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ON CONSTRUCTING f-BIHARMONIC MAPS AND CONFORMAL DEFORMATIONS

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Abstract. In this paper, we present some characterizations of f-biharmonic maps obtained by conformally deforming either the domain metric or the codomain metric. We study, in particular, the case of the identity map and construct new examples of f-biharmonic maps.

1 Introduction.

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. The energy and the bi-energy functional of ϕ are respectively defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g$$

and

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

The map ϕ is said to be harmonic (respectively biharmonic) if it is a critical point of the energy functional (respectively of the bi-energy functional). Equivalently, ϕ is harmonic if and only if

$$\tau(\phi) = \mathrm{Tr}_g \nabla d\phi = 0,$$

and it is biharmonic if and only if

$$\tau_2(\phi) = -\mathrm{Tr}_g \left(\nabla^\phi\right)^2 \tau(\phi) - \mathrm{Tr}_g R^N(\tau(\phi), d\phi)\, d\phi = 0.$$

$\tau(\phi)$ and $\tau_2(\phi)$ are respectively called the tension and the bi-tension field of the map ϕ.

A natural generalization of harmonic and biharmonic maps is defined as follows : Let $f \in C^\infty(M)$ be a positive function, the map ϕ is said to be f-harmonic if it is a critical point of the f-energy functional :

$$E_f(\phi) = \frac{1}{2}\int_M f|d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is f-harmonic if it satisfies the associated Euler-Lagrange equation :

$$\tau_f(\phi) = f\tau(\phi) + d\phi(\mathrm{grad} f) = f\left(\tau(\phi) + d\phi(\mathrm{grad}\ln f)\right) = 0,$$

$\tau_f(\phi)$ is called the f-tension field of ϕ. The map ϕ is said to be f-biharmonic if it is a critical point of the f-bi-energy functional :

$$E_{2,f}(\phi) = \frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g.$$

Equivalently, ϕ is f-biharmonic if it satisfies the associated Euler-Lagrange equation :

$$\tau_{2,f}(\phi) = f\tau_2(\phi) - (\Delta f)\tau(\phi) - 2\nabla_{\operatorname{grad} f}\tau(\phi) = 0,$$

$\tau_{2,f}(\phi)$ is called the f-bitension field of ϕ. Then ϕ is f-biharmonic if and only if (see [6])

$$\tau_2(\phi) - \left(\Delta \ln f + |\operatorname{grad} \ln f|^2\right)\tau(\phi) - 2\nabla_{\operatorname{grad} \ln f}\tau(\phi) = 0. \tag{1.1}$$

It is clear that any harmonic map is f-biharmonic. If the map ϕ is biharmonic ($\tau_2(\phi) = 0$), then ϕ is f-biharmonic if and only if

$$\left(\Delta \ln f + |\operatorname{grad} \ln f|^2\right)\tau(\phi) + 2\nabla_{\operatorname{grad} \ln f}\tau(\phi) = 0.$$

Biharmonic maps have been extensively studied in the literature, with various construction methods leading to numerous examples of biharmonic but non-harmonic maps (see [1], [2], [4], [11], [12], [13], [15]). In [5] and [7], the authors investigate a generalized class of f-harmonic maps, presenting new properties that extend Fuglede-Ishihara's characterization of harmonic morphisms and they discuss the stability of L-harmonic maps. Meanwhile, [6] and [15] explore properties of f-biharmonic maps, focusing on conformal maps between equidimensional manifolds. These works provide new examples of f-biharmonic maps and characterize p-biharmonicity in specific cases. Additionally, [8] establishes conditions for the f-harmonicity of certain maps involving doubly warped product manifolds, while [9] introduces f-biharmonic maps as a natural generalization of biharmonic maps and examines their properties. The author in [10] presents a generalization of the global version of Chen's biharmonic conjecture.

In this work, we present alternative constructions of f-biharmonic maps through conformal deformations of either the domain or codomain metric. First, we characterize f-biharmonicity by conformally deforming the domain metric, which yields new examples and allows us to analyze special cases. Second, we investigate the f-biharmonicity of the identity map under conformal deformations of the codomain metric.

2 The f-biharmonicity of $\phi : \left(M^m, \widetilde{g} = \sigma^{-2}g\right) \longrightarrow (N^n, h)$.

In this section, we consider a smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ and $\widetilde{g} = \sigma^{-2}g$ be a metric conformally equivalent to g. The relation between $\widetilde{\nabla}$ and ∇ is given by (see [3])

$$\widetilde{\nabla}_X Y = \nabla_X Y - X(\ln\sigma)Y - Y(\ln\sigma)X + g(X,Y)\operatorname{grad}\ln\sigma,$$

where ∇ et $\widetilde{\nabla}$ are respectively the connections on M associated with g and $\widetilde{g}$. Let us choose$\{e_i\}_{i=1}^m$ to be an orthonormal frame on (M^m, g), then an orthonormal frame on $\left(M^m, \widetilde{g} = \sigma^{-2}g\right)$ is given by $\{\widetilde{e_i} = \sigma e_i\}_{i=1}^m$. Using the expression for $\widetilde{\nabla}$, we have

$$\widetilde{\nabla}_{e_i} e_i = \nabla_{e_i} e_i + (m-2)\operatorname{grad}\ln\sigma$$

and

$$\widetilde{\nabla}_{\widetilde{e_i}} \widetilde{e_i} = \sigma^2\left(\nabla_{e_i} e_i + (m-1)\operatorname{grad}\ln\sigma\right).$$

The tension field of the map ϕ with respect to $\widetilde{g}$ is given by (see [2])

$$\widetilde{\tau}(\phi) = \sigma^2\left(\tau(\phi) - (m-2)\,d\phi(\operatorname{grad}\ln\sigma)\right).$$

In a first result, we will determine the f-biharmonicity condition of the map $\phi : \left(M^m, \widetilde{g} = \sigma^{-2}g\right) \longrightarrow (N^n, h)$.

