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ON FUZZY KU-IDEALS OF KU-ALGEBRAS

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Abstract.In this article, we introduce the notions of $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-subalgebras and $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-ideals of KU-algebras. We investigate some of their properties and provide several appropriate examples. We establish the characterizations of those fuzzy KU-subalgebra (resp. KU-ideal) in KU-algebras. We prove that a family of $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-ideals of KU-algebras is again an $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-ideal of a KU-algebra. Furthermore, we investigate how the image and pre-image of fuzzy KU-ideals in KU-algebras become fuzzy KU-ideals by examining the image and pre-image of the aforementioned fuzzy KU-ideals in KU-algebras. We discuss the relationships among the fuzzy KU-ideals of KU-algebras with examples. Finally, we determine the concepts of the upper and lower parts of the (δ,k)-characteristic fuzzy subset of a KU-algebra and derive some related results. Moreover, we establish the cartesian product of $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-ideals of a KU-algebra.

1 Introduction

KU-algebra is a novel algebraic structure introduced by Prabpayak and Leerawat [11]. They delved into several related characteristics and defined ideals and homomorphisms of KU algebras. The notion of a fuzzy set was formulated by Zadeh[15], and since then there have been wide-ranging applications of the theory of fuzzy sets, from the design of robots and computer simulation to engineering and water resources planning, etc. The study of fuzzy KU-subalgebras of KU-algebras was first initiated by Mostafa et al.[9]. Alshehri [1] presented the idea of an m-polar fuzzy KU-ideal and investigates its characteristics. Gulistan et al. [5] defined (α,β)-fuzzy KU-subalgebras of KU-algebras and provided some useful characterizations of them. Jun [7] defined the notion of $(\in,\in\vee q_k)$-fuzzy subalgebra in BCK/BCI-algebras and investigated several properties. Gulistan et al. [4] defined the notion of $(\in,\in\vee q_k)$-fuzzy KU-Subalgebras of KU-algebras and provided some related results. Kang [8] introduced the notion of $\left(\in,\in\vee q_\delta^k\right)$-fuzzy subsemigroup in a semigroup and investigated some of its properties. Recently, Ramkumar and Borumand Saeid [13] introduced some types of $(\in,\in\vee q_m^n)$-fuzzy filters of BL-algebras and investigated some interesting results.

In this work, we extend the concept of fuzzy KU-ideals of KU-algebras and provide useful results in terms of $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-subalgebras (KU-ideals) of KU-algebras. We define and study the notions of (δ,k)-upper and lower parts of the (δ,k)-characteristic fuzzy subset of a KU-algebra and obtain some interesting results. Moreover, we present the Cartesian product of $\left(\in,\in\vee q_\delta^k\right)$-fuzzy KU-subalgebras (KU-ideals) of KU-algebras.

2 Preliminaries

Now, we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1. [11] By a *KU-algebra* we mean an algebra with a binary operation " $*$ ", satisfying the following conditions:

(i) $(l * m) * [(m * n) * (l * n)] = 0$

(ii) $l * 0 = 0, \forall\, l \in \mathsf{K}$,

(iii) $0 * l = l, \forall\, l \in \mathsf{K}$,

(iv) $l * m = 0 = m * l$ implies $l = m, \forall\, l, m, n \in \mathsf{K}$.

We call it an algebra $(\mathsf{K}, *, 0)$ of type $(2, 0)$. In further study of this article we denote a KU-algebra by K. We define " $\leq$ " in K as if $l \leq m$ if and only if $m * l = 0$.

Definition 2.2. [11] A subset S of K is called KU-subalgebra of K if $l * m \in S$, whenever $l, m \in S$.

Definition 2.3. [11] A non-empty subset A of K is called a KU-ideal of K if it satisfies the following conditions:

(i) $0 \in A$,

(ii) $l * (m * n) \in A, m \in A$ implies $l * n \in A$, for all $l, m, n \in \mathsf{K}$.

Definition 2.4. [15] Let X be a set, a *fuzzy subset* Ξ in X is a function $\Xi : X \Rightarrow [0, 1]$.

Definition 2.5. [12] Fuzzy point in a K is defined as $\Xi(z) = \begin{cases} t, & \text{if } z = x, \\ 0, & \text{otherwise} \end{cases}$ is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point $[x, t]$ and a fuzzy subset Ξ in a set X, we say that

(i) $[x, t] \in \Xi$ (resp. $[x, t]q\Xi$) if $\Xi(x) \geq t$ (resp. $\Xi(x) + t > 1$). In this case, $[x, t]$ is said to *belong (resp. quasi-coincident)* to a fuzzy subset Ξ (see [12]).

(ii) $[x, t]q_k\Xi$ (resp. $[x, t]q_\delta\Xi$) if $\theta(x) + t > 1 - k$ (resp. $\Xi(x) + t > \delta$), where $k \in [0, 1)$ (resp. $\delta \in (0, 1]$). In this case, $[x, t]$ is said to be *k-quasi-coincident (resp. δ-quasi-coincident* with a fuzzy subset Ξ (see [3, 6]).

(iii) $[x, t]q_\delta^k\Xi$ if $\Xi(x) + t > \delta - k$, where $k, \delta \in [0, 1]$ such that $k < \delta$. In this case, $[x, t]$ is said to be *(δ, k)-quasi-coincident* with a fuzzy subset Ξ (see [8]).

The notation $[x, t] \in \Xi$ means that $\Xi(x) \geq t$ and $[x, t]q_\delta^k\Xi$ means that $\Xi(x) + t > \delta - k$, while the notation $[x, t]\bar{\in}\Xi$ and $[x, t]\bar{q}_\delta^k\Xi \Rightarrow [x, t] \in \Xi$ and $[x, t]q_\delta^k\Xi$ does not hold.

Definition 2.6. [11] A map Ξ from a KU-algebra $(\mathsf{K}, *, 0)$ to a KU-algebra $(\mathsf{L}, *', 0')$ is said to be KU-homomorphism if $\Xi(x_1 * x_2) = \Xi(x_1) *' \Xi(x_2)$, for all $x_1, x_2 \in X$.

3 $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebras and $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideals

This section presents the concepts of $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebras and $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideals within the context of KU-algebras. We examine various fundamental properties that are associated with them.

Definition 3.1. A fuzzy subset Ξ of K is said to be an *$(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra* of K if it satisfy the following conditions:

(i) $[x, t] \in \Xi \Rightarrow [0, t] \in \vee q_\delta^k\Xi$,

(ii) $[x, t_1] \in \Xi, [y, t_2] \in \Xi \Rightarrow [x * y, t_1 \wedge t_2] \in \vee q_\delta^k\Xi \;\; \forall\, x, y \in \mathsf{K}$.

Example 3.2. Consider the KU-algebra $(X, *, 0)$ in which $*$ is defined in Table 1 : Define $\Xi(0) = 0.9, \Xi(l) = 0.8, \Xi(m) = 0.7, \Xi(n) = 0.6, \Xi(p) = 0.5$. Let $t = 0.4$, $\delta = 0.9$ and $k = 0.2$. Then, by routine calculation, it is clear that Ξ is an $(\in, \in \vee q_\delta^k)$ fuzzy KU-subalgebra of K.

Table 1

$*$	0	l	m	n	p
0	0	l	m	n	p
l	0	0	m	n	p
m	0	l	0	n	n
n	0	0	m	0	m
p	0	0	0	0	0

Definition 3.3. A fuzzy subset Ξ of K is an $\left(\in, \in \vee q_\delta^k\right)$*-fuzzy KU-ideal* of K if it satisfy the following conditions:

(i) $[x, t] \in \Xi \Rightarrow [0, t] \in \vee q_\delta^k \Xi$,

(ii) $[x * (y * z), t_1] \in \Xi, [y, t_2] \in \Xi \Rightarrow [x * z, t_1 \wedge t_2] \in \vee q_\delta^k \Xi \;\; \forall\, x, y, z \in \mathsf{K}$. .

