c) \leq 2|\mathcal{A}^c|\overline{d_{KT}}(\sigma, \mathcal{A}).$$

Substituting the value of $\overline{d_{KT}}(\sigma, \mathcal{A}^c)$ into the equation, we get:

$$|\mathcal{A}| \left[\frac{|\mathcal{B}_n|}{2} n(2n-1) - \overline{d_{KT}}(\sigma, \mathcal{A}) \right] \leq 2|\mathcal{A}^c| \, \overline{d_{KT}}(\sigma, \mathcal{A})$$

$$\Rightarrow |\mathcal{A}| \frac{|\mathcal{B}_n|}{2} n(2n-1) \leq (2|\mathcal{A}^c| + |\mathcal{A}|) \, \overline{d_{KT}}(\sigma, \mathcal{A})$$

$$\Rightarrow |\mathcal{A}| \frac{|\mathcal{B}_n|}{2} n(2n-1) \leq (2|\mathcal{B}_n| - |\mathcal{A}|) \, \overline{d_{KT}}(\sigma, \mathcal{A})$$

Substituting the value of $|\mathcal{A}|$ and $\overline{d_{KT}}(\sigma, \mathcal{A})$, we get

$$|\mathcal{B}_n| n(2n-1) \leq (2|\mathcal{B}_n| - |\mathcal{B}_k||\mathcal{S}_{n-k}|) \left[k(2k-1) + (n-k)(n-k-1) \right]$$

$$\Rightarrow |\mathcal{B}_n| \left[n(2n-1) - 2k(2k-1) - 2(n-k)(n-k-1) \right] + |\mathcal{B}_k||\mathcal{S}_{n-k}|[k(2k-1) + (n-k)(n-k-1)] \leq 0$$

$$\Rightarrow (n + 4nk - 6k^2)|\mathcal{B}_n| + (n^2 + 2nk + 3k^2)|\mathcal{B}_k||\mathcal{S}_{n-k}| \leq 0$$

The left-hand side is always positive for any n and $0 < k < n-1$, contradicting our assumption. Therefore, we conclude

$$|\mathcal{A}|\overline{d_{KT}}(\rho, \sigma) > 2\overline{d_{KT}}(\sigma, \mathcal{A}).$$

This completes the proof for

$$\overline{d_{KT}}(\rho, \mathcal{A}) > \overline{d_{KT}}(\sigma, \mathcal{A}), \text{ for all } \rho \in \mathcal{B}_n \setminus \mathcal{A}.$$

$\square$

Theorem 3.13. *Let* $\mathcal{A} = \{\pi = \alpha\beta \mid \alpha \in \mathcal{B}_k \text{ and } \beta \text{ is a permutation of } \{-(k+1), \ldots, -n\}\}$ *for any* $0 \leq k \leq n$, *then* $\mathcal{A}$ *is automedian, i.e.* $\mathcal{M}(\mathcal{A}) = \mathcal{A}$.
Also for any $\sigma \in \mathcal{A}$, *we have*

$$\overline{d_{KT}}(\sigma, \mathcal{A}) = \frac{|S_{n-k}||B_k|}{2} \left[k(2k-1) + (n-k)(n-k-1) \right]$$

Proof. The proof follows the reasoning analogous to that in the previous Theorem 3.12 and is thus omitted for brevity. $\square$

4 Group action and morphism

In this section, we introduce a natural left-group action of the hyperoctahedral group of type $\mathcal{B}_n$ on the power set $\mathcal{P}(\mathcal{B}_n)$ and examine the behavior of the generalized Kendall-τ distance with respect to this action. We then extend this analysis to show that the median operator $\mathcal{M}$ behaves compatibly with this group action, in the sense of being a morphism of actions. These results have important implications for the structure of automedian sets under the action of $\mathcal{B}_n$, as we detail below.

Let $\mathcal{P}(\mathcal{B}_n)$ be a collection of all possible subsets of $\mathcal{B}_n$. The composition operation endows the set of signed permutations $\mathcal{B}_n$ with a group structure, which naturally induces a left action on the power set $\mathcal{P}(\mathcal{B}_n)$, defined as follows:

$$\circ : \mathcal{B}_n \times \mathcal{P}(\mathcal{B}_n) \quad \to \quad \mathcal{P}(\mathcal{B}_n)$$
$$(\rho, \mathcal{A}) \quad \mapsto \quad \rho \circ \mathcal{A},$$

where $\rho \circ \mathcal{A} = \{\rho \circ \alpha \mid \alpha \in \mathcal{A}\}$. For notational convenience, we refer to $\rho \circ \alpha$ by $\rho\alpha$ in the remainder of this article.

We begin with a few observations on the invariance properties of the generalized Kendall-τ distance with regard to this action.