Theorem 2.1. *The map* $\phi : \left(M^m, \widetilde{g} = \sigma^{-2} g\right) \longrightarrow (N^n, h)$ *is* f*-biharmonic if and only if*

$$\begin{aligned}
&\tau_2(\phi) - 2\left(\Delta \ln \sigma - (m-4)\left|grad \ln \sigma\right|^2\right)\tau(\phi) \\
&-\left(\Delta \ln f + \left|grad \ln f\right|^2 - (m-6)\, d\ln f\,(grad \ln \sigma)\right)\tau(\phi) \\
&+(m-6)\nabla^\phi_{grad \ln \sigma}\tau(\phi) - 2\nabla^\phi_{grad \ln f}\tau(\phi) + 2(m-2)\nabla^\phi_{grad \ln f} d\phi(grad \ln \sigma) \\
&+(m-2)\left(\Delta \ln f - (m-6)\, d\ln f\,(grad \ln \sigma) + \left|grad \ln f\right|^2\right) d\phi(grad \ln \sigma) \\
&+(m-2)\left(Tr_g\left(\nabla^\phi\right)^2 d\phi(grad \ln \sigma) - (m-6)\nabla^\phi_{grad \ln \sigma} d\phi(grad \ln \sigma)\right) \\
&+2(m-2)\left(\Delta \ln \sigma - (m-4)\left|grad \ln \sigma\right|^2\right) d\phi(grad \ln \sigma) \\
&+(m-2)\, Tr_g R\left(d\phi(grad \ln \sigma), d\phi\right) d\phi = 0.
\end{aligned} \tag{2.1}$$

Proof. By equation (1.1), the map $\phi : \left(M^m, \widetilde{g} = \sigma^{-2} g\right) \longrightarrow (N^n, h)$ is f-biharmonic if and only if

$$\widetilde{\tau}_2(\phi) - \left(\widetilde{\Delta} \ln f + \left|\widetilde{\text{grad}} \ln f\right|^2\right)\widetilde{\tau}(\phi) - 2\nabla^\phi_{\widetilde{\text{grad}} \ln f}\widetilde{\tau}(\phi) = 0,$$

where

$$\widetilde{\tau}(\phi) = \sigma^2\left(\tau(\phi) - (m-2)\, d\phi(\text{grad} \ln \sigma)\right).$$

A long calculation gives us

$$\begin{aligned}
\widetilde{\tau}_2(\phi) = {}& \sigma^4\tau_2(\phi) - 2\sigma^4\left(\Delta \ln \sigma - (m-4)\left|\text{grad} \ln \sigma\right|^2\right)\tau(\phi) + (m-6)\sigma^4\nabla^\phi_{\text{grad} \ln \sigma}\tau(\phi) \\
&+(m-2)\sigma^4\left(\text{Tr}_g\left(\nabla^\phi\right)^2 d\phi(\text{grad} \ln \sigma) - (m-6)\nabla^\phi_{\text{grad} \ln \sigma} d\phi(\text{grad} \ln \sigma)\right) \\
&+2(m-2)\sigma^4\left(\Delta \ln \sigma - (m-4)\left|\text{grad} \ln \sigma\right|^2\right) d\phi(\text{grad} \ln \sigma) \\
&+(m-2)\sigma^4\text{Tr}_g R\left(d\phi(\text{grad} \ln \sigma), d\phi\right) d\phi,
\end{aligned}$$

$$\widetilde{\text{grad}} \ln f = \sigma^2 \text{grad} \ln f, \quad \left|\widetilde{\text{grad}} \ln f\right|^2 = \sigma^2\left|\text{grad} \ln f\right|^2,$$

$$\widetilde{\Delta} \ln f = \sigma^2\left(\Delta \ln f - (m-2)\, d\ln f\,(\text{grad} \ln \sigma)\right)$$

and

$$\begin{aligned}
\nabla^\phi_{\widetilde{\text{grad}} \ln f}\widetilde{\tau}(\phi) = {}& \sigma^4\nabla^\phi_{\text{grad} \ln f}\tau(\phi) + 2\sigma^4 d\ln f\,(\text{grad} \ln \sigma)\,\tau(\phi) \\
&-2(m-2)\sigma^4 d\ln f\,(\text{grad} \ln \sigma)\, d\phi(\text{grad} \ln \sigma) \\
&-(m-2)\sigma^4\nabla^\phi_{\text{grad} \ln f} d\phi(\text{grad} \ln \sigma).
\end{aligned}$$