Theorem 3.4. *A fuzzy subset Ξ of K is said to be an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of X if and only if it satisfy:*

(i) $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}$,

(ii) $\Xi(x * z) \geq \min\left\{\Xi(x * (y * z)), \Xi(y), \frac{\delta-k}{2}\right\} \;\; \forall\, x, y, z \in \mathsf{K}$.

Proof. Let Ξ of K be an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. Let there exist some x, y, z in K such that

(i) $\Xi(0) < \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}$,

(ii) $\Xi(x * z) < \min\left\{\Xi(x * (y * z)), \Xi(y), \frac{\delta-k}{2}\right\}$.

Now consider (i) and if $\Xi(x) < \frac{\delta-k}{2} \Rightarrow \Xi(0) < \Xi(x)$ and $\Xi(0) < t \leq \Xi(x)$ for some $t \in (0,1) \Rightarrow [x, t] \in \Xi$ but $[0, t]\overline{\in}\Xi$. Moreover $\Xi(0) + t < 2t < \delta - k$ which implies that $[0, t]\overline{q_\delta^k}\Xi$. Hence $[0, t]\overline{\in \vee q_\delta^k}\Xi$, which contradicts the given hypothesis. Now if $\Xi(x) \geq \frac{\delta-k}{2}$ then it will imply that $\left[x, \frac{\delta-k}{2}\right] \in \Xi$ and then $\Xi(0) < \frac{\delta-k}{2} \Rightarrow \left[0, \frac{\delta-k}{2}\right]\overline{\in}\Xi$. Moreover if $\Xi(0) + \frac{\delta-k}{2} < \delta - k \Rightarrow \left[0, \frac{\delta-k}{2}\right]\overline{q_\delta^k}\Xi$ and consequently $\left[0, \frac{\delta-k}{2}\right]\overline{\in \vee q_\delta^k}\Xi$, which contradicts the given hypothesis and thus $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}$.

Now consider (ii) and if $\min\{\Xi(x * (y * z)), \Xi(y)\} < \frac{\delta-k}{2} \Rightarrow \Xi(x * z) < \min\{\Xi(x * (y * z)), \Xi(y)\}$ and for some $t \in (0,1)$ we have $\Xi(x * z) < t \leq \min\{\Xi(x * (y * z)), \Xi(y)\}$, which implies that $[x * (y * z), t] \in \Xi$ and $[y, t] \in \Xi$ but $[x * z, t]\overline{\in}\Xi$. And if $\Xi(x * z) + t < 2t < \delta - k$ and thus $[x * z, t]\overline{q_\delta^k}\Xi$. Consequently $[x * z, t]\overline{\in \vee q_\delta^k}\Xi$ which is contradiction and if $\min\{\Xi(x * (y * z)), \Xi(y)\} \geq \frac{\delta-k}{2}$, we get $[x * z, t]\overline{\in \vee q_\delta^k}\Xi$, which again contradicts the given hypothesis and thus $\Xi(x * z) \geq \min\left\{\Xi(x * (y * z)), \Xi(y), \frac{\delta-k}{2}\right\}$.

Conversely, assume that (i) and (ii) are valid and we have to prove that Ξ of K is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. For this let $[x, t] \in \Xi$ for $x \in \mathsf{K}$ and $t \in [0,1]$. This implies that $\Xi(x) \geq t$. But $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\} \geq \min\left\{t, \frac{\delta-k}{2}\right\}$. Now if $t > \frac{\delta-k}{2}$, then $\Xi(0) \geq \frac{\delta-k}{2}, \Rightarrow \Xi(0) + t > \delta - k \Rightarrow [0, t] q_\delta^k \Xi$ and if $t > \frac{\delta-k}{2}$ then it is obvious that $[0, t] \in \Xi$, thus $[0, t] \in \vee q_\delta^k \Xi$. Hence $[x, t] \in \Xi \Rightarrow [0, t] \in \vee q_\delta^k \Xi$. Similarly, we can show that $[x * (y * z), t_1] \in \Xi, [y, t_2] \in \Xi \Rightarrow [x * z, t_1 \wedge t_2] \in \vee q_\delta^k \Xi$. □

Theorem 3.5. *A fuzzy subset Ξ of K is said to be an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra of X if and only if it satisfy:*

(i) $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}$,

(ii) $\Xi(x * y) \geq \min\left\{\Xi(x), \Xi(y), \frac{\delta-k}{2}\right\} \;\; \forall\, x, y, z \in \mathsf{K}$.

By taking $\delta = 1$ (resp. $k = 0$, ($\delta = 1$ and $k = 0$)) in Theorems 3.4 and 3.5, we can get the below corollary.

Corollary 3.6. *A fuzzy subset Ξ of K is an $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy KU-ideal (resp. KU-subalgebra) of X if and only if Ξ of K is an $(\in, \in \vee q_{k})$-fuzzy (resp. $(\in, \in \vee q_{\delta})$-fuzzy, $(\in, \in \vee q)$-fuzzy) KU-ideal (resp. KU-subalgebra) of X.*

Example 3.7. Consider the KU-algebra $(\mathsf{K}, *, 0)$ in which $*$ is defined in Table 2:

Table 2

$*$	0	l	m	n	p
0	0	l	m	n	p
0	0	l	m	n	p
l	0	0	m	n	p
m	0	l	0	n	n
n	0	0	m	0	m
p	0	0	0	0	0

Define a fuzzy subset Ξ as in Table 3. We can easily verify that Ξ is an $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy

Table 3

x	0	l	m	n	p
$\Upsilon(x)$	0.91	0.72	0.81	0.30	0.60

KU-ideal of K for $\delta = 0.9$ and $k = 0.1$. But

(i) Ξ is not a fuzzy KU-ideal of K, since
$0.30 = \Xi(n) = \Xi(l * n) \ngeq \min\{\Xi(l * (p * n), \Xi(p)\} = \Xi(p) = 0.60$.

(ii) Ξ is not an $(\in, \in \vee q)$-fuzzy KU-ideal of K, since
$0.30 = \Xi(n) = \Xi(m * p) \ngeq \min\{\Xi(m * (p * p), \Xi(p), 0.5\} = 0.5$.

(iii) Ξ is not an $(\in, \in \vee q_{k})$-fuzzy KU-ideal of K, since
$[m * (p * p), 0.68] \in \Xi, [p, 0.45] \in \Xi \Rightarrow [m * p, 0.68 \wedge 0.45] \overline{\in \vee q_{0.15}} \Xi$.

(iv) Ξ is not an $(\in, \in \vee q_{\delta})$-fuzzy KU-ideal of K, since
$[m * (p * p), 0.68] \in \Xi, [p, 0.45] \in \Xi \Rightarrow [m * p, 0.68 \wedge 0.45] \overline{\in \vee q_{0.95}} \Xi$.

Remark 3.8. In the above example, we showed that every fuzzy KU-ideal, $(\in, \in \vee q)$-fuzzy KU-ideal, $(\in, \in \vee q_{k})$-fuzzy KU-ideal and $(\in, \in \vee q_{\delta})$-fuzzy KU-ideal of K is an $(\in, \in \vee q_{\delta}^{k})$-fuzzy ideal of K, but the converses are not necessarily true.

Next, we characterize $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebras (resp. KU-ideals) of K in terms of level sets.

