

ZERO-DIVISOR GRAPHS OF BI-AMALGAMATION

Mohamed Chhiti, Loubna Es-Salhi and Soibri Moindze

Communicated by: Najib Mahdou

MSC 2020 Classifications: 05C15, 05C25.

Keywords and phrases: Bi-amalgamated algebras, Amalgamated algebras, zero divisor graphs.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: M. Chhiti

Abstract. Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and J and J' be two non-zero proper ideals of B and C , respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by:

$$A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j') / a \in A, (j, j') \in J \times J'\}$$

This paper investigates the zero-divisor graphs of bi-amalgamations. Our objective is to characterize when the graph is complete, and compute its diameter and girth for various contexts of bi-amalgamations. The new results recover well-known results on amalgamations, and provide some original examples arising as bi-amalgamations.

1 Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. In 1988, Beck introduced the notion of zero-divisor graphs for his study of the coloring of a commutative ring (cf.[4]). He considered all elements of the ring where vertices of the graphs. D.D.Anderson and Nasser used in 1993 this same notion in [2]. Let R be a ring and let $Z(R)$ denote the set of zero-divisors of R and $Z(R)^* = Z(R) \setminus \{0\}$. The zero-divisor graphs of R , denoted by $\Gamma(R)$, is the graph whose vertices are the elements of $Z(R)^*$ and for distincts $a, b \in Z(R)^*$, there is an edge connecting a and b if and only if $ab = 0$. For two distinct vertices x and y in $\Gamma(R)$, the distance between x and y denoted by $d(x, y)$, is the length of the shortest path connecting x and y , if there is a path, otherwise $d(x, y) = \infty$. The diameter of $\Gamma(R)$ defined as follows: $diam(\Gamma(R)) := \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } \Gamma(R)\}$. The girth of $\Gamma(R)$ denoted by $gr(\Gamma(R))$, is the length of a shortest cycle in $\Gamma(R)$, if there is a cycle in $\Gamma(R)$, otherwise $gr(\Gamma(R)) = \infty$. A graph is connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter less than or equal to one. A singleton graph is connected and of diameter zero. T. G. Lucas in [12], characterized when the diameter of zero-divisor graphs is equal to 0 or 1. On the other hand, it is clear that $\Gamma(R)$ is empty if and only if R is a domain. In [1], D.F.Anderson and P. S. Livingston considered only nonzero zero-divisors as vertices of the graph and proved that the zero-divisor graphs of a (commutative) ring is always connected with diameter less than or equal three. In [14], Mulay, and in [8], DeMeyer and Schneider examined the diameter and the girth of the zero-divisor graph of a ring. In [12], Lucas characterized the diameter of $\Gamma(R)$, $\Gamma(R[X])$ and $\Gamma(R[[X]])$ with some properties of R . For reduced rings, he gave a complete characterization for all three graphs, and for non-reduced rings, he studied the diameter of $\Gamma(R)$ and $\Gamma(R[X])$. He gave an example of a reduced ring R , for which $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 2$ and $diam(\Gamma(R[[X]])) = 3$. He also gave other examples of a reduced and non-reduced ring R , where $diam(\Gamma(R)) = 2$ and $diam(\Gamma(R[X])) = diam(\Gamma(R[[X]])) = 3$. For more details on zero-divisor graphs, see [1, 2, 4, 8, 12, 14].

Let A be a ring and E an A -module, we called the trivial ring extension of A by E , the ring $A \times E$ where the underlying group is $A \times E$ and we defined a multiplication in $A \times E$ by:

$(a, e)(b, f) = (ab, af + be); \forall a, b \in A$, and $e, f \in E$. This notion is introduced by Nagata in [15] (often called the Nagata’s idealization of E over A and denoted $A(+E)$).

In [3], Axtell and Stickles gave a complete characterization of the preservation of the diameter and girth under the Nagata’s idealization. They characterized the girth of the zero-divisor graphs of the Nagata’s idealization and when it is complete and they gave conditions for the zero-divisor graphs to have diameter 2.

Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C , respectively, such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{(f,g)} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}$$

called the *bi-amalgamated algebra of A with (B, C) along (J, J') with respect to (f, g)* . This construction was introduced and studied in [11] as a natural generalization of duplications and amalgamated algebra of rings along an ideal. Given a ring homomorphism $f : A \rightarrow B$ and an ideal J of B , the bi-amalgamation $A \bowtie^{f,i} (J, f^{-1}(J))$ coincides with the amalgamated algebra of A with B along J with respect to f introduced and studied by D’Anna, Finocchiaro and Fontana in [5, 6, 9] as the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

When $A = B$, $J = I$ an ideal of A and $f = id_A$, then $A \bowtie^{id_A} I$ is called the amalgamated duplication of A along I and denoted $A \bowtie I$ introduced and studied by D’Anna and Fontana in [7]. This construction can be seen as a bi-amalgamation as follows:

$$A \bowtie I = A \bowtie^{id_A, id_A} (I, I)$$

In [10], Kabbaj and Mimouni studied the notion of zero-divisor graphs of amalgamations, they characterized when the zero-divisor graphs of amalgamations is complete, and they computed the diameter and the girth of amalgamations. In [13], Maimani and Yassemi studied the diameter and girth of the graph of $A \bowtie I$.

In this paper, we investigate the zero-divisor graph of bi-amalgamations. Our aim is to characterize when the graph is complete and compute its diameter and girth for different contexts of bi-amalgamations. Precisely, sections 2 and 3 study, respectively, the cases when $diam(\Gamma(A \bowtie^{(f,g)} (J, J'))) = 2$, $diam(\Gamma(A \bowtie^{(f,g)} (J, J'))) = 3$ and $\Gamma(A \bowtie^{(f,g)} (J, J'))$ is complete. Section 4 will study the girth of $\Gamma(A \bowtie^{(f,g)} (J, J'))$. The new results are compared to previous works carried out in various settings of duplications and amalgamations [10, 13]. All results are backed with new and illustrative examples arising as bi-amalgamations.

2 When is $diam(\Gamma(A \bowtie^{f,g} (J, J')))=2$ or 3

Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and let J and J' be two proper ideals of B and C , respectively, such that $f^{-1}(J) = g^{-1}(J')$. We consider: $R := A \bowtie^{(f,g)} (J, J') = \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}$ called the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . The aim of this section is to investigate the diameter of $\Gamma(R)$, precisely when $diam(\Gamma(R)) = 2$ or 3. Throughout, let $Z(J)$ denote the set of zero-divisors on J as an ideal of $f(A) + J$; that is, the elements of $f(A) + J$ that annihilate some non-zero element of J and $Z(J')$ denote the set of zero-divisors on J' as an ideal of $g(A) + J'$; that is, the elements of $g(A) + J'$ that annihilate some non-zero element of J' .

Consider the following conditions:

- (C₁): For all $x \in Z(J)$, and all $j \in J : jx \neq 0$, then $ij = ix = 0$, for some $0 \neq i \in J$.
- (C₂): For all $x \in Z(J')$ and all $j' \in J' : j'x \neq 0$, then $i'j' = i'x = 0$, for some $0 \neq i' \in J'$.
- (C₃): For all $x, y \in Z(J); x \neq y$, then $ix = iy = 0$, for some $0 \neq i \in J$.
- (C₄): For all $x, y \in Z(J'); x \neq y$, then $i'x = i'y = 0$, for some $0 \neq i' \in J'$.