Returning to the first equation in the proof of this theorem, we deduce that the map $\phi : \left(M^m, \widetilde{g} = \sigma^{-2} g\right) \longrightarrow (N^n, h)$ is f-biharmonic if and only if

$$\begin{aligned}
&\tau_2(\phi) - 2\left(\Delta \ln \sigma - (m-4)\left|\text{grad} \ln \sigma\right|^2\right)\tau(\phi) \\
&-\left(\Delta \ln f + \left|\text{grad} \ln f\right|^2 - (m-6)\, d\ln f\,(\text{grad} \ln \sigma)\right)\tau(\phi) \\
&+(m-6)\nabla^\phi_{\text{grad} \ln \sigma}\tau(\phi) - 2\nabla^\phi_{\text{grad} \ln f}\tau(\phi) + 2(m-2)\nabla^\phi_{grad \ln f} d\phi(\text{grad} \ln \sigma) \\
&+(m-2)\left(\Delta \ln f - (m-6)\, d\ln f\,(\text{grad} \ln \sigma) + \left|\text{grad} \ln f\right|^2\right) d\phi(\text{grad} \ln \sigma) \\
&+(m-2)\left(\text{Tr}_g\left(\nabla^\phi\right)^2 d\phi(\text{grad} \ln \sigma) - (m-6)\nabla^\phi_{\text{grad} \ln \sigma} d\phi(\text{grad} \ln \sigma)\right) \\
&+2(m-2)\left(\Delta \ln \sigma - (m-4)\left|\text{grad} \ln \sigma\right|^2\right) d\phi(\text{grad} \ln \sigma) \\
&+(m-2)\,\text{Tr}_g R\left(d\phi(\text{grad} \ln \sigma), d\phi\right) d\phi = 0.
\end{aligned}$$

□

An immediate consequence of this theorem is given by the following corollary :

Corollary 2.2. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be an harmonic map and let $\widetilde{g} = \sigma^{-2} g$. The map $\phi : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ is f-biharmonic if and only if*

$$\begin{aligned}
&Tr_g \left(\nabla^\phi\right)^2 d\phi\left(grad \ln\sigma\right) - (m-6)\,\nabla^\phi_{grad\ln\sigma} d\phi\left(grad\ln\sigma\right) + 2\nabla^\phi_{grad\ \ln f} d\phi\left(grad\ln\sigma\right)\\
&+ \left(\Delta \ln f - (m-6)\, d\ln f\left(grad\ln\sigma\right) + |grad\ \ln f|^2\right) d\phi\left(grad\ln\sigma\right)\\
&+ 2\left(\Delta\ln\sigma - (m-4)\,|grad\ln\sigma|^2\right) d\phi\left(grad\ln\sigma\right)\\
&+ Tr_g R\left(d\phi\left(grad\ln\sigma\right), d\phi\right) d\phi = 0.
\end{aligned}$$

Moreover if $\phi = Id$, we conclude that the identity map $Id : \left(M^m, \widetilde{g} = \sigma^{-2} g\right) \longrightarrow (M^m, g)$ is f-biharmonic if and only if

$$\begin{aligned}
&grad\Delta\ln\sigma - \frac{(m-6)}{2} grad\left(|grad\ln\sigma|^2\right) + 2\nabla_{grad\ln f}\, grad\ln\sigma\\
&+ \left(\Delta\ln f - (m-6)\, d\ln f\left(grad\ln\sigma\right) + |grad\ \ln f|^2\right) grad\ln\sigma\\
&+ 2\left(\Delta\ln\sigma - (m-4)\,|grad\ln\sigma|^2\right) grad\ln\sigma\\
&+ 2Ricci\left(grad\ln\sigma\right) = 0.
\end{aligned}$$

As a first application, we construct an example of a f-biharmonic map where the initial map is harmonic.

Example 2.3. Let $\phi : \left(\mathbb{R}^{2n}, g_{\mathbb{R}^{2n}}\right) \to \left(\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}}\right)$ be the Hopf map defined by

$$\phi(x, y) = \left(|x|^2 - |y|^2, 2x\overline{y}\right) \in \mathbb{R} \oplus \mathbb{K} = \mathbb{R}^{n+1},$$

where $n = 2, 4$ or 8 and $x, y \in \mathbb{K} \simeq \mathbb{R}^n$ with $\mathbb{K} = \mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions), or $\mathbb{O}$ (octonions), respectively. Let us write a point $(x, y) \in \mathbb{R}^{2n}$ in the form

$$(x, y) = r\left(\cos\theta.p, \sin\theta.q\right), \quad \left(r \in [0, +\infty)\,,\ \ \theta \in \left[0, \frac{\pi}{2}\right],\ \ p, q \in \mathbb{S}^{n-1}\right)$$

and those of $\mathbb{R}^{n+1}$ in the form

$$s(\cos t, \sin t.w), \quad \left(s \in [0, +\infty)\,,\ \ \theta \in [0, \pi]\,,\ \ w \in \mathbb{S}^{n-1}\right).$$