Theorem 3.9. *A fuzzy subset Ξ of K is an $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy KU-ideal of K if and only if the set $U[\Xi, t] = \{x \in \mathsf{K} \mid \Xi(x) \geq t\}$ is a KU-ideal of K where $t \in \left(0, \frac{\delta - k}{2}\right]$.*

Proof. Assume that Ξ of K is an $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy KU-ideal of K and let $x \in U[\Xi, t]$ which implies that $\Xi(x) \geq t$ for some $t \in \left(0, \frac{\delta - k}{2}\right]$. But $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta - k}{2}\right\} \geq \min\left\{t, \frac{\delta - k}{2}\right\} = t$, which implies that $0 \in U[\Xi, t]$. Again, let $(x * (y * z)) \in U[\Xi, t]$ and $y \in U[\Xi, t]$, then by definition, we get $\Xi(x * (y * z)) \geq t$ and $\Xi(y) \geq t$ but $\Xi(x * z) \geq \min\left\{\Xi(x * (y * z)), \Xi(y), \frac{\delta - k}{2}\right\} \geq \min\left\{t, t, \frac{\delta - k}{2}\right\} = t$, which implies that $x * z \in U[\Xi, t]$. Hence $U[\Xi, t]$ is a KU-ideal of K where $t \in \left(0, \frac{\delta - k}{2}\right]$.

Conversely, let $U[\Xi, t]$ be a KU-ideal of K where $t \in \left(0, \frac{\delta - k}{2}\right]$ and we show that Ξ of K is an $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy KU-ideal of K. For this let there exist some $t \in \left(0, \frac{\delta - k}{2}\right]$ such that $\Xi(0) < t \leq \min\left\{\Xi(x), \frac{\delta - k}{2}\right\}$ which implies that $x \in U[\Xi, t]$ but $0 \bar{\in} U[\Xi, t]$, which is a contradiction and hence $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta - k}{2}\right\}$. Similarly, we can prove that $\Xi(x * z) \geq \min\left\{\Xi(x * (y * z)), \Xi(y), \frac{\delta - k}{2}\right\}$. □

Table 4

$*$	0	l	m	n	p	q
0	0	l	m	n	p	q
l	0	0	m	m	p	q
m	0	0	0	l	p	q
n	0	0	0	0	p	q
p	0	0	0	l	0	q
q	0	0	0	0	0	0

Example 3.10. Consider the KU-algebra $(\mathsf{K}, *, 0)$ in which $*$ is defined in Table 4 :

Define a fuzzy subset Ξ of K as $\Xi(0) = 0.9, \Xi(l) = 0.8, \Xi(m) = 0.75, \Xi(n) = 0.7, \Xi(p) = 0.65, \Xi(q) = 0.3$. Then

$$U[\Xi, t] = \begin{cases} \mathsf{K} & \text{if } t \in (0, 0.3] \text{ for } \delta = 0.9, k = 0.3, \\ \{0, l, m, n, p\} & \text{if } t \in (0.3, 0.4] \text{ for } \delta = 0.8, k = 0.4. \end{cases}$$

As K and $\{0, l, m, n, p\}$ are KU-ideals of K, so by Theorem 3.9, Ξ of K is an $\left(\in, \in \vee q_\delta^k\right)$ fuzzy KU-ideal of K.

Corollary 3.11. *A fuzzy subset Ξ of K is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra of K if and only if $U[\Xi, t] = \{x \in \mathsf{K} \mid \Xi(x) \geq t\}$ is a KU-subalgebra of K where $t \in \left(0, \frac{\delta - k}{2}\right]$.*

Theorem 3.12. *Let $\mathcal{M}$ be a KU-subalgebra (resp. KU-ideal) of K and let Ξ be a fuzzy subset in K such that, $\Xi(x) = \rho$ if $x \in A$ and $\Xi(x) = 0$ otherwise, where $\rho \in [0, 1]$. Then Ξ is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra (resp. KU-ideal) of K.*

Proof. Let $x, y, z \in X$ such that $x * (y * z) \in \mathcal{M}$, $y \in \mathcal{M}$. Then $(x * z) \in \mathcal{M}$ and so $\Xi(x * z) = \rho = \min\{\Xi(x * (y * z)), \Xi(y), \frac{\delta - k}{2}\}$. And if $x * (y * z) \notin \mathcal{M}, y \notin \mathcal{M} \Rightarrow \Xi(x * (y * z)) = 0$ or $\Xi(y) = 0$. Thus $\Xi(x*z) \geq 0 = \min\{\Xi(x*(y*z)), \Xi(y), \frac{\delta - k}{2}\}$. Therefore Ξ is a $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. □

Theorem 3.13. *Every $(\in, \in)$-fuzzy KU-subalgebra (resp., ideal) of K is also an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra (resp., ideal) of K.*

In the following definitions, we define the notions of an $\left(\in, q_\delta^k\right)$-fuzzy KU-subalgebra and $\left(\in, q_\delta^k\right)$-fuzzy KU-ideal of a KU-algebra.

Definition 3.14. A fuzzy subset Ξ of K is said to be an $\left(\in, q_\delta^k\right)$-fuzzy KU-subalgebra of K if it satisfy the following conditions:

(i) $[x, t] \in \Xi \Rightarrow [0, t] \in q_\delta^k \Xi$,

(ii) $[x * y, t_1] \in \Xi, [y, t_2] \in \Xi \Rightarrow [x, t_1 \wedge t_2] \in q_\delta^k \Xi$.

Definition 3.15. A fuzzy subset Ξ of K is said to be an $\left(\in, q_\delta^k\right)$-fuzzy KU-ideal of K if it satisfy the following conditions:

(i) $[x, t] \in \Xi \Rightarrow [0, t] \in q_\delta^k \Xi$,

(ii) $[x * (y * z), t_1] \in \Xi, [y, t_2] \in \Xi \Rightarrow [x * z, t_1 \wedge t_2] \in q_\delta^k \Xi$.

Theorem 3.16. *Every $\left(\in, q_\delta^k\right)$-fuzzy KU-subalgebra (resp., KU-ideal) of K is also an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra (resp., KU-ideal) of K.*

Theorem 3.17. *Any $(\in \vee q_\delta^k, \in \vee q_\delta^k)$-fuzzy KU-ideal is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K.*

The converses of the above theorems are not true, as shown in the following example.

Example 3.18. Consider the KU-algebra $(\mathsf{K}, *, 0)$ in which $*$ is defined in Table 5 : Define a fuzzy subset Ξ as in Table 6, then Ξ is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K for $\delta = 0.9$ $k = 0.5$, but Ξ is not an $\left(\in, q_\delta^k\right)$-fuzzy KU-ideal of K because $l_{0.7} \in \Xi$ but $0_{0.7} \notin \Xi$.

Table 5

$*$	0	l	m	n	p
0	0	l	m	n	p
l	0	0	m	n	p
m	0	l	0	n	n
n	0	0	m	0	m
p	0	0	0	0	0

Table 6

x	0	l	m	n	p
$\Upsilon(x)$	0.68	0.71	0.59	0.59	0.30

Theorem 3.19. *Let $\emptyset \neq A \subset \mathsf{K}$. Then the characteristic function Ξ_A of A is an $\left(\in, \in \vee q_\delta^k\right)$ fuzzy KU-subalgebra (resp., KU-ideal) of K if and only if A is a KU-subalgebra (resp., KU-ideal) of K.*

Proof. Let A be a KU-ideal of K. Then it is obviously an $(\in, \in)$-fuzzy KU-ideal of K which implies that A is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K.

