Palestine Journal of Mathematics

Vol 15(1)(2026) , 876–884

A New Combination of k-Jacobsthal Numbers and Some of Its Properties

Jirawat Kantalo, Fongchan Ban A-Pai and Supunnee Sompong

Communicated by: Ayman Badawi

MSC 2010 Classifications: Primary 11B99; Secondary 11B57, 15A15.

Keywords and phrases: k-Jacobsthal number, Hessenberg matrix, Permanent.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

The corresponding author was very thankful to Thailand Science Research and Innovation (TSRI) as allocated a budget through National Science, Research, and Innovation Fund (NSRF), Sakon Nakhon Rajabhat University [Contact No. FF-SNRU_9/2025] for their necessary support and facility.

Corresponding Author: Supunnee Sompong

Abstract In this paper, we present a new sequence derived from the terms of the k-Jacobsthal sequence and present its recurrence relation along with some of its properties. Moreover, we investigate its properties using the matrix method and define some square lower Hessenberg matrices to obtain the determinant and the permanent that give the odd and even terms of this sequence.

1 Introduction

Special sequences of integers have been widely studied by many mathematicians in various ways, such as generalizations and properties of sequences. The Jacobsthal sequence $J_n{}_{n=0}^{\infty}$, which is well-known sequences, was newly presented by A.F. Horadam in 1996 [1] and is defined by

$$J_{n+1} = J_n + 2J_{n-1}, \tag{1.1}$$

for $n \geq 1$ and $J_0 = 0, J_1 = 1$.

In recent years, the Jacobsthal sequences and their generalizations have many interesting properties and applications. For example, In [2] Koken and Bozkurt deduce some properties and Binet like formula for the Jacobsthal sequence by matrix method. In 2022 [3], S. Vasanthi and B. Sivakumar investigate some properties of these matrices formed by Jacobsthal sequence. In the same year, Adegoke et al. [4] studied finite sums involving Jacobsthal sequence as a special case of Horadam and generalized Tribonacci sequences.

More recently, the Jacobsthal sequence has been generalized for any positive real number k namely $k-$Jacobsthal sequence $\{J_{k,n}\}_{n=0}^{\infty}$ which is defined by Falcon in 2014 [5], is defined by

$$J_{k,n+1} = J_{k,n} + kJ_{k,n-1},\ n \geq 1 \tag{1.2}$$

with initial conditions $J_{k,0} = 0$ and $J_{k,1} = 1$. It is obvious that if $k = 2$, the k-Jacobsthal sequence reduces to the Jacobsthal sequence. See more studies of generalizations and some identities of Jacobsthal sequence in [6], [7], [8], [9] and [10].

Additionally, some authors have generalized a new sequence through a combination of sequences. For example, in 2006, Satana and Diaz-Barrero [11] introduced the sequence $a_n = P_{2n} + P_{2n+1}$, where P_n is the Pell sequence, which is defined $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$ with the initial conditions $P_0 = 0$ and $P_1 = 1$. They then generalized the sequence as $a_{n+1} = 6a_n - a_{n-1}$ with the initial conditions $a_0 = 1$ and $a_1 = 7$. And in 2018, Gozeri introduced several sequences in the same way, such as $V_n = P_{n+1} + P_n$ and $Y_n = P_{n+2} - P_n$ with more details in [12].

Table 1. The first 10 terms of the $k-$Jacobsthal sequence are as follows:

n	$J_{k,n}$
0	0
1	1
2	1
3	$1+k$
4	$1+2k$
5	$1+3k+k^2$
6	$1+4k+3k^2$
7	$1+5k+6k^2+k^3$
8	$1+6k+10k^2+4k^3$
9	$1+7k+15k^2+10k^3+k^4$

Recently, many authors have attempted to generalize the well-known sequences such as the Pell sequence, the Lucas sequence and the Narayana sequence. In 2023, S.H.J. Petroudi et al. [13] introduced and analyzed the Pell-Narayana sequence. In 2024, S. Kapoor and P. Kumar [14] presented the Narayana-Lucas hybrinomial sequence, which is a generalization of the Narayana-Lucas hybrid numbers.

In this paper, we introduce a new sequence that is a combination of terms from the $k-$Jacobsthal sequence and present some identities of the sequence. Moreover, we explore some properties using the matrix method and provide the Hessenberg matrices to investigate the determinant and the permanent, which are related to the odd and even terms of this sequence.

2 Preliminary

In this section, the definitions and lemmas for the determinants and the permanents of the lower Hessenberg matrix, which are useful in this study, are presented as follows:

Definition 2.1. [15] A matrix $A_n = [a_{r,s}]_{n\times n}$ is called a lower Hessenberg matrix if $a_{r,s} = 0$ when $s - r > 1$, i.e.,

$$A_n = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}. \tag{2.1}$$

Lemma 2.2. *[16] Let A_n be a lower Hessenberg matrix. The following determinant formula for A_n is given by*

$$\det A_n = a_{n,n}\det A_{n-1} + \sum_{t=1}^{n-1}\left((-1)^{n-t} a_{n,t}\left[\prod_{j=t}^{n-1} a_{j,j+1}\right]\det A_{t-1}\right), \tag{2.2}$$

for $n \geq 2$, where $\det A_0 = 1$ *and* $\det A_1 = a_{1,1}$.