Observation 1. Consider the signed permutations α, ρ, ψ in $\mathcal{B}_n$. Then

$$\overline{d_{KT}}(\alpha\rho, \alpha\psi) = \overline{d_{KT}}(\rho, \psi).$$

The above observation can also be seen as the following:
For α, ρ, $\psi \in \mathcal{B}_n$, we have

$$\overline{d_{KT}}(\rho, \alpha\psi) = \overline{d_{KT}}(\alpha^{-1}\rho, \psi) \tag{4.1}$$

$$\overline{d_{KT}}(\alpha\rho, \psi) = \overline{d_{KT}}(\rho, \alpha^{-1}\psi). \tag{4.2}$$

We next generalize this invariance property to the case where one of the arguments is a subset of permutations.

Observation 2. Consider the subset $\mathcal{A} \subseteq \mathcal{B}_n$ consisting of signed permutations, and let α, ρ be any elements of $\mathcal{B}_n$. Then

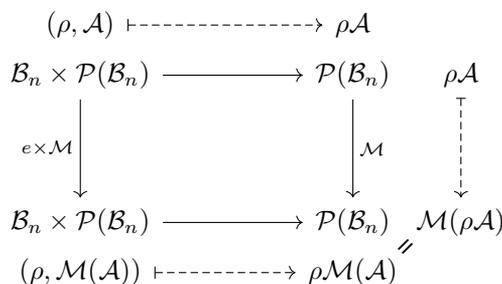$$\overline{d_{KT}}(\alpha\rho, \alpha\mathcal{A}) = \overline{d_{KT}}(\rho, \mathcal{A}).$$

Similarly, this observation admits equivalent formulations analogous to (4.1) and (4.2), namely: for $\mathcal{A} \subseteq \mathcal{B}_n$ and α, ρ any signed permutations of $\mathcal{B}_n$. Then

$$\overline{d_{KT}}(\rho, \alpha\mathcal{A}) = \overline{d_{KT}}(\alpha^{-1}\rho, \mathcal{A})$$
$$\overline{d_{KT}}(\alpha\rho, \mathcal{A}) = \overline{d_{KT}}(\rho, \alpha^{-1}\mathcal{A})$$

Having established these foundational properties, we are now ready to demonstrate that the median operator $\mathcal{M}$ interacts compatibly with this group action.

Theorem 4.1. *Define the map* $\mathcal{M} : \mathcal{P}(\mathcal{B}_n) \to \mathcal{P}(\mathcal{B}_n)$ *such that, for any subset* $\mathcal{A} \subseteq \mathcal{B}_n$, $\mathcal{M}$ *takes* $\mathcal{A}$ *to its median set* $\mathcal{M}(\mathcal{A})$. *Then this map* $\mathcal{M}$ *respects the group action, that is, it satisfies* $\rho\mathcal{M}(\mathcal{A}) = \mathcal{M}(\rho\mathcal{A})$ *for any* ρ *in* $\mathcal{B}_n$, *as illustrated in the commutative diagram below:*

Proof. We aim to prove that $\rho\mathcal{M}(\mathcal{A}) = \mathcal{M}(\rho\mathcal{A})$ for any signed permutation $\rho \in \mathcal{B}_n$ and any subset $\mathcal{A} \subseteq \mathcal{B}_n$. To prove this, we apply the equivalent statement of Observation 2 given above, that is, for any $\sigma \in \mathcal{B}_n$,

$$\overline{d_{KT}}(\sigma, \rho\mathcal{A}) = \overline{d_{KT}}(\rho^{-1}\sigma, \mathcal{A}).$$

Now, recall that $\mathcal{M}(\mathcal{A})$ consists of elements from $\mathcal{B}_n$ that minimize the distance to the set $\mathcal{A}$. Thus, a signed permutation σ is in $\mathcal{M}(\mathcal{A})$ if and only if $\overline{d_{KT}}(\sigma, \mathcal{A})$ is minimal among all signed permutations in $\mathcal{B}_n$. So,

$$\sigma \in \mathcal{M}(\rho\mathcal{A}) \Leftrightarrow \sigma \text{ minimizes the distance to } \rho\mathcal{A}$$
$$\Leftrightarrow \overline{d_{KT}}(\sigma, \rho\mathcal{A}) \leq \overline{d_{KT}}(\rho, \rho\mathcal{A}) \quad \forall \rho \in \mathcal{B}_n$$
$$\Leftrightarrow \overline{d_{KT}}(\rho^{-1}\sigma, \mathcal{A}) \leq \overline{d_{KT}}(\rho^{-1}\rho, \mathcal{A}) \quad \forall \rho \in \mathcal{B}_n$$
$$\Leftrightarrow \rho^{-1}\sigma \text{ minimizes the distance to } \mathcal{A}$$
$$\Leftrightarrow \rho^{-1}\sigma \in \mathcal{M}(\mathcal{A})$$
$$\Leftrightarrow \sigma \in \rho\mathcal{M}(\mathcal{A}).$$

Therefore, $\sigma \in \mathcal{M}(\rho\mathcal{A})$ if and only if $\sigma \in \rho\mathcal{M}(\mathcal{A})$. This completes the proof that $\rho\mathcal{M}(\mathcal{A}) = \mathcal{M}(\rho\mathcal{A})$ for any signed permutation $\rho \in \mathcal{B}_n$ and subset $\mathcal{A} \subseteq \mathcal{B}_n$. $\square$

We define the **orbit** of a subset $\mathcal{A}$ with respect to the action $\circ$ to be the collection of all sets $\sigma\mathcal{A}$, where σ varies over $\mathcal{B}_n$.