Conversely, assume that A is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K and we show that A is a KU-ideal of K. For this let $x * (y * z) \in A, y \in A$ then by definition $\Xi_A(x * (y * z)) = 1$ and $\Xi_A(y) = 1 \Rightarrow [(x * (y * z)), 1] \in \Xi_A$ and $[y, 1] \in \Xi_A$. But by hypothesis $\Xi_A(x * z) \geq \min\left\{\Xi_A(x * (y * z)), \Xi_A(y), \frac{\delta - k}{2}\right\} = \min\left\{1, 1, \frac{\delta - k}{2}\right\} = \frac{\delta - k}{2}$ and as $0 \leq k < \delta \leq 1$ so $\frac{\delta - k}{2} \neq 0$ and hence $\Xi_A(x*z) \geq 1 \Rightarrow x*z \in A$. Moreover in the same way $\Xi_A(0) \geq \min\left\{\Xi_A(x), \frac{\delta - k}{2}\right\} = 1 \Rightarrow 0 \in A$. Hence A is a KU-ideal of K. The other cases can be seen similarly. □

In the next theorem, we proved that the union of the family of $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebras (resp. KU-ideals) is again an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra (resp. KU-ideal) of a KU-algebra.

Theorem 3.20. *Let $\{\Xi_i : i \in \wedge\}$ be a family of an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-subalgebra (resp., KU-ideal) of K. Then so is their intersection $\Xi = \cap_{i \in \Lambda} \Xi_i$.*

Proof. Let $\{\Xi_i : i \in \wedge\}$ be a family of $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K and we have to show that $\Xi = \underset{i \in \Lambda}{\cap} \Xi_i$ is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. For this let $[x, t] \in \Xi$ and we have to show that $[0, t] \in \vee q_\delta^k \Xi$. Assume that $[0, t] \in \vee q_\delta^k \Xi \Rightarrow \Xi(0) < t$ and $\Xi(0) + t < \delta - k$. Which implies that $\Xi(0) < \frac{\delta - k}{2}$. Now let $\Delta_1 = \{i \in \wedge \mid [0, t] \in \Xi_i\}$ and $\Delta_2 = \left\{i \in \wedge \mid [0, t] q_\delta^k \Xi_i\right\} \cap \{i \in \wedge \mid [0, t] \overline{\in} \Xi_i\}$ then we have $\wedge = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Let us suppose that if $\Delta_2 = \emptyset$, then $[0, t] \in \Xi_i \ \forall\ i \in \wedge \Rightarrow \Xi_i(0) \geq t, \forall\ i \in \wedge \Rightarrow \Xi(0) = \bigcap\limits_{i \in \wedge} \Xi_i(0) \geq t$ which contradicts the assumption and so $\Delta_2 \neq \emptyset$. Thus for each $i \in \Delta_2$ we have $[0, t] + t \geq \delta - k$ and $[0, t] < t$, it implies that $t > \frac{\delta - k}{2}$. Now since $[x, t] \in \Xi \Rightarrow \Xi(x) \geq t$ and we can write it as $\Xi(x) \geq t > \frac{\delta - k}{2}$ $\forall\ i \in \wedge$. Next assume that $\Xi_i(0) < \frac{\delta - k}{2} = t_1$ and let $t_1 < r < \frac{\delta - k}{2}$ which implies $[x, r] \in \Xi_i$, but $[0, t] \in \bar{q}_k \Xi_i$, which contradicts the fact that Ξ_i is given to be an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. Thus $\Xi_i(0) \geq \frac{\delta - k}{2}\ \forall\ i \in \wedge$ and hence $\Xi(0) \geq \frac{\delta - k}{2} \Rightarrow [0, t] \in \vee q_\delta^k \Xi$. Similarly we can show that if $[x * (y * z), t_1] \in \Xi, [y, t_2] \in \Xi$, then it implies that $[x * z, t_1 \wedge t_2] \in \vee q_\delta^k \Xi$. Which shows that $\Xi = \bigcap_{i \in \wedge} \Xi_i$ is $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. The other cases can be seen similarly. □

By taking $k = 0$ (resp. $\delta = 1$, ($\delta = 1$ and $k = 0$)) in Theorem 3.20, we have the following corollary.

Corollary 3.21. *If $\{\Xi_i : i \in \wedge\}$ is a family of an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy (resp. $(\in, \in \vee q_k)$-fuzzy, $(\in, \in \vee q)$-fuzzy) KU-ideal of K, then so is their intersection $\Xi = \cap_{i \in \Lambda} \Xi_i$.*

Theorem 3.22. *Let A be a KU-subalgebra (resp. KU-ideal) of K and λ be a fuzzy subset of K such that*

$$\Xi(x) = \begin{cases} \eta & if \quad x \in A, \\ 0 & otherwise, \end{cases}$$

where $\eta \geq \frac{\delta-k}{2}$*. Then* λ *is an* $(\in, \in \vee q_\delta^k)$*-fuzzy* KU*-subalgebra (resp.* KU*-ideal) of* K*.*

Proof. Let $x \in \mathsf{K}$ and $t \in (0,1]$ be such that $[x,t] \in \Xi$. Then $\Xi(x) \geq t > 0$. But $\Xi(0) \geq \Xi(x)$. This implies that $\Xi(0) \geq \Xi(x) = \eta$. So $0 \in A$. Therefore $[0,t] \in \Xi$ and so $\Xi(0) = \eta$. If $t \leq \frac{\delta-k}{2}$, then $\Xi(0) = \eta \geq \frac{\delta-k}{2} \geq t$. Hence $[0,t] \in \Xi$. If $t > \frac{\delta-k}{2}$, then $\Xi(0)+t \geq \frac{\delta-k}{2}+t > \frac{\delta-k}{2}+\frac{\delta-k}{2} = \delta - k$ and so $[0,t]q_\delta^k\,\Xi$. Therefore $[0,t] \in \vee q_\delta^k\,\Xi$. Let $x,y,z \in \mathsf{K}$ and $t,r \in (0,1]$ be such that $y_t \in \Xi$ and $[(x*(y*z)),r] \in \Xi$. Then $\Xi(y) \geq t > 0$ and $\Xi(x*(y*z)) \geq r > 0$. This implies that $y \in A$ and $x*(y*z) \in A$, and so $x*z \in A$. Thus $\Xi(x*z) = \eta$. If $\min\{t,r\} \leq \frac{\delta-k}{2}$, then $\Xi(x*z) = \eta \geq \frac{\delta-k}{2} \geq \min\{t,r\}$. Hence $[(x*z),\{t \wedge r\}] \in \Xi$. If $\min\{t,r\} > \frac{\delta-k}{2}$, then $\Xi(x*z) + \min\{t,r\} \geq \frac{\delta-k}{2} + \min\{t,r\} > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$ and so $(x*z)_{\min\{t,r\}}q_\delta^k\,\Xi$. Therefore $[(x*z),\{t \wedge r\}] \in \vee q_\delta^k\,\Xi$. Hence Ξ is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K. □

Definition 3.23. Let $S \subseteq \mathsf{K}$. Then the fuzzy subset Ξ_S in K defined by

$$\Xi_S(x) = \begin{cases} \delta - k & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

where $x \in \mathsf{K}$, is called a (δ,k)*-characteristic fuzzy subset* of S in K.

Theorem 3.24. *A non-empty subset* S *of* K *is a* KU*-subalgebra (resp.* KU*-ideal) of* K *if and only if the* (δ,k)*-characteristic fuzzy subset* Ξ_S *of* S *is an* $(\in, \in \vee q_\delta^k)$*-fuzzy* KU*-subalgebra (resp.* KU*-ideal) of* K*.*

Proof. Let S be a KU-ideal of K. Then by Theorem 3.22, Ξ_S is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K.