We start by giving necessary and sufficient conditions for the diameter of the zero-divisor graphs of the bi-amalgamation to be equal to 2 when $f(A) + J$ or $g(A) + J'$ is a domain.

Theorem 2.1. Under the above notation, assume that J and J' are nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Then:

- (1) If $f(A) + J$ is a domain, then $\text{diam}(\Gamma(R)) = 2$ if and only if (C_2) and (C_4) hold.
- (2) If $g(A) + J'$ is a domain, then $\text{diam}(\Gamma(R)) = 2$ if and only if (C_1) and (C_3) hold.

Before proving Theorem 2.1, we establish the following lemma.

Lemma 2.2. Under the above notation, suppose that J and J' are nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Let

$$V_1 := \{(0, j')/0 \neq j' \in J'\}.$$

$$V_2 := \{(j, 0)/0 \neq j \in J\}.$$

$$V_3 := \{(f(a) + j, g(a) + j')/0 \neq g(a) + j' \in Z(J')\}.$$

$$V_4 := \{(f(a) + j, g(a) + j')/0 \neq f(a) + j \in Z(J)\}.$$
 Then :

- (i) If $f(A) + J$ is a domain, then $Z(R)^* = V_1 \cup V_2 \cup V_4$.
- (ii) If $g(A) + J'$ is a domain, then $Z(R)^* = V_1 \cup V_2 \cup V_3$.

Proof. (1) Assume that $f(A) + J$ is a domain. Let $x := (f(a) + j, g(a) + j') \in Z(R)^*$ and $0 \neq y := (f(b) + i, g(b) + i') \in R$ such that $xy = 0$, for all $(0, 0) \neq (i, i') \in J \times J'$. Then $(f(a) + j)(f(b) + i) = (g(a) + j')(g(b) + i') = 0$. If $f(a) + j = 0$, then $0 \neq g(a) + j' \in J'$ and so $x \in V_1$. If $f(a) + j \neq 0$, then $f(b) + i = 0$, therefore $0 \neq g(b) + i' \in J'$ and $(g(a) + j')(g(b) + i') = 0$, then $g(a) + j' = 0$ or $g(a) + j' \in Z(J')$, then $x \in V_2$ or $x \in V_4$. Hence $Z(R)^* \subseteq V_1 \cup V_2 \cup V_4$. In the otherwise, we have $V_1 \cup V_2 \subseteq Z(R)^*$. Let $x = (f(a) + i, g(a) + i') \in V_4$, then $0 \neq g(a) + i' \in Z(J')$, then there is $0 \neq k' \in J'$ such that $k'(g(a) + i') = 0$, then $(f(a) + i, g(a) + i')(0, k') = 0$ therefore $x \in Z(R)^*$. Hence $Z(R)^* = V_1 \cup V_2 \cup V_4$.

(2) For the same reasoning, it is easy to prove that if $g(A) + J'$ is a domain, then $Z(R)^* = V_1 \cup V_2 \cup V_3$. □

Proof of Theorem 2.1

(1) Assume that C_2 and C_4 hold. Notice that every vertex in V_1 is adjacent to every vertex in V_2 . Next, let us handle five other possible distinct cases.

- Let $x_1 := (0, i') \neq (0, j') =: y_1 \in V_1$ and let $0 \neq j \in J$, then $x_1 - (j, 0) - y_1$ is a path in $\Gamma(R)$, then $d(x_1, y_1) \leq 2$.
- Let $x_2 := (i, 0) \neq (j, 0) =: y_2 \in V_2$ and let $0 \neq j' \in J'$, then $x_2 - (0, j') - y_2$ is a path in $\Gamma(R)$, then $d(x_2, y_2) \leq 2$.
- Let $x_1 := (0, i') \in V_1$ and $x_4 := (f(a) + j, g(a) + j') \in V_4$. If $i'(g(a) + j') = 0$, then $x_1 x_4 = 0$, then $d(x_1, x_4) = 1$ (x_1 and x_4 are adjacent). If $i'(g(a) + j') \neq 0$, then by (C_2) there is $0 \neq k' \in J'$ such that $k'i' = k'(g(a) + j') = 0$. Thus $x_1 - (0, k') - x_4$ is a path in $\Gamma(R)$ and hence $d(x_1, x_4) = 2$.
- Let $x_2 := (j, 0) \in V_2$ and let $x_4 := (f(a) + j, g(a) + j') \in V_4$. The fact $x_4 \in V_4$, implies that there is $0 \neq k' \in J'$ such that $k'(g(a) + j') = 0$. Then $x_2 - (0, k') - x_4$ is a path in $\Gamma(R)$, so $d(x_2, x_4) = 2$.
- Let $x_4 := (f(a) + i, g(a) + i') \neq (f(b) + j, g(b) + j') =: y_4 \in V_4$. If $g(a) + i' = g(b) + j'$, then there exists $0 \neq k' \in J'$ such that $k'(g(a) + i') = k'(g(b) + j') = 0$, then $x_4 - (0, k') - y_4$ is a path in $\Gamma(R)$, then $d(x_4, y_4) = 2$. If $g(a) + i' \neq g(b) + j'$ by C_4 , there exists $0 \neq k' \in J'$ such that $k'(g(a) + i') = k'(g(b) + j') = 0$, then $x_4 - (0, k') - y_4$ is a path in $\Gamma(R)$, so $d(x_4, y_4) = 2$.

Consequently $\text{diam}(\Gamma(R)) = 2$. Conversely, assume that $\text{diam}(\Gamma(R)) = 2$. Let $0 \neq x := g(a) + j' \in Z(J')$ and $i' \in J'$ with $i'(g(a) + j') \neq 0$. Let $0 \neq j \in J$ such that $f(a) + j \neq 0$, then $(f(a) + j, x)$ and $(0, i')$ are two non adjacent vertices in $\Gamma(R)$ or $\text{diam}(\Gamma(R)) = 2$, then there is a path in $\Gamma(R)$: $(f(a) + j, x) - (f(b) + k, g(b) + k') - (0, i')$, for some $(k, k') \in J \times J', b \in A$. Since $f(A) + J$ is a domain and $f(a) + j \neq 0$, then $f(b) + k = 0$, then $0 \neq g(b) + k' \in J'$ and $(g(b) + k')x = (g(b) + k')i' = 0$, so (C_2) holds. Let $x := g(a) + j' \neq g(b) + i' =: y \in Z(J')$. Let $0 \neq i, j \in J$ such that $0 \neq f(a) + j$, then $(f(a) + j, x)$ and $(f(b) + i, y)$ are two non adjacent vertices in $\Gamma(R)$ or $\text{diam}(\Gamma(R)) = 2$, then there is a path in $\Gamma(R)$: $(f(a) + j, x) - (f(c) +$

$r, g(c) + r') - (f(b) + i, y)$, for some $c \in A, (r, r') \in J \times J'$. Since $f(A) + J$ is a domain and $f(a) + j \neq 0$, then $(f(c) + r) = 0$, then $0 \neq g(c) + r' \in J'$ and $(g(c) + r')x = (g(c) + r')y = 0$, so (C_4) holds.