Corollary 4.2. *Consider the subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{B}_n$ such that both lie in the same orbit of the action $\circ : \mathcal{B}_n \times \mathcal{P}(\mathcal{B}_n) \to \mathcal{P}(\mathcal{B}_n)$. Then $\mathcal{A}$ is automedian if and only if $\mathcal{B}$ is automedian.*

Proof. For some $\rho \in \mathcal{B}_n$, we can infer that $\mathcal{B} = \rho\mathcal{A}$ since the subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{B}_n$ lie in the same orbit. Thus, we have

$$\mathcal{M}(\mathcal{B}) = \mathcal{B} \Leftrightarrow \mathcal{M}(\rho\mathcal{A}) = \rho\mathcal{A}$$
$$\Leftrightarrow \rho\mathcal{M}(\mathcal{A}) = \rho\mathcal{A} \ [\text{ by Theorem 4.1}]$$
$$\Leftrightarrow \mathcal{M}(\mathcal{A}) = \mathcal{A}$$

Thus, $\mathcal{A}$ is automedian if and only if $\mathcal{B}$ is automedian. $\square$

Remark 4.3. Consider the subset $\mathcal{A}$ of $\mathcal{B}_n$. Then $\mathcal{A}$ is automedian if and only if $neg(\mathcal{A})$ is automedian. This is straightforward from Corollary 4.2 by choosing $\pi = -1 \ -2 \cdots -n$ such that $neg(\mathcal{A}) = \pi\mathcal{A}$. In particular, by taking $\mathcal{A} = \mathcal{S}_n$, we see that $neg(\mathcal{S}_n)$ is an automedian set.

The following corollaries are immediate consequences of the orbit-invariance property established in Corollary 4.2.

Corollary 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}_n$ in the same orbit of the action $\circ : \mathcal{B}_n \times \mathcal{P}(\mathcal{B}_n) \to \mathcal{P}(\mathcal{B}_n)$. Then $\mathcal{B}$ is a singleton set, and so $\mathcal{B}$ is automedian if and only if $\mathcal{A}$ is automedian and a singleton set.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are in the same orbit, there exists an element $\pi \in \mathcal{B}_n$, such that $\mathcal{B} = \pi\mathcal{A}$. If $\mathcal{B}$ is a singleton set, then it is automedian (by Theorem 3.3). Then by Theorem 4.1 and Corollary 4.2, we see that $\mathcal{A}$ is a singleton set and also an automedian. $\square$

Corollary 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of signed permutations of $\mathcal{B}_n$ in the same orbit of the action $\circ : \mathcal{B}_n \times \mathcal{P}(\mathcal{B}_n) \to \mathcal{P}(\mathcal{B}_n)$. Then $\mathcal{M}(\mathcal{B}) = \mathcal{B}_n \Leftrightarrow \mathcal{M}(\mathcal{A}) = \mathcal{B}_n$.*

Proof. For some $\rho \in \mathcal{B}_n$, we can infer that $\mathcal{B} = \rho\mathcal{A}$ since the subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{B}_n$ lie in the same orbit. Therefore

$$
\begin{aligned}
\mathcal{M}(\mathcal{B}) = \mathcal{B}_n &\Leftrightarrow \mathcal{M}(\rho\mathcal{A}) = \mathcal{B}_n \\
&\Leftrightarrow \rho\mathcal{M}(\mathcal{A}) = \mathcal{B}_n \quad [\because \text{ by Theorem 4.1}] \\
&\Leftrightarrow \mathcal{M}(\mathcal{A}) = \rho^{-1}\mathcal{B}_n \\
&\Leftrightarrow \mathcal{M}(\mathcal{A}) = \mathcal{B}_n.
\end{aligned}
$$

$\square$

5 Conclusion remarks

In this work, we investigated automedian sets in the hyperoctahedral group $\mathcal{B}_n$ under the generalized Kendall-τ distance. We identified and characterized various classes of such sets, including singleton sets, sets closed under negation or transposition, and those formed through cyclic shifts or structured concatenations. Our results showed that both $\mathcal{S}_n$ and $\mathcal{B}_n$ themselves are automedian, and we established invariance properties under natural group actions that underlie their symmetry.

Preliminary observations suggest that the problem of recognizing or constructing automedian sets may be solvable in polynomial time. However, an explicit algorithm remains to be developed, which we identify as an important direction for future work. These findings contribute to the theoretical understanding of signed permutation spaces and offer potential applications in ranking, clustering, and computational biology. Further research may also explore extensions to other distance metrics or permutation groups, as well as efficient algorithmic frameworks for detecting automedian structures.

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