Conversely, assume that Ξ_S is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K. Let $a,b,c \in S$. Then $\Xi_S(a*c) \geq \min\{\Xi_S(a*(b*c)), \Xi_S(b), \frac{\delta-k}{2}\} = \min\{\delta-k, \delta-k, \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$. This implies that $a*c \in S$. Hence A is a KU-ideal of K. □

For any fuzzy subset Ξ in K and $t \in (0,1]$, we denote $\Xi_t = \left\{x \in \mathsf{K} \mid [x,t]q_\delta^k\Xi\right\}$ and $[\Xi]_t = \left\{x \in \mathsf{K} \mid [x,t] \in \vee q_\delta^k\Xi\right\}$ then it is clear that $[\Xi]_t = U[x,t] \cup \Xi_t$.

Theorem 3.25. *Let* Ξ *be a fuzzy subset of* K*. Then* Ξ *is an* $\left(\in, \in \vee q_\delta^k\right)$*-fuzzy* KU*-subalgebra (resp.,* KU*-ideal) of* K *if and only if* $[\Xi]_t$ *is a* KU*-subalgebra (resp.,* KU*-ideal) of* K *for all* $t \in (0,1]$*.*

Proof. Let us assume that Ξ is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. We aim to prove that $[\Xi]_t$ is a KU-ideal of K, for all $t \in (0,1]$. For this let $x \in [\Xi]_t = U[x,t] \cup \Xi_t$, which implies that $[x,t] \in \vee q_\delta^k\Xi \Rightarrow \Xi(x) \geq t$ or $\Xi(x) + t > \delta - k$. As $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}$, so we have the following cases.

(i) If $\Xi(x) \geq t$ and $t > \frac{\delta-k}{2}$, then $\Xi(0) \geq \frac{\delta-k}{2} \Rightarrow \Xi(0) + t > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$, which implies that $[0,t]q_\delta^k\Xi$ and if $t \leq \frac{\delta-k}{2}$ then $\Xi(0) \geq t \Rightarrow [0,t] \in \Xi$. Hence $[0,t] \in \vee q_\delta^k\Xi$.

(ii) If $\Xi(x) + t > \delta - k$ and $t > \frac{\delta-k}{2}$, then $\Xi(0) \geq (\delta - k - t) \wedge \frac{\delta-k}{2} \Rightarrow \Xi(0) \geq \delta - k - t$, which implies that $[0,t]q_\delta^k\Xi$ and if $t \leq \frac{\delta-k}{2}$, then $\Xi(0) \geq (\delta - k - t) \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2} = t$, which implies that $[0,t] \in \Xi$. Hence $[0,t] \in \vee q_\delta^k\Xi$. Thus form both cases we get $0 \in [\Xi]_t$.

Again let $(x*(y*z)) \in [\Xi]_t$ and $y \in [\Xi]_t \Rightarrow [x*(y*z),t] \in \vee q_\delta^k\Xi$ and $[y,t] \in \vee q_\delta^k\Xi \Rightarrow [x*(y*z),t] \in \Xi$ or $[x*(y*z),t]q_\delta^k\Xi$ and $[y,t] \in \Xi$ or $[y,t]q_\delta^k\Xi \Rightarrow \Xi(x*(y*z)) \geq t$ or $\Xi(x*(y*z)) + t + k > 1$ and $\Xi(y) \geq t$ or $\Xi(y) + t + k > 1$. So we discuss the following cases.

(i) If $\Xi(x*(y*z)) \geq t$ and $\Xi(y) \geq t$. So $\Xi(x*z) \geq \min\left\{t, t, \frac{\delta-k}{2}\right\}$ and if $t > \frac{\delta-k}{2} \Rightarrow \Xi(x) \geq \frac{\delta-k}{2}$ and hence $\Xi(x*z)+t > \delta-k \Rightarrow [x*z,t]q_k\Xi$ and if $t \leq \frac{\delta-k}{2}$ then $\Xi(x*z) \geq t \Rightarrow [x*z,t] \in \Xi$. Hence $[x*z,t] \in \vee q_\delta^k\Xi$.

Similarly, we get $[x*z,t] \in \vee q_\delta^k\Xi$. Which shows that $[\Xi]_t$ is a KU-ideal of K for all $t \in (0,1]$ from all other cases.

Conversely, assume that $[\Xi]_t$ is a KU-ideal of K for all $t \in (0,1]$ and we have to show that Ξ is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. Suppose there exist some $t \in (0,1]$ such that $\Xi(0) < t \leq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}, \Xi(x*z) < t \leq \min\left\{\Xi(x*(y*z)), \Xi(y), \frac{\delta-k}{2}\right\} \Rightarrow x \in U[\Xi,t] \subseteq [\Xi]_t \Rightarrow 0 \in [\Xi]_t$ by hypothesis. $\Xi(0) \geq t$ or $\Xi(0) + t + k > 1$, this is a contradiction. Similarly, $\Xi(x*z) < t \leq \min\left\{\Xi(x*(y*z)), \Xi(y), \frac{\delta-k}{2}\right\}$ leads to a contradiction. Thus $\forall\, x,y,z \in \mathsf{K}$ we have $\Xi(0) \geq \min\left\{\Xi(x), \frac{\delta-k}{2}\right\}$ and $\Xi(x*z) \geq \min\left\{\Xi(x*y)*z, \Xi(y), \frac{\delta-k}{2}\right\}$ which shows that Ξ is an $\left(\in, \in \vee q_\delta^k\right)$-fuzzy KU-ideal of K. The other case can be seen in a similar way. □

By taking $k = 0$ (resp. $\delta = 1$, ($\delta = 1$ and $k = 0$)) in Theorem 3.25, we have the following corollary.

Corollary 3.26. *Let Ξ be a fuzzy subset of K. Then Ξ is an $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy (resp. $(\in, \in \vee q_{k})$-fuzzy, $(\in, \in \vee q)$-fuzzy) KU-ideal of K if and only if $[\Xi]_t$ is a KU-ideal of K for all $t \in (0, 1]$.*

Theorem 3.27. *Let there be an $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebra (resp., KU-ideal) of K such that $\{\Xi(x) \,|\, \Xi(x) < \frac{\delta-k}{2}\} \geq 2$. Then Ξ can be expressed as the union of two proper non-equivalent $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebra (KU-ideal) of K.*

Proof. Define the fuzzy sets as

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in [\Xi]_{t_1} \\ t_2 & \text{if } x \in [\Xi]_{t_2} \backslash [\Xi]_{t_1} \\ \vdots & \vdots \\ t_r & \text{if } x \in [\Xi]_{t_r} \backslash [\Xi]_{t_{r-1}} \end{cases}$$

and

$$\lambda(x) = \begin{cases} \Xi(x) & \text{if } x \in [\Xi]_{\frac{\delta-k}{2}} \\ t_2 & \text{if } x \in [\Xi]_{t_2} \backslash [\Xi]_{\frac{\delta-k}{2}} \\ \vdots & \vdots \\ t_r & \text{if } x \in [\Xi]_{t_r} \backslash [\Xi]_{t_{r-1}} \end{cases}$$

for $[\Xi]_{\frac{\delta-k}{2}} \subseteq [\Xi]_{t_1} \subseteq \ldots \subseteq [\Xi]_{t_r} = \mathsf{K}$ and $\left\{\Xi(x) \,\middle|\, \Xi(x) < \frac{\delta-k}{2}\right\} = \{t_1, t_2, \ldots, t_r\}$ for $t_1 > t_2 > \ldots > t_r$ with $r \geq 2$. Then, by level cut theorem μ and λ are $\left(\in, \in \vee q_{\delta}^{k}\right)$-fuzzy KU-ideal of K and the chain of $\left(\in, \in \vee q_{\delta}^{k}\right)$-level KU-ideals μ and λ are given by respectively as $[\Xi]_{t_1} \subseteq [\Xi]_{t_2} \subseteq \ldots \subseteq [\Xi]_{t_r}$ and $[\Xi]_{\frac{\delta-k}{2}} \subseteq [\Xi]_{t_2} \subseteq \ldots \subseteq [\Xi]_{t_r}$. They are non-equivalent and $\Xi = \mu \cup \lambda$. This completes the proof. The other cases can be seen in a similar way. □