(2) Assume that (C_1) and (C_3) hold. In view of the proof of (1), we envisage only three cases:

- Let $x_1 := (0, i') \in V_1$ and $x_3 := (f(a) + j, g(a) + j') \in V_3$, then there is $0 \neq k \in J$ such that $k(f(a) + j) = 0$, then $x_1 - (k, 0) - x_3$ is a path in $\Gamma(R)$, then $d(x_1, x_3) = 2$
- Let $x_2 := (i, 0) \in V_2$ and $x_3 := (f(a) + j, g(a) + j') \in V_3$. If $i(f(a) + j) = 0$, then $x_2x_3 = 0$, then $d(x_2, x_3) = 1$. If $i(f(a) + j) \neq 0$, then by (C_1) , there is $0 \neq k \in J$ such that $k(f(a) + j) = ki = 0$, then $x_2 - (k, 0) - x_3$ is a path in $\Gamma(R)$, then $d(x_2, x_3) = 2$.
- Let $x_3 := (f(a) + i, g(a) + i') \neq (f(b) + j, g(b) + j') =: y_3 \in V_3$. If $f(a) + i = f(b) + j$, then there is $0 \neq k \in J$ such that $k(f(a) + i) = k(f(b) + j) = 0$, then $x_3 - (k, 0) - y_3$ is a path in $\Gamma(R)$, then $d(x_3, y_3) = 2$. If $f(a) + i \neq f(b) + j$, then by (C_3) , there is $0 \neq k \in J$ such that $k(f(a) + i) = k(f(b) + j) = 0$, then $x_3 - (k, 0) - y_3$ is a path in $\Gamma(R)$, then $d(x_3, y_3) = 2$.

Consequently $diam(\Gamma(R)) = 2$. Conversely, Assume that $diam(\Gamma(R)) = 2$. Let $0 \neq x := f(a) + i \in Z(J)$ and $j \in J$ with $j(f(a) + i) \neq 0$. Let $0 \neq i' \in J'$ such that $g(a) + i' \neq 0$. We have $(f(a) + i, g(a) + i')$ and $(j, 0)$ are two non adjacent vertices in $\Gamma(R)$, or $diam(\Gamma(R)) = 2$, then there is a path $(x, g(a) + i') - (f(b) + k, g(b) + k') - (j, 0)$. Since $g(A) + J'$ is a domain and $g(a) + i' \neq 0$, then $g(b) + k' = 0$, then $0 \neq f(b) + k \in J$ and $(f(b) + k)x = (f(b) + k)j = 0$, so (C_1) holds. Let $x := f(a) + i \neq f(b) + j =: y \in Z(J)$. Let $0 \neq i', j' \in J'$ such that $g(a) + i' \neq 0$. We have $(x, g(a) + i')$ and $(y, g(b) + j')$ are two non adjacent vertices in $\Gamma(R)$, or $diam(\Gamma(R)) = 2$, then there is a path: $(x, g(a) + i') - (f(c) + r, g(c) + r') - (y, g(b) + j')$. Since $g(A) + J'$ is a domain and $g(a) + i' \neq 0$, then $g(c) + r' = 0$, so $0 \neq f(c) + r \in J$ and $(f(c) + r)x = (f(c) + r)y = 0$. Thus (C_3) holds.

The special case where both $f(A) + J$ and $g(A) + J'$ are domains is given in the next corollary.

Corollary 2.3. *Let $f : A \rightarrow B, g : A \rightarrow C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$ and R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . Assume that $f(A) + J$ and $g(A) + J'$ are domains. Then $diam(\Gamma(R)) = 2$.*

Proof. If $f(A) + J$ and $g(A) + J'$ are domains, then C_1, C_2, C_3 and C_4 always hold since $Z(J) = Z(J') = 0$. Then by Theorem 2.1, $diam(\Gamma(R)) = 2$. □

Corollary 2.4. [10, Corollary 3.2]: *Let $f : A \rightarrow B$ be a ring homomorphism, J a non-zero proper ideal of B such that $0 \neq f^{-1}(J)$. If A and $f(A) + J$ are domains, then $diam(\Gamma(A \bowtie^f (J))) = 2$*

Next, consider the following conditions:

(C_5) : For all $a \in Z(f^{-1}(J))$ and all $b \in f^{-1}(J) : ab \neq 0$, then $ac = bc = 0$ for some $0 \neq c \in f^{-1}(J)$.

(C_6) : For all $a, b \in Z(f^{-1}(J)) : a \neq b$, then $ac = bc = 0$ for some $0 \neq c \in f^{-1}(J)$. The next result is a direct consequence of Theorem 2.1.

Corollary 2.5. [10, Theorem 3.1]: *Let $f : A \rightarrow B$ be a ring homomorphism, J a non-zero proper ideal of B , assume $0 \neq f^{-1}(J)$.*

(i) *If A is a domain, then $diam(\Gamma(A \bowtie^f (J))) = 2$ if and only if (C_1) and (C_3) hold.*

(ii) *If $f(A) + J$ is a domain, then $diam(\Gamma(A \bowtie^f (J))) = 2$ if and only if (C_5) and (C_6) hold.*

Next, we investigate conditions under which $diam(\Gamma(A \bowtie^{f,g} (J, J'))) = 3$. For this purpose, let us negate the aforementioned conditions $(C1), (C2), (C3), (C4), (C5), (C6)$, to get the following:

$(\overline{C_1})$: There exist $x \in Z(J)$ and $j \in J$ with $jx \neq 0$ and for all $i \in J, ij \neq 0$ or $ix \neq 0$.

- (\overline{C}_2) : There exist $x \in Z(J')$ and $j' \in J'$ with $j'x \neq 0$ and for all $i' \in J'$, $i'j' \neq 0$ or $i'x \neq 0$.
 - (\overline{C}_3) : There exist $x, y \in Z(J)$ with $x \neq y$ and for all $i \in J$, $ix \neq 0$ or $iy \neq 0$.
 - (\overline{C}_4) : There exist $x, y \in Z(J')$ with $x \neq y$ and for all $i' \in J'$, $i'x \neq 0$ or $i'y \neq 0$.
 - (\overline{C}_5) : There exist $a \in Z(f^{-1}(J))$ and $b \in f^{-1}(J)$ with $ab \neq 0$ and for all $c \in f^{-1}(J)$, $ac \neq 0$ or $bc \neq 0$.
 - (\overline{C}_6) : There exist $a, b \in Z(f^{-1}(J))$ with $a \neq b$ and for all $c \in f^{-1}(J)$, $ac \neq 0$ or $bc \neq 0$.
- As a straightforward application of Theorem 2.1, we get the next “dual” result.

Proposition 2.6. *Under the above notation, suppose that J and J' are nonzero proper ideals of B and C , respectively, with $0 \neq f^{-1}(J) = g^{-1}(J')$.*

- (i) *If $f(A) + J$ is a domain, then $\text{diam}(\Gamma(R)) = 3$ if and only if (\overline{C}_2) or (\overline{C}_4) hold.*
- (ii) *If $g(A) + J'$ is a domain, then $\text{diam}(\Gamma(R)) = 3$ if and only if (\overline{C}_1) or (\overline{C}_3) hold.*

Corollary 2.7. [10, Proposition 3.8]: *Let $f : A \rightarrow B$ be a ring homomorphism, J a non-zero proper ideal of B such that $0 \neq f^{-1}(J)$.*

- (i) *If A is a domain, then $\text{diam}(\Gamma(A \bowtie^f(J))) = 3$ if and only if (\overline{C}_1) or (\overline{C}_3) hold.*
- (ii) *If $f(A) + J$ is a domain, then $\text{diam}(\Gamma(A \bowtie^f(J))) = 3$ if and only if (\overline{C}_5) or (\overline{C}_6) hold.*

Proposition 2.8. *Let $f : A \rightarrow B, g : A \rightarrow C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . Then :*

- (i) *If J and J' are regulars, $I_0 = f^{-1}(J) = g^{-1}(J')$ is radical and $Z(f(A) + J)$ or $Z(g(A) + J')$ is not an ideal, then $\text{diam}(\Gamma(R)) = 3$.*
- (ii) *If $Z(A)$ is not an ideal, f injective, $I_0 = f^{-1}(J) = g^{-1}(J')$ is radical, $f^{-1}(Z(B)) \subseteq Z(A)$ and $g^{-1}(Z(C)) \subseteq Z(A)$, then $\text{diam}(\Gamma(R)) = 3$.*

Proof. (1) Claim 1: $Z(R)$ is not an ideal.