The map ϕ takes the form

$$\phi(r\cos\theta.p, r\sin\theta.q) = (s\cos t, s\sin t.p\overline{q}),$$

where $s = r^2$ and $t = 2\theta$. It is well known that this map is harmonic. The metrics on $\mathbb{R}^{2n}$ and $\mathbb{R}^{n+1}$ have respectively the expressions :

$$g_{\mathbb{R}^{2n}} = dr^2 + r^2 d\theta^2 + r^2\cos^2\theta.g_{\mathbb{S}^{n-1}} + r^2\sin^2\theta.g_{\mathbb{S}^{n-1}}$$

and

$$g_{\mathbb{R}^{n+1}} = ds^2 + s^2 dt^2 + s^2\sin^2 t.g_{\mathbb{S}^{n-1}},$$

An orthonormal basis of $\mathbb{R}^{2n}$ is given by :

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{1}{r}\frac{\partial}{\partial\theta}, e_j = \frac{1}{r\cos\theta}\xi_j, \quad e_k = \frac{1}{r\sin\theta}\xi_k,$$

with

$$\xi_j = r(\cos\theta.X_j, 0), \quad j = 3, ..., n+1$$

and

$$\xi_k = r(0, \sin\theta.X_k), \quad k = n+2, ..., 2n,$$

where the vectors $\{X_j\}_{3\leq j\leq n+1}$ et $\{X_k\}_{n+2\leq k\leq 2n}$ are unit vectors tangent to the sphere $\mathbb{S}^{n-1}$. We have

$$\nabla_{e_i}e_i = \frac{1-2n}{r}\frac{\partial}{\partial r} + \frac{n-1}{r^2}(\tan\theta - \cot\theta)\frac{\partial}{\partial\theta}.$$

We consider $\widetilde{g}_{\mathbb{R}^{2n}} = \sigma^{-2}g_{\mathbb{R}^{2n}}$ conformal to the Euclidean metric $g_{\mathbb{R}^{2n}}$, where we suppose that the functions σ and f depend only on r. A straightforward calculation gives us

$$d\phi\,(\operatorname{grad}\ln\sigma) = 2r\alpha\frac{\partial}{\partial s}, \quad \nabla^{\phi}_{\operatorname{grad}\ln\sigma}d\phi\,(\operatorname{grad}\ln\sigma) = \left(2r\alpha\alpha' + 2\alpha^2\right)\frac{\partial}{\partial s},$$

$$\operatorname{Tr}_g\left(\nabla^{\phi}\right)^2 d\phi\,(\operatorname{grad}\ln\sigma) = 2\left(r\alpha'' + (2n+1)\,\alpha' + \frac{(2n-1)}{r}\alpha\right)\frac{\partial}{\partial s},$$

$$\Delta\ln\sigma = \alpha' + \frac{(2n-1)}{r}\alpha, \quad \Delta\ln f = \beta' + \frac{(2n-1)}{r}\beta,$$

$$|\operatorname{grad}\ln\sigma|^2 = \alpha^2, \quad |\operatorname{grad}\ln f|^2 = \beta^2$$

and

$$\nabla^{\phi}_{\operatorname{grad}\ln f}d\phi\,(\operatorname{grad}\ln\sigma) = (2r\beta\alpha' + 2\alpha\beta)\frac{\partial}{\partial s},$$

where

$$\alpha = (\ln\sigma)' \text{ and } \beta = (\ln f)'.$$

Then the f-biharmonicity of the Hopf map is equivalent to the following ordinary differential equation of second order :

$$\begin{aligned} & r\alpha'' - 2\,(n-4)\,r\alpha\alpha' + (2n+1)\,\alpha' + 2r\beta\alpha' + r\alpha\beta' \\ & + \frac{(2n-1)}{r}\alpha + 2\,(n+2)\,\alpha^2 - 2r\,(2n-4)\,\alpha^3 \\ & + (2n-1)\,\alpha\beta + 2\alpha\beta + r\alpha\beta^2 - (2n-6)\,r\alpha^2\beta = 0. \end{aligned}$$

We deduce the special solutions of the form $\alpha = \frac{a}{r}$ and $\beta = \frac{b}{r}$ where $a, b \in \mathbb{R}$. Eliminating the trivial solutions $a = 0$ and $b = 0$, we obtain

$$b = -2a \text{ and } b = 2\,(n-2)\,a - 2\,(n-1)\,.$$

(i) When $n = 2$, we obtain $b = -2a$. Then

$$\sigma\,(r) = C_1 r^a \text{ and } f\,(r) = C_2 r^{-2a} \text{ where } a \in \mathbb{R}^*.$$

(ii) For $n = 4$, we find $b = -2a$ or $b = 2\,(2a-3)$. It follows that

$$\sigma\,(r) = C_1 r^a \text{ and } f\,(r) = C_2 r^{-2a}, \quad a \in \mathbb{R}^*$$

or

$$\sigma\,(r) = C_1 r^a \text{ and } f\,(r) = C_2 r^{2(2a-3)}, \quad a \in \mathbb{R}^*.$$

(iii) In the case where $n = 8$, we obtain $b = -2a$ or $b = 2\,(6a-7)$. Which gives

$$\sigma\,(r) = C_1 r^a \text{ and } f\,(r) = C_2 r^{-2a}, \quad a \in \mathbb{R}^*$$

or

$$\sigma\,(r) = C_1 r^a \text{ and } f\,(r) = C_2 r^{2(6a-7)}, \quad a \in \mathbb{R}^*.$$

In these cases, the Hopf map $\phi : \left(\mathbb{R}^{2n}, \widetilde{g}_{\mathbb{R}^{2n}} = \sigma^{-2}g_{\mathbb{R}^{2n}}\right) \to \left(\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}}\right)$ is f-biharmonic.