Definition 3.28. [14] Let $\mathfrak{g}$ be a mapping from a set X into a set Y. If η is a fuzzy subset of X, then the *image of η under* $\mathfrak{g}$, denoted by $\mathfrak{g}(\eta)$, is a fuzzy subset of Y defined by

$$\mathfrak{g}(\eta)(y) = \begin{cases} \sup\limits_{x \in \mathfrak{g}^{-1}(y)} \eta(x) & if \quad \mathfrak{g}^{-1}(y) \neq \emptyset, \\ 0 & otherwise, \end{cases}$$

for every $y \in Y$, where $\mathfrak{g}^{-1}(y) = \{x \in X | \mathfrak{g}(x) = y\}$.

Definition 3.29. [14] A fuzzy subset η of a set X is said to have *sup property* if, for any subset A of X, there exists $a_0 \in A$ such that $\eta(a_0) = \sup\limits_{a \in A} \eta(a)$.

In the next theorem, we proved that the homomorphic image of $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebra (resp., KU-ideal) of KU-algebra is also an $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebra (resp., KU-ideal) of KU-algebra.

Theorem 3.30. *Let $\mathfrak{g}$ be a KU-homomorphism between KU-algebra K to KU-algebra L. Then, for every $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebra (resp., KU-ideal) η of K, the image of η under $\mathfrak{g}$ is also an $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU-subalgebra (resp., KU-ideal) of L.*

Proof. By definition $\eta(c_1) = \mathfrak{g}(\eta)(c_1) = \sup_{x \in \mathfrak{g}-1(c_1)} \eta(x)$, for all $c_1 \in \mathsf{L}$ and $\sup \Phi = 0$. Since $\mathfrak{g} : \mathsf{K} \to \mathsf{L}$ is a homomorphism and since η is an $(\in, \in \vee q_{\delta}^{k})$-fuzzy KU -ideal of K so $\eta(0) \geq \min\{\eta(x), \frac{\delta-k}{2}\}$ $\forall\ x \in \mathsf{K}$. Let the image of η under $\mathfrak{g}$ is equal to φ (say). In order to prove the theorem we have to show that $\varphi(0_1) \geq \inf\left\{\varphi(b_1), \frac{\delta-k}{2}\right\}$ and $\varphi(b_1 * d_1) \geq \min\left\{\varphi(b_1 * (c_1 * d_1)), \varphi(c_1), \frac{\delta-k}{2}\right\}$. Consider $\varphi(0_1) = \sup_{t \in \mathfrak{g}^{-1}(0)} \eta(t) = \eta(0) \geq \inf\{\eta(x), \frac{\delta-k}{2}\}$ $\forall\ x \in \mathsf{K}$ Which implies that $\varphi(0_1) \geq \sup_{t \in \mathfrak{g}^{-1}(b_1)} \eta(t) = \varphi(b_1) \wedge \frac{\delta-k}{2}$ for any $b_1 \in \mathsf{L}$.

Now for any $b_1, c_1, d_1 \in \mathsf{K}$ and let $b_0 \in \mathfrak{g}^{-1}(b_1), c_0 \in \mathfrak{g}^{-1}(c_1), d_0 \in \mathfrak{g}^{-1}(d_1)$ be such that $\eta(b_0 * d_0) = \sup_{t\in\mathfrak{g}^{-1}(b_1*d_1)} \eta(t), \eta(c_0) = \sup_{t\in\mathfrak{g}^{-1}(c_1)} \eta(t)$ and

$$\begin{aligned}\eta(b_0 * (c_0 * d_0)) &= \varphi\{\mathfrak{g}(b_0 * (c_0 * d_0))\} = \varphi(b_1 * (c_1 * d_1)) \\ &= \sup_{(b_0*(c_0*d_0))\in\mathfrak{g}^{-1}(b_1*(c_1*d_1))} \eta(b_0 * (c_0 * d_0)) \\ &= \sup_{t\in\mathfrak{g}^{-1}(b_1*(c_1*d_1))} \eta(t).\end{aligned}$$

Then

$$\begin{aligned}\varphi(b_1 * d_1) &= \sup_{t\in\mathfrak{g}^{-1}(b_1*d_1)} \eta(t) = \eta(b_0 * d_0) \\ &\geq \min\left\{\eta(b_0 * (c_0 * d_0)), \eta(c_0), \tfrac{\delta-k}{2}\right\} \\ &= \min\left\{\sup_{t\in\mathfrak{g}^{-1}(b_1*(c_1*d_1))} \eta(t), \sup_{t\in\mathfrak{g}^{-1}(c_1)} \eta(t), \tfrac{\delta-k}{2}\right\} \\ &= \min\left\{\varphi(b_1 * (c_1 * d_1)), \varphi(c_1), \tfrac{\delta-k}{2}\right\}\end{aligned}$$

Hence $\mathfrak{g}(\eta)$ is $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of L. □

Definition 3.31. [14] Let $\mathfrak{g}$ be a mapping from a set X into a set Y. If η is a fuzzy subset of Y, then the *pre-image of* η *under* $\mathfrak{g}$, denoted by $\mathfrak{g}^{-1}(\eta)$, is a fuzzy subset of X defined by $\mathfrak{g}^{-1}(\eta)(x) = \eta(\mathfrak{g}(x))$ for all $x \in X$.

Theorem 3.32. *Let* $\mathfrak{g} : \mathsf{K} \to \mathsf{L}$ *be a* KU*-homomorphism and* Ξ *be an* $(\in, \in \vee q_\delta^k)$*-fuzzy* KU*-subalgebra (resp.,* KU*-ideal) of* L*. Then* $\mathfrak{g}^{-1}(\Xi)$ *which we call it as the pre-image of* Ξ *is an* $(\in, \in \vee q_\delta^k)$*-fuzzy* KU*-subalgebra (resp.,* KU*-ideal) of* K.

Proof. Let $\mathfrak{g}^{-1}(\Xi)(0) = \Xi(\mathfrak{g}(0)) \geq \Xi(\mathfrak{g}(l)) \wedge \frac{\delta-k}{2} = \mathfrak{g}^{-1}(\Xi)(l) \wedge \frac{\delta-k}{2}$. Now consider for $x, y, z \in \mathsf{K}$, $\mathfrak{g}^{-1}(\Xi)(x * z) = \Xi(\mathfrak{g}(x * z)) \geq \Xi(\mathfrak{g}(x * (y * z))) \wedge \Xi(\mathfrak{g}(y)) \wedge \frac{\delta-k}{2} = \mathfrak{g}^{-1}(\Xi(x * (y * z))) \wedge \mathfrak{g}^{-1}(\Xi(y)) \wedge \frac{\delta-k}{2}$. Hence proved. □

4 Cartesian product of $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal

This section presents results concerning $(\in, \in \vee q_\delta^k)$-fuzzy relations in KU-algebras and the $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideals associated with KU-algebras.