Let $(0, 0) \neq (j, j') \in J \times J'$, it's clear that $(0, j'), (j, 0) \in Z(R)$ but (j, j') is regular if J and J' are regulars, as desired.

Claim 2: If L_1 is a minimal prime ideal in $f(A) + J$, then $\overline{L}_1 = (L_1 \times (g(A) + J')) \cap R$ is minimal prime ideal in R . Let L_1 be a proper ideal of $f(A) + J$, assume that $L_1 \in \text{spec}(f(A) + J)$ is minimal prime in $f(A) + J$. Let $P \in \text{spec}(R)$ such that $P \subseteq \overline{L}_1$. If $J \times J' \subseteq P$, then there exists $L \in \text{spec}(f(A) + J), J \subseteq L$ and $L' \in \text{spec}(g(A) + J'), J' \subseteq L'$ such that $P = \overline{L} = \overline{L'}$ by [11, Proposition 5.3]. Let $f(a) + j \in L \in \text{spec}(f(A) + J), a \in A, j \in J$, then $(f(a) + j, g(a) + j') \in R$ for all $j' \in J'$ and $(f(a) + j, g(a) + j') \in L \times (g(A) + J')$, then $(f(a) + j, g(a) + j') \in (L \times (g(A) + J')) \cap R = \overline{L} = P \subseteq \overline{L}_1$, then $(f(a) + j, g(a) + j') \in \overline{L}_1 = (L_1 \times (g(A) + J')) \cap R$, then $(f(a) + j, g(a) + j') \in L_1 \times (g(A) + J')$, so $f(a) + j \in L_1 \Rightarrow L \subseteq L_1$. Then by minimality, $L = L_1$, then $\overline{L} = \overline{L}_1 = P$. If $J \times J' \not\subseteq P$, then there exists $L \subseteq \text{spec}(f(A) + J)$, (or in $\text{spec}(g(A) + J')$) such that $J \not\subseteq L$, (or $J' \not\subseteq L$) and $P = \overline{L}$ by [11, Proposition 5.3]. Say $L \subseteq \text{spec}(f(A) + J), J \not\subseteq L$ and $P = \overline{L} = (L \times (g(A) + J')) \cap R \subseteq \overline{L}_1$. Let $0 \neq f(a) + j \in L; a \in A, j \in J$, then $(f(a) + j, g(a) + j') \in R$ for all $j' \in J'$ and $(f(a) + j, g(a) + j') \in L \times (g(A) + J')$, then $(f(a) + j, g(a) + j') \in (L \times (g(A) + J')) \cap R = \overline{L} \subseteq \overline{L}_1$, then $(f(a) + j, g(a) + j') \in L_1 \times (g(A) + J')$, then $f(a) + j \in L_1$, therefore $L \subseteq L_1$. Then by minimality $L = L_1$, then $\overline{L} = \overline{L}_1$. Hence $P = \overline{L}_1$. In both cases, we have $P = \overline{L}_1$, then \overline{L}_1 is minimal prime in R .

Claim 3: If L_2 is minimal prime in $g(A) + J'$, then $\overline{L}_2 = (f(A) + J) \times L_2 \cap R$ is minimal prime in R . To prove this, use the same reasoning as in claim 2.

Now, by claim 1, $Z(R)$ is not an ideal. If R is a non-reduced ring, then $\text{diam}(\Gamma(R)) = 3$ by [12, Corollary 2.5]. Assume that R is reduced, since I_0 is radical, $f(A) + J$ is reduced, $\text{Nil}(C) \cap J' = 0, g(A) + J'$ is reduced, and $\text{Nil}(B) \cap J = 0$. Suppose that $f(A) + J$ is reduced, and $\text{Nil}(C) \cap J' = 0$. Assume that $\text{diam}(\Gamma(R)) \leq 2$, then by [12, Theorem 2.2], R has exactly two minimal prime ideals. If $\text{diam}(\Gamma(f(A) + J)) = 3$, then by [12, Theorem 2.6(4)] $f(A) + J$ has more than two minimal prime ideals, then claim 2 yields that R has more than two minimal prime ideals, a contradiction. Hence $\text{diam}(\Gamma(f(A) + J)) \leq 2, Z(f(A) + J)$ is not an ideal of $f(A) + J$ reduced, then $f(A) + J$ has exactly two distinct minimal primes L_1, L_2 and by claim 2 R has two distinct minimal primes $\overline{L}_1, \overline{L}_2$. On the other hand, R reduced equivalent that $\alpha \times_{A/I_0} \beta$

is reduced, with $\alpha : f(A) + J \rightarrow A/I_0$ and $\beta : g(A) + J' \rightarrow A/I_0$ be two ring homomorphisms and $I_0 = f^{-1}(J) = g^{-1}(J')$, $\ker(f) = J$ and $\ker(g) = J'$. The fact that $\alpha \times_{A/I} \beta$ is reduced, implies that $\text{Nilp}(f(A) + J) \cap J = \text{Nilp}(g(A) + J') \cap J' = 0$ by [5, Proposition 4.9]. Since $\text{Nilp}(g(A) + J') \cap J' = 0$, then there is one minimal prime ideal L in $g(A) + J'$, with $J' \not\subseteq L$, then by claim 3, $\bar{L} = ((f(A) + J) \times L) \cap R$ is a minimal prime ideal in R ; hence R has three minimal prime ideals, a contradiction. It completes the proof of (1).

(2) We claim that if $Z(A)$ is not an ideal, f injective, $f^{-1}(Z(B)) \subseteq Z(A)$, and $g^{-1}(Z(C)) \subseteq Z(A)$, then $Z(R)$, $Z(f(A) + J)$ and $Z(g(A) + J')$ are not ideals. Let $0 \neq a, b \in Z(A)$ such that $a - b \notin Z(A)$. The fact that $a, b \in Z(A)$, yields that there are $0 \neq s, t \in A$ such that $as = 0 = bt$, then $(f(a), g(a)), (f(b), g(b)) \in Z(R)$, $f(a), f(b) \in Z(f(A) + J)$ and $g(a), g(b) \in Z(g(A) + J')$ (since f is injective). If $f(a - b) = 0$, then $f(a - b) \in Z(B)$, then $a - b \in f^{-1}(Z(B)) \subseteq Z(A)$. If $g(a - b) = 0$, for the same reasoning we have $a - b \in Z(A)$. Hence we can suppose that $0 \neq f(a - b)$ and $0 \neq g(a - b)$. If $(f(a - b), g(a - b)) \in Z(R)$, then there exists $0 \neq (f(s) + k, g(s) + k') \in R$ such that $f(a - b)(f(s) + k) = 0 = g(a - b)(g(s) + k')$. If $f(s) + k = 0$, then $0 \neq g(s) + k' \in J'$ and $g(a - b)(g(s) + k') = 0$, then $g(a - b) \in Z(C)$, then $a - b \in Z(A)$. If $g(s) + k' = 0$ for the same reasoning we have also $a - b \in Z(A)$. On the otherwise we have also $a - b \in Z(A)$. If $f(a - b) \in Z(f(A) + J)$, then there exists $0 \neq f(s) + k \in f(A) + J$ such that $f(a - b)(f(s) + k) = 0$, then $f(a - b) \in Z(B)$, therefore $a - b \in Z(A)$. For the same reasoning, if $g(a - b) \in Z(g(A) + J')$, then $a - b \in Z(A)$. Consequently $Z(R)$, $Z(f(A) + J)$ and $Z(g(A) + J')$ are not ideals. After this, we use the same reasoning as (1) and prove that $\text{diam}(\Gamma(R)) = 3$. □