For the identity map, we find the following results and examples.

Corollary 2.4. *If we consider* $x = (t, x_2, ..., x_m) \in \mathbb{R}_+^* \times \mathbb{R}^{m-1}$ *and we suppose that the functions* σ *and* f *depend only on* t. *The identity map* $Id : \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, \widetilde{g} = \sigma^{-2} g_0\right) \to \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, g_0\right)$ *(*$m \neq 2$*) is* f*-biharmonic if and only if the functions* $\alpha = (\ln \sigma)'$ *and* $\beta = (\ln f)'$ *are solutions of the following differential equation*

$$\alpha'' - (m-8)\,\alpha\alpha' + 2\beta\alpha' + \alpha\beta' - (m-6)\,\alpha^2\beta$$
$$+\,\alpha\beta^2 - 2\,(m-4)\,\alpha^3 = 0.$$

One method for solving this differential equation is to determine particular solutions, which we will discuss in the following example :

Example 2.5. Let us look for the solutions which are written in the form $\alpha = \frac{a}{t}$ and $\beta = \frac{b}{t}$ where $a, b \in \mathbb{R}^*$. We deduce that the identity map $Id : \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, \widetilde{g} = \sigma^{-2} g_0\right) \to \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, g_0\right)$ ($m \neq 2$) is f-biharmonic if and only if a and b are solutions of the following algebraic equation

$$(2a + b - 1)\,(4a + b - am - 2) = 0.$$

This equation has two solutions given by

$$b = -2a + 1, \qquad b = (m-4)\,a + 2.$$

It follows that for all $a \in \mathbb{R}^*$, we obtain

$$f\,(t) = C_1 t^{-2a+1}, \qquad \sigma\,(t) = C_2 t^a,$$

or

$$f\,(t) = C_1 t^{(m-4)a+2}, \qquad \sigma\,(t) = C_2 t^a.$$

For this two cases, the identity map $Id : \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, \widetilde{g}\right) \to \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, g_0\right)$ is f-biharmonic.

Remark 2.6. We can look for other solutions by setting

$$\alpha = (\ln \sigma)' = a \text{ and } \beta = (\ln f)' = b \text{ where } a, b \in \mathbb{R}^*,$$

we obtain

$$b^2 - (m-6)\,ab - 2\,(m-4)\,a^2 = 0,$$

which gives

(i) For $m = 4$, we find $b = -2a$, then $\sigma\,(t) = C_1 e^{at}$ and $f\,(t) = C_2 e^{-2at}$.

(ii) For $m \neq 4$, we obtain $b = -2a$ or $b = (m-4)\,a$. It follows that

$$\sigma\,(t) = C_1 e^{at} \text{ and } f\,(t) = C_2 e^{-2at}$$

or

$$\sigma\,(t) = C_1 e^{at} \text{ and } f\,(t) = C_2 e^{(m-4)at}.$$

Remark 2.7. Considering the case where $\sigma = f$, i.e. $\alpha = \beta$, we have the following possibilities :

(i) If $\alpha = a \in \mathbb{R}^*$, we find $m = 5$ and $\sigma\,(t) = f\,(t) = Ce^{at}$.

(ii) If $\alpha = \frac{a}{t}, a \in \mathbb{R}^*$, we obtain :

- For $m = 5$, we find $a = \frac{1}{3}$ and $\sigma\,(t) = f\,(t) = Ct^{\frac{1}{3}}$.
- For $m \neq 5$, we obtain $a = \frac{1}{3}$ or $a = -\frac{2}{m-5}$, then

$$\sigma\,(t) = f\,(t) = Ct^{\frac{1}{3}} \text{ or } \sigma\,(t) = f\,(t) = Ct^{-\frac{2}{m-5}}.$$

Corollary 2.8. *Now we will study the f-biharmonicity of the identity map $Id : \left(\mathbb{R}^m \setminus \{0\}, \widetilde{g} = \sigma^{-2} g_0\right) \longrightarrow \left(\mathbb{R}^m \setminus \{0\}, g_0\right)$ $(m \neq 2)$ where we suppose that the functions σ and f depend only on $r = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_m^2}$. The identity map is f-biharmonic if and only if the functions $\alpha = (\ln \sigma)'$ and $\beta = (\ln f)'$ are solutions of the following differential equation :*

$$\begin{aligned} &\alpha'' - (m-8)\,\alpha\alpha' + \frac{(m-1)}{r}\alpha' + 2\beta\alpha' + \alpha\beta' \\ &- \frac{(m-1)}{r^2}\alpha + \frac{(m-1)}{r}\alpha\beta + \frac{2\,(m-1)}{r}\alpha^2 + \alpha\beta^2 \\ &- (m-6)\,\alpha^2\beta - 2\,(m-4)\,\alpha^3 = 0. \end{aligned}$$