Definition 4.1. Let K be a KU-algebra and ϱ is a fuzzy subset of K. Then ϱ is said to be fuzzy KU-relation on K if $\varrho : \mathsf{K} \times \mathsf{K} \to [0, 1]$.

Definition 4.2. Let ϱ be a fuzzy KU-relation on the set K and ξ be a fuzzy subset of K. Then ϱ is said to be $(\in, \in \vee q_\delta^k)$-fuzzy KU-relation on ξ if $r_{a_1} \in \xi, s_{a_2} \in \xi \Rightarrow (r, s)_{\min\{a_1,a_2\}} \in \vee q_\delta^k \varrho$.

Definition 4.3. Let ς and ξ be the fuzzy subset of the KU-algebra K. Then the cartesian product of these fuzzy subsets is defined by $(\varsigma \times \xi)(r, s) = \min\{\varsigma(r), \xi(s)\}, \forall\, r, s \in \mathsf{K}$.

Definition 4.4. Let ξ be the fuzzy subset of the KU-algebra K. Then we define strongest (α, β)-fuzzy KU-relation as ϱ_ξ [called as $(\in, \in \vee q_\delta^k)$-fuzzy KU-relation on ξ] given by $r_{a_1} \in \xi, s_{a_2} \in \xi \Rightarrow (r, s)_{\min\{a_1,a_2\}} \in \vee q_\delta^k \xi$.

Proposition 4.5. *Let* ϱ_ς *be the strongest* $(\in, \in \vee q_\delta^k)$*-fuzzy* KU*-relation on the* KU*-algebra* K *for a given fuzzy subset* ς *of* K*. If* ϱ_ς *is the* $(\in, \in \vee q_\delta^k)$*-fuzzy* KU*-ideal on* $\mathsf{K} \times \mathsf{K}$ *then* $\varsigma(0) \geq \min\{\varsigma(r), \frac{\delta-k}{2}\}\ \forall\, r \in \mathsf{K}$.

Proof. Let ϱ_ς is the $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal on $\mathsf{K} \times \mathsf{K}$. Then, by definition

$$\varrho_\varsigma(r, r) = \min\{\varsigma(r), \varsigma(r)\} \leq \{0, 0, \tfrac{\delta-k}{2}\} = \min\{\varsigma(0), \varsigma(0), \tfrac{\delta-k}{2}\},$$

where $(0, 0) \in \mathsf{K} \times \mathsf{K}$. Which implies that $\varsigma(0) \geq \min\{\varsigma(r), \frac{\delta-k}{2}\}\ \forall\, r \in \mathsf{K}$. □

Theorem 4.6. *Let ϱ and ς be two $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of* K*. Then the cartesian product $\varrho \times \varsigma$ is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of* $\mathsf{K} \times \mathsf{K}$.

Proof. Let for any $(t,t) \in \mathsf{K} \times \mathsf{K}$, we have

$$\begin{aligned}(\varrho \times \varsigma)(0,0) &= \min\{\varrho(0), \varsigma(0)\} \\ &\geq \min\{\varrho(t), \varsigma(t)\} \\ &= \min\{(\varrho \times \varsigma)(t,t), \tfrac{\delta-k}{2}\}\end{aligned}$$

Again let $(t,v),(u,w),(s,o) \in \mathsf{K} \times \mathsf{K}$.

$$\begin{aligned}(\varrho \times \varsigma)(t*s, v*o) &= \min\{\varrho(t*s), \varsigma(t*o)\} \\ &\geq \min\{\min\{\varrho(t*(u*s)), \varrho(u), \tfrac{\delta-k}{2}\}, \\ &\quad \min\{\varsigma(v*(w*o)), \varsigma(w), \tfrac{\delta-k}{2}\}\} \\ &= \min\{\min\{\varrho(t*(u*s)), \varsigma(v*(w*o))\}, \min\{\varrho(u), \varsigma(w)\}, \tfrac{\delta-k}{2}\} \\ &= \min\{(\varrho \times \varsigma)(t*(u*s), (v*(w*o))), (\varrho \times \varsigma)(u,w), \tfrac{\delta-k}{2}\}.\end{aligned}$$

□

Theorem 4.7. *Let ϱ_ς be the strongest $(\in, \in \vee q_\delta^k)$-fuzzy KU-relation on the KU-algebra* K *for a given fuzzy subset ς of* K*. Then ς is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of* K *if and only if ϱ_ς is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of* $\mathsf{K} \times \mathsf{K}$.

Proof. Let us assume that ς is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K. Consider,

$$\begin{aligned}\varrho_\varsigma(0,0) &= \min\{\varsigma(0), \varsigma(0)\} \geq \min\{\varsigma(t), \varsigma(m), \tfrac{\delta-k}{2}\} \\ &= \min\{\min\{\varsigma(t), \varsigma(m)\}, \tfrac{\delta-k}{2}\} = \min\{\varrho_\varsigma(t,v), \tfrac{\delta-k}{2}\}\end{aligned}$$

Now consider

$$\begin{aligned}\varrho_\varsigma(t*s, v*o) &= \min\{\varsigma(t*s), \varsigma(t*o)\} \\ &\geq \min\{\min\{\varsigma(t*(u*s)), \varsigma(u), \tfrac{\delta-k}{2}\}, \min\{\varsigma(v*(w*o)), \varsigma(w), \tfrac{\delta-k}{2}\}\} \\ &= \min\{\min\{\varsigma(t*(u*s)), \varsigma(v*(w*o))\}, \min\{\varsigma(u), \varsigma(w)\}, \tfrac{\delta-k}{2}\} \\ &= \min\left\{\varrho_\varsigma(t*(u*s), (v*(w*o))), \varrho_\varsigma(u,w), \tfrac{\delta-k}{2}\right\}.\end{aligned}$$

Hence ϱ_ς is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of $\mathsf{K} \times \mathsf{K}$.
Conversely assume that ϱ_ς is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of $\mathsf{K} \times \mathsf{K}$. Let for all $(t,t) \in \mathsf{K} \times \mathsf{K}$. Now consider

$$\min\{\varsigma(0), \varsigma(0)\} = \varrho_\varsigma(t,t) = \min\{\varsigma(t), \varsigma(t), \tfrac{\delta-k}{2}\} \Rightarrow \varsigma(0) \geq \min\{\varsigma(t), \tfrac{\delta-k}{2}\}$$

Now let $(t,v),(u,w),(s,o) \in \mathsf{K} \times \mathsf{K}$ and consider

$$\begin{aligned}\min\{\varsigma(t*s), \varsigma(t*o)\} &= \varrho_\varsigma(t*s, v*o) \\ &\geq \min\{\varrho_\varsigma(t,v)*((u,w)*(s,o)), \varrho_\varsigma(u,w), \tfrac{\delta-k}{2}\} \\ &= \min\{\varrho_\varsigma(t*(u*s)), \varrho_\varsigma(v*(w*o)), \varrho_\varsigma(u,w), \tfrac{\delta-k}{2}\} \\ &= \min\left\{\begin{array}{c}\min\{\{\varsigma(t*(u*s)), \varsigma(v*(w*o))\}\}, \\ \min\{\varsigma(u), \varsigma(w)\}, \tfrac{\delta-k}{2}\end{array}\right\} \\ &= \min\left\{\begin{array}{c}\min\{\min\{\varsigma(t*(u*s)), \varsigma(u)\}, \tfrac{\delta-k}{2}\}, \\ \min\{\varsigma(v*(w*o)), \varsigma(w), \tfrac{\delta-k}{2}\}\end{array}\right\}.\end{aligned}$$

If we choose $v = w = o = 0$, then

$$\varsigma(t*s) \geq \min\{\varsigma(t*(u*s)), \varsigma(u), \tfrac{\delta-k}{2}\}$$

which shows that ς is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K. □

5 (δ, k)-upper and lower parts of $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideals

In this section, we define the notions of (δ, k)-upper and lower parts of the (δ, k)-characteristic fuzzy subset of a KU-algebra and obtain some related results.