Example 2.9. Let $A := Z_4, B := Z_2[X], J := XB, C := Z_4$ and $J' := 2Z_4$. Consider two ring homomorphisms $f : A \rightarrow B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g : A \rightarrow C$ defined by: $g(a) = a$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $f(A) + J$ is a domain and $g(A) + J' := Z_4$ is a domain, $Z(J') = \{0, 2\}$. For all $x \in Z(J')$ and all $j' \in J'$, we have $xj' = 0$, then (C_2) holds. For all $x, y \in Z(J')$ with $x \neq y$, we have $2x = 2y = 0$ ($2 \in J'$), then (C_4) holds. Therefore, by Theorem 2.1(1), the diameter of the zero-divisor graphs of R is 2.

Example 2.10. Let $A := Z_4, B := Z_2[X]/(X^2), J := \{0, x\}, C := Z_3, J' := 2Z_3$, where x denotes the class of X modulo X^2 . Consider two ring homomorphisms $f : A \rightarrow B$ and $g : A \rightarrow C$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $g(A) + J' = Z_3$ is a domain and $Z(J) = \{0, 1, x, 1 + x\}$. We have $1 + x \in Z(J)$, $x \in J$ with $x(1 + x) = x \neq 0$ and for all $0 \neq j \in J, j(1 + x) \neq 0$, then \bar{C}_1 holds. We have also $1, 1 + x \in Z(J), 1 \neq 1 + x$ and for all $0 \neq j \in J, j1 \neq 0$ and $j(1 + x) \neq 0$, then \bar{C}_3 holds. Then by Proposition 2.6(2), the diameter of the zero-divisor graphs of R is 3.

3 When $\Gamma(A \bowtie^{f,g} (J, J'))$ is complete

Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms, J and J' two proper ideals of B and C respectively such that $f^{-1}(J) = g^{-1}(J')$. Let $R := A \bowtie^{(f,g)} (J, J') = \{(f(a) + j, g(a) + j')/a \in A, (j, j') \in J \times J'\}$ be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . Our aim in this section is to establish necessary and sufficient conditions for $\Gamma(R)$ to be complete. Throughout, let:

$$Z(A) \bowtie^{(f,g)} (J, J') := \{(f(a) + j, g(a) + j')/a \in Z(A), (j, j') \in J \times J'\}$$

and for any $(j, j') \in J \times J'$, let:

$$(0 : j) = \{x \in f(A) + J/xj = 0\} \quad \text{and} \quad (0 : j') = \{x \in g(A) + J'/xj' = 0\}$$

In what follows, we give necessary and sufficient conditions for $\Gamma(R)$ to be complete.

Theorem 3.1. Let $f : A \rightarrow B, g : A \rightarrow C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C respectively such that $0 \neq f^{-1}(J) = g^{-1}(J')$ and R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . If f is injective and $Z(f(A) + J) \subseteq J$ or $Z(g(A) + J') \subseteq J'$ then the following conditions are equivalent:

- (i) $\Gamma(R)$ is complete;
- (ii) $Z(A)^2 = 0, J^2 = J'^2 = 0, Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$ for all $(0, 0) \neq (j, j') \in J \times J'$;
- (iii) $Z(R)^2 = 0$.

Moreover, if any one of the three condition holds, then $Z(R) = Z(A) \bowtie^{((f,g))} (J, J')$ and $gr(\Gamma(R)) = 3$.

For the proof of Theorem 3.1, we need the lemma given below.

Lemma 3.2. Under the above notation, suppose that J and J' are nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Let:

- $V_1 := \{(0, j')/0 \neq j' \in J'\}$.
 - $V_2 := \{(j, 0)/0 \neq j \in J\}$.
 - $V_3 := \{(f(a) + j, g(a) + j')/0 \neq g(a) + j' \in Z(J')\}$.
 - $V_4 := \{(f(a) + j, g(a) + j')/0 \neq f(a) + j \in Z(J)\}$.
 - $V_5 := Z(f(A) + J)^* \times Z(g(A) + J')^*$.
- Then : $Z(R) \subseteq V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$.

Proof. Let $x = (f(a) + j, g(a) + j') \in Z(R), a \in A, (j, j') \in J \times J'$, then there is $0 \neq y = (f(s) + i, g(s) + i') \in R$ such that $xy = 0, s \in A, (i, i') \in J \times J'$. Therefore $(f(a) + j)(f(s) + i) = 0$ and $(g(a) + j')(g(s) + i') = 0$. If $f(a) + j = 0$, then $f(a) \in J$, then $a \in f^{-1}(J) = g^{-1}(J')$ then $g(a) \in J'$, so $0 \neq g(a) + j' \in J'$. Hence $x \in V_1$. If $g(a) + j' = 0$, then $0 \neq f(a) + j \in J$. Hence $x \in V_2$. Assume that $0 \neq f(a) + j$ and $0 \neq g(a) + j'$. If $f(s) + i = 0$, then $0 \neq g(s) + i' \in J'$ and $(g(a) + j')(g(s) + i') = 0$, then $g(a) + j' \in Z(J')$. Thus $x \in V_3$. If $g(s) + i' = 0$, then $0 \neq f(s) + i \in J$ and $(f(a) + j)(f(s) + i) = 0$, then $f(a) + j \in Z(J)$. Thus $x \in V_4$. On the other hand $f(a) + j \in Z(f(A) + J)^*$ and $g(a) + j' \in Z(g(A) + J')^*$. Thus $x \in V_5$, as desired. \square