By looking for solutions of the form $\alpha = \frac{a}{r}$ and $\beta = \frac{b}{r}$ where $a, b \in \mathbb{R}^*$, we find

$$b = \frac{1}{2}\left((m-6)\,a - (m-4) \pm \sqrt{(m-2)^2 a^2 - 2\,(m-2)^2 a + m^2}\right).$$

Then we obtain

$$\sigma(r) = C_1 r^a \text{ and } f(r) = C_2 r^b.$$

For example if we consider $m = 6$ and $a = 2$, we obtain $b = -4$ or $b = 2$, it follows that the identity map $Id : \left(\mathbb{R}^6 \setminus \{0\}, \widetilde{g} = C_1 r^{-4} g_0\right) \longrightarrow \left(\mathbb{R}^6 \setminus \{0\}, g_0\right)$ is f-biharmonic where $f(r) = C_2 r^{-4}$ or $f(r) = C_2 r^2$.

The case of a non-biharmonic map is discussed in the following example :

Example 2.9. Let the map $\phi : (\mathbb{R}^3, g_{\mathbb{R}^3}) \to (\mathbb{R}^2, g_{\mathbb{R}^2})$ defined by

$$\phi(x_1, x_2, x_3) = \left(\sqrt{x_1^2 + x_2^2}, x_3\right).$$

On using the cylindrical coordinates (r, θ, x_3) in $\mathbb{R}^3$ and standard Euclidean coordinates in $\mathbb{R}^2$, the map ϕ takes the form

$$\phi(r\cos\theta, r\sin\theta, x_3) = (y_1, y_2),$$

where

$$y_1 = r = \sqrt{x_1^2 + x_2^2}, \qquad y_2 = x_3.$$

A simple calculation give

$$\tau(\phi) = \frac{1}{r}\frac{\partial}{\partial y_1} \text{ and } \tau_2(\phi) = -\frac{1}{r^3}\frac{\partial}{\partial y_1}.$$

The metrics on $\mathbb{R}^3$ and $\mathbb{R}^2$ have respectively the expressions

$$g_{\mathbb{R}^3} = dr^2 + r^2 d\theta^2 + dx_3^2.$$

$$g_{\mathbb{R}^2} = dy_1^2 + dy_2^2.$$

Let $\widetilde{g}_{\mathbb{R}^3} = \sigma^{-2} g_{\mathbb{R}^3}$ where we suppose that the functions σ and f depend only on r. By a simple calculation, note that $\phi : (\mathbb{R}^3, g_{\mathbb{R}^3}) \to (\mathbb{R}^2, g_{\mathbb{R}^2})$ is f-harmonic if and only if $f(r) = \frac{C}{r}$. Using Theorem 2.1 , we deduce that $\phi : (\mathbb{R}^3, \widetilde{g}_{\mathbb{R}^3}) \to (\mathbb{R}^2, g_{\mathbb{R}^2})$ is f-biharmonic if and only if $\alpha = (\ln \sigma)'$ and $\beta = (\ln f)'$ satisfy the following differential equation

$$\begin{aligned} &\alpha'' + 5\alpha\alpha' - \frac{1}{r}\alpha' + 2\beta\alpha' + \alpha\beta' - \frac{1}{r}\beta' - \frac{1}{r^2}\alpha + \frac{2}{r}\alpha^2 \\ &+ 2\alpha^3 - \frac{1}{r}\alpha\beta + 3\alpha^2\beta + \alpha\beta^2 - \frac{1}{r}\beta^2 - \frac{1}{r^3} = 0. \end{aligned}$$

We will look for special solutions of the form $\alpha(r) = \frac{a}{r}$ and $\beta(r) = \frac{b}{r}$, where $a, b \in \mathbb{R}^*$, we distinguish two cases :

(i) For $a = 1$, we obtain $\sigma(r) = C_1 r$ and $f(r) = C_2 r^b$ for all $b \in \mathbb{R}^*$.

(ii) If $a \notin \left]-1, 3\right[$, we obtain $b = \frac{1}{2}\left(-3a + 1 \pm \sqrt{(a+1)(a-3)}\right)$. Then $\sigma(r) = C_1 r^a$ and $f(r) = C_2 r^b$.

In these cases, the map $\phi : (\mathbb{R}^3, \widetilde{g}_{\mathbb{R}^3}) \to (\mathbb{R}^2, g_{\mathbb{R}^2})$ is f-biharmonic.

3 The f-biharmonicity of $Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = \sigma^{-2} g\right)$.

For the conformal deformation of the codomain metric, we only consider the case of the identity map.