Definition 5.1. Let Ξ be a fuzzy subset of K. Define the (δ, k)-*upper part* Ξ^+ and the (δ, k)-*lower part* Ξ^- of Ξ as follows:

$$\Xi^+ = \max\{\Xi(x), \tfrac{\delta-k}{2}\} \quad \text{and} \quad \Xi^- = \min\{\Xi(x), \tfrac{\delta-k}{2}\} \quad \text{for all } x \in X.$$

In Example 3.7, we have shown that an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K is not a fuzzy KU-ideal of K. In the following theorem, we show that the lower part Ξ^- of an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal Ξ is a fuzzy KU-ideal of K.

Theorem 5.2. *If Ξ is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra (resp., KU-ideal) of K, then the lower part Ξ^- of Ξ is a fuzzy KU-subalgebra (resp., KU-ideal) of K.*

Proof. Let Ξ be an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K and $x, y, z \in N$. Then

$$\begin{aligned}\Xi^-(x * z) &= \min\{\Xi(x * z), \tfrac{\delta-k}{2}\}\\ &\geq \min\{\min\{\Xi(x * (y * z)), \Xi(y), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}\\ &= \min\{\min\{\Xi(x * (y * z)), \tfrac{\delta-k}{2}\}, \min\{\Xi(y), \tfrac{\delta-k}{2}\}\}\\ &= \min\{\Xi^-(x * (y * z)), \Xi^-(y)\}.\end{aligned}$$

Therefore Ξ^- is a fuzzy KU-ideal of K. □

Combining Theorem 5.2 and Remark 3.8, we have the following theorem.

Theorem 5.3. *If Ξ is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra (resp., KU-ideal) of K, then the lower part Ξ^- of Ξ is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra (resp., KU-ideal) of K.*

Combining Theorem 5.3 and Theorem 3.25, we have the following corollary.

Corollary 5.4. *Let Ξ be an $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra (resp., KU-ideal) of K. Then the non-empty level subset $[\Xi_t^-]_\delta^k = \{x \in N | \Xi^-(x) + t > \delta - k\}$ of Ξ is a KU-subalgebra (resp., KU-ideal) of K for all $t \in (\frac{\delta-k}{2}, 1]$.*

Theorem 5.5. *Let $\{\Xi_i\}_{i\in I}$ be a family of $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra (resp., KU-ideal) of K. Then $\Xi^- = \bigcap_{i\in I} \Xi_i^-$ is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra (resp., KU-ideal) of K where $(\bigcap_{i\in I} \Xi_i^-)(x) = \inf_{i\in I}\{\Xi_i^-(x)\}$.*

Remark 5.6. We proved that the converse of Figure 1 is true for the lower part Ξ^- of an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K.

Definition 5.7. Let A be a non-empty subset of K. Then the (δ, k)-*upper part* Ξ_A^+ and the (δ, k)-*lower part* Ξ_A^- of the (δ, k)-characteristic fuzzy subset Ξ_A of A are defined by

$$\Xi_A^+(x) = \begin{cases} \delta - k & if \quad x \in A, \\ \frac{\delta-k}{2} & if \quad x \notin A, \end{cases} \quad \text{and} \quad \Xi_A^-(x) = \begin{cases} \frac{\delta-k}{2} & if \quad x \in A, \\ 0 & if \quad x \notin A. \end{cases}$$

In the above definition, if we take $\Xi_A^+(x) = 1$ when $x \in A$, we obtain the (δ, k)-*upper part* χ_A^+ and the (δ, k)-*lower part* χ_A^- of the *characteristic function* χ_A of A.

Lemma 5.8. *Let A and B be any two non-empty subsets of K. Then the following statements are hold: $(\Xi_A \cap \Xi_B)^- = \Xi_{A\cap B}^-$, $(\Xi_A \cup \Xi_B)^- = \Xi_{A\cup B}^-$ and $(\Xi_A \circ \Xi_B)^- = \Xi_{AB}^-$.*

Theorem 5.9. *Let A be a non-empty subset of K. Then the (δ, k)-lower part Ξ_A^- of the (δ, k)-characteristic fuzzy subset Ξ_A of A is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K if and only if A is an KU-ideal of K.*

Corollary 5.10. *Let A be a non-empty subset of K. Then the (δ, k)-lower part χ_A^- of the characteristic function χ_A of A is an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K if and only if A is an KU-ideal of K.*

Corollary 5.11. *Let ϱ and ς be two $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of K. Then the cartesian product of the lower part of ϱ and ς, $\varrho^- \times \varsigma^-$ is also an $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideal of $\mathsf{K} \times \mathsf{K}$.*

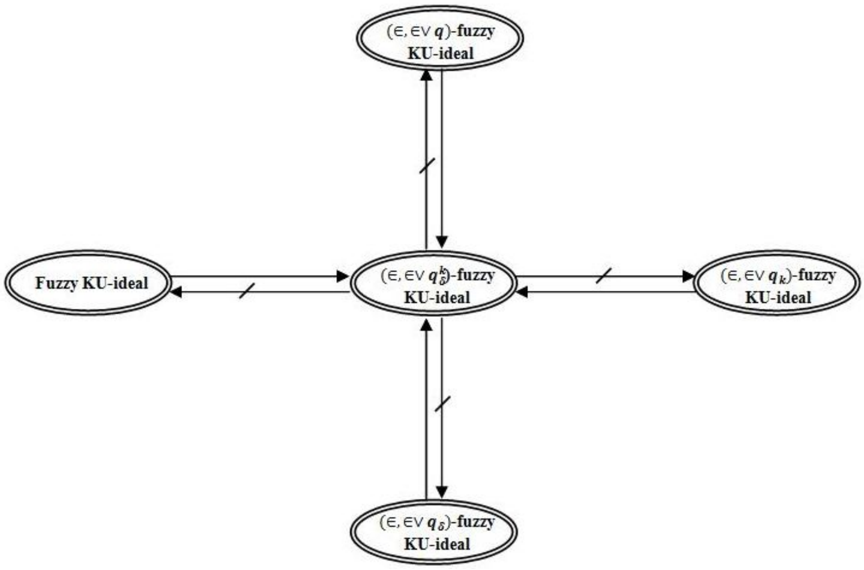

Figure 1: Relation between all types of fuzzy KU-ideals

6 Conclusions

In this article, the notions of $(\in, \in \vee q_\delta^k)$-fuzzy KU-subalgebra and $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideals of KU-algebras were introduced. Some characterizations of $(\in, \in \vee q_\delta^k)$-fuzzy KU-ideals of KU-algebras with suitable examples were discussed. Also, we discussed a few results of those fuzzy KU-ideals in KU-algebras under homomorphism, the image and the pre-image of fuzzy KU-ideal under homomorphism of KU-algebras are defined. Finally, the notions of (δ, k)-upper and lower parts of the (δ, k)-characteristic fuzzy subset of a KU-algebra was discussed. We discussed the relationships among the fuzzy KU-ideals of KU-algebras with examples and proved that the converses are true for the lower part of those fuzzy KU-subalgebras (KU-ideals) of KU-algebras. In the future, one can apply this article's notions to other ideals of KU-algebras such as implicative ideal, positive implicative ideal, complete ideal, E-ideal and complete E-ideal, etc.

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