Proof of Theorem 3.1

(1) \Rightarrow (2): Assume that $\Gamma(R)$ is complete. We claim that $Z(A)^2 = 0$. Assume that there exists $0 \neq a, b \in Z(A)$ such that $ab \neq 0$. Since f is injective, then $f(ab) \neq 0$. The fact that $a \in Z(A)$ implies that there exists $0 \neq s \in A : as = 0$, then $f(a)f(s) = g(a)g(s) = 0$. But f is injective, then $0 \neq f(s)$, then $(f(s), g(s)) \in R^*$ and $(f(a), g(a))(f(s), g(s)) = 0$, therefore $(f(a), g(a)) \in Z(R)$. The fact that $b \in Z(A)$ yields for the same reasoning that $(f(b), g(b)) \in Z(R)$. Since $\Gamma(R)$ complete, then $(f(a), g(a))(f(b), g(b)) = 0$ equivalent that $(f(ab), g(ab)) = 0$ equivalent that $f(ab) = g(ab) = 0$, a contradiction since f is injective and $0 \neq ab$. Hence $Z(A)^2 = 0$. Next We claim that $J^2 = J'^2 = 0$. Let $(0, 0) \neq (j, j') \in J \times J'$, it's clear that $(j, 0), (0, j') \in Z(R)$. If $|J| \geq 3$ and $|J'| \geq 3$, then for all $0 \neq i \in J, i \neq j$ and all $0 \neq i' \in J', i' \neq j'$, we have $(i, 0)(j, 0) = 0$ and $(0, i')(0, j') = 0$, ($\Gamma(R)$ is complete), then $ij = i'j' = 0$. Hence $(j, j')(i, 0) = (j, j')(0, i') = 0$, then $(j, j') \in Z(R)$. Since $\Gamma(R)$ complete, then $(j, j')(j, 0) = (j, j')(0, j') = 0$, then $j^2 = j'^2 = 0$. If $|J| = |J'| = 2$. Let $0 \neq i \in J$ and $0 \neq i' \in J'$. If $i^2 = i$ and $i'^2 = i'$, then $i(1 - i) = i'(1 - i') = 0$, then $(1 - i, 1 - i')(i, i') = 0$. Since J and J' are proper ideals of B and C , respectively, then $0 \neq 1 - i$ and $0 \neq 1 - i'$. Then $(i, i') \in Z(R)$, then $(i, i')(i, 0) = (i, i')(0, i') = 0$ since $\Gamma(R)$ is complete. Then $i^2 = i'^2 = 0$, equivalent that $i = i' = 0$, a contradiction. In both cases we have $J^2 = J'^2 = 0$ if the zero-divisor graphs is complete. Next, We claim that $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$, for all $0 \neq j \in J, 0 \neq j' \in J'$. Let $0 \neq a \in Z(A)$, then there exists $0 \neq s \in A : as = 0$, then $f(a)f(s) = g(a)g(s) = 0$. Since f is injective, then $0 \neq (f(s), g(s)) \in R$, then $(f(a), g(a)) \in Z(R)$ and we have $(j, j')(0, j') = 0$, then $(j, j') \in Z(R)^*$. Then $(f(a), g(a))(j, j') = 0$, since $\Gamma(R)$ is complete. Then $jf(a) = j'g(a) = 0$ for any $(0, 0) \neq (j, j') \in J \times J'$, then $f(a) \in (0 : j)$ and $g(a) \in (0 : j')$. Then $a \in f^{-1}(0 : j)$ and $a \in g^{-1}(0 : j')$. Hence $Z(A) \subseteq f^{-1}(0 : j)$ and $Z(A) \subseteq g^{-1}(0 : j')$. Suppose that for any $(0, 0) \neq (j, j') \in J \times J'$, we have $jf(a) = j'g(a) = 0$, for $0 \neq a \in A$, then $(f(a), g(a))(j, 0) = (f(a), g(a))(0, j') = 0$. Let $0 \neq c \in f^{-1}(J) = g^{-1}(J')$ such that $c \notin \ker(f) \cup \ker(g)$, then $(f(a), g(a))(f(c), 0) = (f(a), g(a))(0, g(c)) = 0$, then $f(ac) = g(ac) = 0$. Since f is injective, then $ac = 0$, then $a \in Z(A)$. Hence $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$, for any $(0, 0) \neq (j, j') \in J \times J'$.

(2) \Rightarrow (3): Assume that $Z(A)^2 = 0, J^2 = J'^2 = 0$ and $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$, for all

$(j, j') \in J \times J'$. Let $x = (f(a) + j, g(a) + j') \in Z(R)^*$, $a \in A, (j, j') \in J \times J'$, then by lemma 3.2, $x \in V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$. If $x \in V_1 \cup V_2$, it's easy to show that $a \in Z(A)$. If $x \in V_3$, then $0 \neq g(a) + j' \in Z(J')$, then there exists $0 \neq k' \in J'$ such that $k'(g(a) + j') = 0$, then $kg(a) = 0$, so $a \in g^{-1}(0 : k') = Z(A)$. If $x \in V_4$, then $0 \neq f(a) + j \in Z(J)$ there exists $0 \neq k \in J$ such that $k(f(a) + j) = 0$, then $kf(a) = 0$, then $a \in f^{-1}(0 : k) = Z(A)$. If $x \in V_5$, then $f(a) + j \in Z(f(A) + J)^*$ and $g(a) + j' \in Z(g(A) + J')^*$. Suppose that $Z(f(A) + J) \subseteq J$, since $J^2 = 0$, then for all $0 \neq i \in J, i(f(a) + j) = 0$, then $a \in f^{-1}(0 : i) = Z(A)$. Hence $Z(R) \subseteq Z(A) \bowtie^{(f,g)} (J, J')$. On the other hand, let $(f(a) + j, g(a) + j') \in Z(A) \bowtie^{(f,g)} (J, J')$, then $a \in Z(A), (j, j') \in J \times J'$. Then there is $0 \neq b \in Z(A)$ such that $ab = 0$ since $Z(A)^2 = 0$, then $f(ab) = g(ab) = 0$. Then $(f(a) + j, g(a) + j')(f(b), g(b)) = (f(ab) + jf(b), g(ab) + j'g(b)) = (jf(b), j'g(b))$. The fact that $b \in Z(A)$, implies that for all $(0, 0) \neq (j, j') \in J \times J'; jf(b) = j'g(b) = 0$. Then $(f(a) + j, g(a) + j')(f(b), g(b)) = 0$, then $(f(a) + j, g(a) + j') \in Z(R)$. Hence $Z(R) = Z(A) \bowtie^{(f,g)} (J, J')$. Let $x \neq y \in Z(A) \bowtie^{(f,g)} (J, J')$, it's easy to prove that $xy = 0$, then $Z(R)^2 = 0$.

(3) \Rightarrow (1): It's trivial by the Anderson, Livingston definition. The "moreover" is proved by (2) \Rightarrow (3) and the fact that if $0 \neq a \in f^{-1}(J) = g^{-1}(J')$, then $(f(a), 0) - (f(a), g(a)) - (0, g(a)) - (f(a), 0)$ is a cycle in $\Gamma(R)$ of length 3, then $gr(\Gamma(R)) = 3$.

Example 3.3. Let $A := Z_4, B := Z_4 \times Z_4, J := 2Z_4 \times 2Z_4$ be an ideal of $B, f : A \rightarrow B$ be the ring homomorphism defined by: $f(a) = (a, a)$, for all $a \in A$, then f is injective, $f^{-1}(J) = \{0, 2\}$. Let $C = Z_2 \times Z_2[X]/(X), J' := ((0, \bar{X}))$ be an ideal of $C, g : A \rightarrow C$ be the ring homomorphism defined by: $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. It's easy to show that $g^{-1}(J') = \{0, 2\} = f^{-1}(J)$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $Z(A) = \{0, 2\} = f^{-1}(0 : (0 : \bar{X})) = g^{-1}(0 : j), 0 \neq j \in J, J^2 = 0 = J'^2, Z(A)^2 = 0$ and $Z(f(A) + J) = \{(0, 0), (0, 2), (2, 0), (2, 2)\} = J$. Thus by Theorem 3.1, $\Gamma(R)$ is complete.