Theorem 3.1. *The identity map $Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = \sigma^{-2} g\right)$ ($m \neq 2$) is f-biharmonic if and only if*

$$\begin{aligned} & grad\Delta \ln \sigma + \frac{(m-6)}{2} grad\left(\left|grad \ln \sigma\right|^2\right) + 2\left(\Delta \ln \sigma\right) grad \ln \sigma \\ & - (m-2)\left|grad \ln \sigma\right|^2 grad \ln \sigma + 2Ricci\left(grad \ln \sigma\right) \\ & + 2\nabla_{grad \ln f}\, grad \ln \sigma - \left|grad \ln \sigma\right|^2 grad \ln f \\ & + \left(\Delta \ln f + \left|grad \ln f\right|^2\right) grad \ln \sigma = 0. \end{aligned} \tag{3.1}$$

Proof. The tension field of $Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = \sigma^{-2} g\right)$ is given by (see [13])

$$\widetilde{\tau}(Id) = (m-2)\,\mathrm{grad} \ln \sigma.$$

Using the f-biharmonicity equation and the fact that $m \neq 2$, we deduce that the identity map $Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = \sigma^{-2} g\right)$ is f-biharmonic if and only if

$$\begin{aligned} & \mathrm{Tr}_g \widetilde{\nabla}^2 \mathrm{grad} \ln \sigma + \left(\Delta \ln f + \left|\mathrm{grad} \ln f\right|^2\right) \mathrm{grad} \ln \sigma \\ & + 2\widetilde{\nabla}_{\mathrm{grad} \ln f}\, \mathrm{grad} \ln \sigma + \mathrm{Tr}_g \widetilde{R}\left(\mathrm{grad} \ln \sigma, \cdot\right) \cdot = 0. \end{aligned}$$

After a long calculation, we obtain the following formulas

$$\begin{aligned} \mathrm{Tr}_g \widetilde{\nabla}^2 \mathrm{grad} \ln \sigma = {} & \mathrm{grad}\Delta \ln \sigma - 2\mathrm{grad}\left(\left|\mathrm{grad} \ln \sigma\right|^2\right) + \left(\Delta \ln \sigma\right) \mathrm{grad} \ln \sigma \\ & - (m-2)\left|\mathrm{grad} \ln \sigma\right|^2 \mathrm{grad} \ln \sigma + \mathrm{Ricci}\left(\mathrm{grad} \ln \sigma\right), \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_g \widetilde{R}\left(\mathrm{grad} \ln \sigma, \cdot\right) \cdot = {} & \mathrm{Ricci}\left(\mathrm{grad} \ln \sigma\right) + \left(\Delta \ln \sigma\right) \mathrm{grad} \ln \sigma \\ & + \frac{(m-2)}{2} \mathrm{grad}\left(\left|\mathrm{grad} \ln \sigma\right|^2\right) \end{aligned}$$

and

$$\widetilde{\nabla}_{\mathrm{grad} \ln f}\, \mathrm{grad} \ln \sigma = \nabla_{\mathrm{grad} \ln f}\, \mathrm{grad} \ln \sigma - \left|\mathrm{grad} \ln \sigma\right|^2 \mathrm{grad} \ln f.$$

Then we conclude that the identity map $Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = \sigma^{-2} g\right)$ is f-biharmonic if and only if

$$\begin{aligned} & \mathrm{grad}\Delta \ln \sigma + \frac{(m-6)}{2} \mathrm{grad}\left(\left|\mathrm{grad} \ln \sigma\right|^2\right) + 2\left(\Delta \ln \sigma\right) \mathrm{grad} \ln \sigma \\ & - (m-2)\left|\mathrm{grad} \ln \sigma\right|^2 \mathrm{grad} \ln \sigma + 2\mathrm{Ricci}\left(\mathrm{grad} \ln \sigma\right) \\ & + 2\nabla_{\mathrm{grad} \ln f}\, \mathrm{grad} \ln \sigma - \left|\mathrm{grad} \ln \sigma\right|^2 \mathrm{grad} \ln f \\ & + \left(\Delta \ln f + \left|\mathrm{grad} \ln f\right|^2\right) \mathrm{grad} \ln \sigma = 0. \end{aligned}$$

□

Corollary 3.2. *If we consider $x = (t, x_2, ..., x_m) \in \mathbb{R}_+^* \times \mathbb{R}^{m-1}$ and we suppose that the functions σ and f depend only on t. The identity map $Id : \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, g_0\right) \rightarrow \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, \widetilde{g} = \sigma^{-2} g_0\right)$ ($m \neq 2$) is f-biharmonic if and only if the functions $\alpha = (\ln \sigma)'$ and $\beta = (\ln f)'$ are solutions of the following differential equation*

$$\alpha'' + (m-4)\alpha\alpha' + 2\beta\alpha' + \alpha\beta' - (m-2)\alpha^3 - \alpha^2\beta + \alpha\beta^2 = 0.$$

Example 3.3. As solutions of the differential equation obtained, we distinguish the following cases :

(i) Let's look for solutions which are written in the form $\alpha = \frac{a}{t}$ and $\beta = \frac{b}{t}$ where $a, b \in \mathbb{R}^*$, we deduce $Id : \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, g_0\right) \to \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, \widetilde{g} = \sigma^{-2} g_0\right)$ is f-biharmonic if and only if a and b are solutions to the following algebraic equation :

$$(m-2)\,a^2 - b^2 + ab + (m-4)\,a + 3b - 2 = 0.$$

Solving this last equation gives us

$$b = \frac{1}{2}a + \frac{3}{2} \pm \frac{1}{2}\sqrt{(4m-7)\,a^2 + 2\,(2m-5)\,a + 1},$$

which gives

$$\sigma(t) = C_1 t^a \text{ and } f(t) = C_2 t^b.$$

For example, if $m = 4$, we find $b = 2\,(a+1)$ or $b = -a + 1$.