4 On the girth of the zero-divisors graph of bi-amalgamations

This section deals with the girth of the zero-divisor graphs of bi-amalgamations $R := A \bowtie^{(f,g)} (J, J')$ for some various settings of $f(A) + J$ and $g(A) + J'$.

Consider the following conditions:

- w_1 : For any distinct elements $0 \neq i, j \in J, ij \neq 0$.
- w_2 : For any distinct elements $0 \neq i', j' \in J'; i'j' \neq 0$.
- w_3 : For any elements $0 \neq a, b \in f^{-1}(J), a \neq b; ab \neq 0$.
- \bar{w}_1 : There are two distinct elements $0 \neq i, j \in J$ with $ij = 0$.
- \bar{w}_2 : There are two distinct elements $0 \neq i', j' \in J'$ with $i'j' = 0$.
- \bar{w}_3 : There are two distinct elements $0 \neq a, b \in f^{-1}(J)$ with $ab = 0$.

Theorem 4.1. Under the above notation, suppose that J and J' are nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$.

- (i) If \bar{w}_1 or \bar{w}_2 hold, then $gr(\Gamma(R)) = 3$.
- (ii) If $f(A) + J$ is a domain and $|J'| = 2$ or $g(A) + J'$ is a domain and $|J| = 2$, then $gr(\Gamma(R)) = \infty$.
- (iii) If $f(A) + J$ is a domain, $|J'| \geq 3$ and w_2 holds or $g(A) + J'$ is a domain, $|J| \geq 3$ and w_1 holds, then $gr(\Gamma(R)) = 4$.

Proof. (1): Let $0 \neq i, j \in J, i \neq j$ with $ij = 0$, let $0 \neq j' \in J'$, clearly, $(i, 0) - (0, j') - (j, 0) - (i, 0)$ is a cycle in $\Gamma(R)$ of length 3, then $gr(\Gamma(R)) = 3$. Let $0 \neq i', j' \in J', i' \neq j'$ with $i'j' = 0$, let $0 \neq j \in J$. Then $(0, j') - (j, 0) - (0, i') - (0, j')$ is a cycle in $\Gamma(R)$ of length 3, then $gr(\Gamma(R)) = 3$.

(2): Assume $f(A) + J$ is a domain and $|J'| = 2$. Suppose that $\Gamma(R)$ contain a cycle of length n , say: $x_1 - x_2 - x_3 - \dots - x_n - x_1$ with $x_k := (f(a_k) + i_k, g(a_k) + i'_k)$, for $k = 1, \dots, n$. If $f(a_1) + i_1 \neq 0$, then $f(a_2) + i_2 = f(a_n) + i_n = 0$, so $0 \neq g(a_2) + i'_2 \in J'$ and $0 \neq g(a_n) + i'_n \in J'$ with $g(a_2) + i'_2 \neq g(a_n) + i'_n$, it's a contradiction with the hypothesis. If $f(a_1) + i_1 = 0$,

then $0 \neq g(a_1) + i_1 \in J'$. Since $(f(a_2) + i_2)(f(a_3) + i_3) = 0$, then $f(a_2) + i_2 = 0$ or $f(a_3) + i_3 = 0$, then $0 \neq g(a_2) + i'_2 \in J'$ or $0 \neq g(a_3) + i'_3 \in J'$ with $g(a_1) + i'_1 \neq g(a_2) + i'_2$ or $g(a_1) + i'_1 \neq g(a_3) + i'_3$; it's contradiction. With the same reasoning, in both cases we have a contradiction with our hypothesis, then $gr(\Gamma(R)) = \infty$. Assume that $g(A) + J'$ is a domain and $|J| = 2$. Suppose that $\Gamma(R)$ contains a cycle of length n , say: $x_1 - x_2 - \dots - x_n - x_1$ with $x_k := (f(a_k) + i_k, g(a_k) + i'_k)$, for $k = 1, \dots, n$. If $g(a_1) + i'_1 \neq 0$, then $g(a_2) + i'_2 = g(a_n) + i'_n = 0$, then $0 \neq f(a_2) + i_2 \in J$ and $0 \neq f(a_n) + i_n \in J$ with $f(a_2) + i_2 \neq f(a_n) + i_n$, it's a contradiction. If $g(a_1) + i'_1 = 0$, then $0 \neq f(a_1) + i_1 \in J$. Since $(g(a_2) + i'_2)(g(a_3) + i'_3) = 0$ and $g(A) + J'$ is a domain, then $g(a_2) + i'_2 = 0$ or $g(a_3) + i'_3 = 0$, then $f(a_1) + i_1 \neq f(a_2) + i_2 \in J$ or $f(a_1) + i_1 \neq f(a_3) + i_3 \in J$, a contradiction. Then $gr(\Gamma(R)) = \infty$.

(3): Assume that $f(A) + J$ is a domain, $|J'| \geq 3$ and (C_2) holds. Let $0 \neq i', j' \in J', i' \neq j'$, let $0 \neq i \in J$. clearly, $(0, j') - (i, 0) - (0, i') - (i^2, 0) - (0, j')$ is a cycle in $\Gamma(R)$ of length 4, then $gr(\Gamma(R)) \leq 4$.

Suppose that $gr(\Gamma(R)) = 3$, and let $x_1 - x_2 - x_3 - x_1$ a cycle in $\Gamma(R)$ of length 3 with $x_k := (f(a_k) + i_k, g(a_k) + i'_k)$, for $k = 1, 2, 3$. If $f(a_1) + i_1 \neq 0$, then $f(a_2) + i_2 = f(a_3) + i_3 = 0$, then $0 \neq g(a_2) + i'_2 \in J'$ and $0 \neq g(a_3) + i'_3 \in J'$ with $g(a_2) + i'_2 \neq g(a_3) + i'_3$ and $(g(a_2) + i'_2)(g(a_3) + i'_3) = 0$, a contradiction. If $f(a_1) + i_1 = 0$, then $0 \neq g(a_1) + i'_1 \in J'$. Since $(f(a_2) + i_2)(f(a_3) + i_3) = 0$ then $f(a_2) + i_2 = 0$ or $f(a_3) + i_3 = 0$, then $0 \neq g(a_2) + i'_2 \in J'$ or $0 \neq g(a_3) + i'_3 \in J'$ with $g(a_1) + i'_1 \neq g(a_2) + i'_2$ and $(g(a_1) + i'_1)(g(a_2) + i'_2) = 0$ or $g(a_1) + i'_1 \neq g(a_3) + i'_3$ and $(g(a_1) + i'_1)(g(a_3) + i'_3) = 0$, a contradiction. Then $gr(\Gamma(R)) = 4$. Assume that $g(A) + J'$ is a domain, $|J| \geq 3$ and (C_1) holds. Let $0 \neq i, j \in J, i \neq j$, let $0 \neq j' \in J'$, then $(i, 0) - (0, j') - (j, 0) - (0, j'^2) - (i, 0)$ is a cycle in $\Gamma(R)$ of length 4, then $gr(\Gamma(R)) \leq 4$. Suppose that $gr(\Gamma(R)) = 3$ and let $x_1 - x_2 - x_3 - x_1$ a cycle in $\Gamma(R)$ of length 3 with $x_k := (f(a_k) + i_k, g(a_k) + i'_k)$, for $k = 1, 2, 3$. If $g(a_1) + i'_1 \neq 0$, then $g(a_2) + i'_2 = g(a_3) + i'_3 = 0$, then $0 \neq (f(a_2) + i_2), (f(a_3) + i_3) \in J$ with $f(a_2) + i_2 \neq f(a_3) + i_3$ and $(f(a_2) + i_2)(f(a_3) + i_3) = 0$, a contradiction. If $g(a_1) + i'_1 = 0$, then $0 \neq f(a_1) + i_1 \in J$. Since $(g(a_2) + i'_2)(g(a_3) + i'_3) = 0$ and $g(A) + J'$ is a domain, then $g(a_2) + i'_2 = 0$ or $g(a_3) + i'_3 = 0$, then $0 \neq f(a_2) + i_2 \in J$ or $0 \neq f(a_3) + i_3 \in J$ with $f(a_1) + i_1 \neq f(a_2) + i_2$ and $(f(a_1) + i_1)(f(a_2) + i_2) = 0$ or $(f(a_1) + i_1)(f(a_3) + i_3) = 0$, a contradiction. Hence in both cases we have $gr(\Gamma(R)) = 4$. □

Corollary 4.2. [10, Theorem 4.1] *Let $f : A \rightarrow B$ be a ring homomorphism, J a non-zero proper ideal of B such that $0 \neq f^{-1}(J)$.*

- (i) *If $\overline{w_1}$ or $\overline{w_3}$ hold, then $gr(\Gamma(A \bowtie^f(J))) = 3$.*
- (ii) *If A is a domain and $|J| = 2$ or $|f^{-1}(J)| = 2$ and $f(A) + J$ is a domain, then $gr(\Gamma(A \bowtie^f(J))) = \infty$.*
- (iii) *If A is a domain, $|J| \geq 3$ and (w_1) holds or $f(A) + J$ is a domain, $|f^{-1}(J)| \geq 3$ and (w_3) holds, then $gr(\Gamma(A \bowtie^f(J))) = 4$.*

Next we give a result when $f(A) + J$ and $g(A) + J'$ are domains.

Corollary 4.3. *Under the above notation, suppose that J and J' are nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Assume that $f(A) + J$ and $g(A) + J'$ are domains. Then $gr(\Gamma(R)) = 4$.*

Corollary 4.4. [10, Corollary 4.2] *Let $f : A \rightarrow B$ a ring homomorphism, J a nonzero proper ideal of B . Assume that both A and $f(A) + J$ are domains with $0 \neq f^{-1}(J)$. Then $gr(\Gamma(A \bowtie^f(J))) = 4$.*

The special case, where neither $f(A) + J$ nor $g(A) + J'$ is domain, is given below.

Proposition 4.5. *Let $f : A \rightarrow B, g : A \rightarrow C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C , respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. If $Z(A) \neq 0$ and f and g are injectives. Then $gr(\Gamma(A \bowtie^{(f,g)}(J, J'))) = 3$.*

Proof. Assume that $Z(A) \neq 0$, f and g are injectives. Let $0 \neq a, b \in A$ such that $ab = 0$ ($a = b$ is a possibility), let $0 \neq c \in f^{-1}(J) = g^{-1}(J')$. If $ca = 0$ or $cb = 0$, then $c \in f^{-1}(J) \cap Z(A)$. If $ca \neq 0$ and $cb \neq 0$, then $ca \in f^{-1}(J) \cap Z(A)$. Then without loss generality, we assume that

$f^{-1}(J) \cap Z(A) \neq \{0\}$. Let $0 \neq x \in f^{-1}(J) \cap Z(A)$ and let $0 \neq y \in A$ such that $xy = 0$, then $f(x)f(y) = g(x)g(y) = 0$. Since f and g injectives, then $0 \neq f(x), f(y)$ and $0 \neq g(x), g(y)$. Hence, $(f(y), g(y)) - (f(x), 0) - (0, g(x)) - (f(y), g(y))$ is a cycle in $\Gamma(A \bowtie^{(f,g)} (J, J'))$ of length 3, then $gr(\Gamma(A \bowtie^{(f,g)} (J, J'))) = 3$. \square

Corollary 4.6. [10, Proposition 4.4] Let $f : A \rightarrow B$ be a ring homomorphism, J a non-zero proper ideal of B . Assume that $Z(A) \neq 0$ and f injective with $0 \neq f^{-1}(J)$. Therefore $gr(\Gamma(A \bowtie^f (J))) = 3$.

Example 4.7. Let $A := Z_6, B := Z_6 \times Z_6, C := Z_8 \times Z_8, J := 2Z_6 \times Z_6$ and $J' := 2Z_8 \times Z_8$. Consider the ring homomorphisms $f : A \rightarrow B$ defined by: $f(a) = (a, a)$ for all $a \in A$ and $g : A \rightarrow C$ defined by: $g(a) = (a, a)$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $\overline{w_1}$ holds, then by Theorem 4.1(1), $gr(\Gamma(R)) = 3$.

Example 4.8. Let $A := Z_4, B := Z_2[X], J := XB, C := Z_4, J' := 2Z_4$. Consider the ring homomorphisms $f : A \rightarrow B$ and $g : A \rightarrow C$ defined by: $f(0) = f(2) = 0, f(1) = f(3) = 1$ and $g(a) = a$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $f(A) + J$ is a domain and $|J'| = 2$, then by Theorem 4.1(2), $gr(\Gamma(R)) = \infty$.

Example 4.9. Let $A := Z_4, B := Z_2[X]/(X^2), J := \{0, x\} (|J| = 2)$, where x denotes the class of X modulo (X^2) and $f : A \rightarrow B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Let $C := Z_3, J' := 2Z_3$ and $g : A \rightarrow C$ defined by: $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $g(A) + J' = Z_3$ is a domain and $|J| = 2$, then by Theorem 4.1(2), $gr(\Gamma(R)) = \infty$.

Example 4.10. Let $A := Z_4, B := Z_2[X], J := XB, C := Z_6$ and $J' := 2Z_6$. Consider the ring homomorphisms $f : A \rightarrow B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g : A \rightarrow C$ defined by: $g(a) = a$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $|J'| \geq 3, f(A) + J$ is a domain and (w_2) holds, then by Theorem 4.1(3), $gr(\Gamma(R)) = 4$.

Example 4.11. Consider the bi-amalgamation R defined as in Example 4.7. Clearly $Z(A) \neq 0$ and f and g are injectives, then by Proposition 4.5, $gr(\Gamma(R)) = 3$.

Example 4.12. Let $A := Z_4, B := Z_2[X], J := XB, C := Z_5, J' := 2Z_5$. Consider the ring homomorphisms $f : A \rightarrow B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g : A \rightarrow C$ defined by: $g(a) = a$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) . It's easy to show that $f(A) + J$ and $g(A) + J'$ are domains, then by Corollary 4.3, $gr(\Gamma(R)) = 4$.

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Author information

Mohamed Chhiti, Modelling and Mathematical Structures Laboratory
Department of Mathematics
Faculty of Economics of Fez, University S.M. Ben Abdellah Fez, Morocco, Morocco.
E-mail: chhiti.med@hotmail.com

Loubna Es-Salhi, Modelling and Mathematical Structures Laboratory
Department of Mathematics
Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco, Morocco.
E-mail: loubna.essalhi@usmba.ac.ma

Soibri Moindze, Laboratory of Mathematics, Statistics, Computer Science and applications
Faculty of Science and Technology of Morani, Comoros, University of the Comoros, Comoros.
E-mail: moindzesoibri@gmail.com

Received: 2025-05-21

Accepted: 2026-01-12