(ii) If we assume that α and β are non-zero constant functions ($\alpha = a, \beta = b, a, b \in \mathbb{R}^*$), we find

$$b = \frac{1}{2}\left(1 \pm \sqrt{4m-7}\right) a, m \geq 3.$$

Then we obtain

$$\sigma(t) = C_1 e^{at} \text{ and } f(t) = C_2 e^{\frac{1}{2}\left(1 \pm \sqrt{4m-7}\right) at}.$$

In all these cases the identity $Id : \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, g_0\right) \to \left(\mathbb{R}_+^* \times \mathbb{R}^{m-1}, \widetilde{g} = \sigma^{-2} g_0\right)$ is f-biharmonic.

Corollary 3.4. *By imposing the fact that the functions σ and f depend only on $r = |x|$, we deduce that the identity map $Id : (\mathbb{R}^m \backslash \{0\}, g_0) \longrightarrow \left(\mathbb{R}^m \backslash \{0\}, \widetilde{g} = \sigma^{-2} g_0\right)$ $(m \neq 2)$ is f-biharmonic if and only if the functions $\alpha = (\ln \sigma)'$ and $\beta = (\ln f)'$ are solutions of the following differential equation :*

$$\alpha'' + \frac{(m-1)}{r}\alpha' + (m-4)\,\alpha\alpha' + 2\beta\alpha' + \alpha\beta' - \frac{(m-1)}{r^2}\alpha + \alpha\beta^2$$
$$+ \frac{(m-1)}{r}\alpha\beta + \frac{2\,(m-1)}{r}\alpha^2 - (m-2)\,\alpha^3 - \alpha^2\beta = 0.$$

Example 3.5. By looking for solutions of the form $\alpha = \frac{a}{r}$ and $\beta = \frac{b}{r}$ where $a, b \in \mathbb{R}^*$, we conclude that $Id : (\mathbb{R}^m \backslash \{0\}, g_0) \longrightarrow \left(\mathbb{R}^m \backslash \{0\}, \widetilde{g} = \sigma^{-2} g_0\right)$ $(m \neq 2)$ is f-biharmonic if and only if a and b are solutions of the following algebraic equation

$$b^2 + (m-4-a)\,b - (m-2)\,a^2 + (m+2)\,a - 2\,(m-2) = 0.$$

We obtain $\sigma(t) = Ct^a$ and $f(t) = Ct^b$ where

$$b = \frac{1}{2}\left(a - m + 4 \pm \sqrt{(4m-7)\,a^2 - 6ma + m^2}\right), \quad a \in \mathbb{R}^*.$$

Remark 3.6. Treating the case where $\sigma = f$, i.e. $\alpha = \beta = \frac{a}{r}, a \in \mathbb{R}^*$, we obtain the following algebraic equation :

$$(m-2)\,a^2 - 2\,(m-1)\,a + 2\,(m-2) = 0.$$

This algebraic equation has real solutions if $m \in \{3, 4\}$.

(i) For $m = 3$, we obtain $a = 2 - \sqrt{2}$ or $a = \sqrt{2} + 2$.

(ii) For $m = 4$, we obtain $a = 1$ or $a = 2$.

4 Other examples.

In this case we consider a biharmonic non-harmonic maps and we give two examples of f-biharmonic maps.

(i) Let the inversion $\phi : \mathbb{R}^m \backslash \{0\} \longrightarrow \mathbb{R}^m \backslash \{0\}$ $(m > 2)$ defined by $\phi(x) = \frac{x}{|x|^2}$ where we suppose that the function f depends only on $r = |x|$, after a long calculation, we distinguish two cases :

- For $m = 4$ the inversion $\phi : \mathbb{R}^4 \backslash \{0\} \longrightarrow \mathbb{R}^4 \backslash \{0\}$ is biharmonic non-harmonic and we prove that the inversion is f-biharmonic if and only if $f(r) = ar^4 + b$.
- If $m \neq 4$, the inversion $\phi : \mathbb{R}^m \backslash \{0\} \longrightarrow \mathbb{R}^m \backslash \{0\}$ is not biharmonic. In this case, we deduce that the inversion is f-biharmonic if and only if $f(r) = ar^4 + \frac{mb}{r^{m-4}}$.

(ii) We consider the map $\phi : \mathbb{R}^m \backslash \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{m-1}$ $(m > 2)$, given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$, for $r > 0, \theta \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$. We suppose that function f depends only on $r = |x|$, we obtain :

- For $m = 4$, this map is biharmonic non-harmonic. We conclude that the map ϕ is f-biharmonic if and only if $f(r) = ar^2 + b$.
- In the case where $m \neq 4$, this map is not biharmonic. We prove that the map ϕ is f-biharmonic if and only if $f(r) = ar^2 + \frac{(m-2)b}{r^{m-4}}$.

5 Conclusion

This paper aims to develop new methods for constructing f-biharmonic maps via conformal deformations of the domain or codomain metric. Our results provide several examples and enable the study of special cases. Future research may explore additional construction techniques to further advance this field.

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