Definition 2.3. Let A_n be an $n\times n$ matrix, the permanent of A_n is defined by

$$\operatorname{per} A_n = \sum_{\sigma\in\mathbb{S}_n}\prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{2.3}$$

where $\mathbb{S}_n$ denotes the set of permutations of $\{1, 2, \ldots, n\}$.

Lemma 2.4. *[17] Let A_n be a lower Hessenberg matrix. The following permanent formula for A_n is given by*

$$\operatorname{per} A_n = a_{n,n} \operatorname{per} A_{n-1} + \sum_{t=1}^{n-1} \left(a_{n,t} \left[\prod_{j=t}^{n-1} a_{j,j+1} \right] \operatorname{per} A_{t-1} \right), \tag{2.4}$$

for $n \geq 2$, where $\operatorname{per} A_0 = 1$ *and* $\operatorname{per} A_1 = a_{1,1}$.

3 Main Result

In this section, we introduce a new sequence $X_{k,n}$, along with the determinants and the permanents of lower Hessenberg matrices associated with the sequence $X_{k,n}$.

3.1 Some Identities of the Sequence $X_{k,n}$

Firstly, we provide a new sequence $X_{k,n}$, which is a combination of terms from the k-Jacobsthal sequence.

Definition 3.1. For any non-negative integer n and positive real number k, define the sequence $X_{k,n}$ by

$$X_{k,n} = 2kJ_{k,n} + J_{k,n+1}, \ n \geq 0. \tag{3.1}$$

Table 2. The first 10 terms of the sequence $X_{k,n}$

n	$X_{k,n}$
0	1
1	$1+2k$
2	$1+3k$
3	$1+4k+2k^2$
4	$1+5k+5k^2$
5	$1+6k+9k^2+2k^3$
6	$1+7k+14k^2+7k^3$
7	$1+8k+20k^2+16k^3+2k^4$
8	$1+9k+27k^2+30k^3+9k^4$
9	$1+10k+35k^2+50k^3+25k^4+2k^5$

Theorem 3.2. *For any positive integer n and real number k, we have the recurrence relation*

$$X_{k,n+1} = X_{k,n} + kX_{k,n-1}, \tag{3.2}$$

with initial conditions $X_{k,0} = 1$ and $X_{k,1} = 1 + 2k$.

Proof.

$$\begin{aligned} X_{k,n+1} &= 2kJ_{k,n+1} + J_{k,n+2} \\ &= 2k(J_{k,n} + kJ_{k,n-1}) + J_{k,n+1} + kJ_{k,n} \\ &= 2kJ_{k,n} + J_{k,n+1} + k(2kJ_{k,n-1} + J_{k,n}) \\ &= X_{k,n} + kX_{k,n-1} \end{aligned}$$

□

The characteristic equation of the recurrence relation $X_{k,n+1} = X_{k,n} + kX_{k,n-1}$ has distinct roots $\sigma_1 = \frac{1+\sqrt{4k+1}}{2}$ and $\sigma_2 = \frac{1-\sqrt{4k+1}}{2}$. We derive the binet formula of the sequence $X_{k,n}$ as follows:

Theorem 3.3. *(Binet formula) For any non-negative integer n and positive real number k,*

$$X_{k,n} = \sigma_1^{n+1} + \sigma_2^{n+1} \tag{3.3}$$

Proof. From $X_{k,n} = 2kJ_{k,n} + J_{k,n+1}$, we have

$$\begin{aligned} X_{k,n} &= 2k\left(\frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}\right) + \left(\frac{\sigma_1^{n+1} - \sigma_2^{n+1}}{\sigma_1 - \sigma_2}\right) \\ &= \frac{\sigma_1^{n+1}(2k\sigma_1^{-1} + 1) + \sigma_2^{n+1}(-2k\sigma_2^{-1} - 1)}{\sigma_1 - \sigma_2} \\ &= \frac{\sigma_1^{n+1}\left(\sqrt{4k+1}\right) + \sigma_2^{n+1}\left(\sqrt{4k+1}\right)}{\sqrt{4k+1}} \\ &= \sigma_1^{n+1} + \sigma_2^{n+1} \end{aligned}$$

□

Theorem 3.4. *For any non-negative integer n, r and real number k, we have*

$$X_{k,n-r}X_{k,n+r} - X_{k,n}^2 = (-k)^{n-r+1}X_{k,r-1}^2 - 4(-k)^{n+1} \tag{3.4}$$

Proof.

$$\begin{aligned} X_{k,n-r}X_{k,n+r} - X_{k,n}^2 &= (\sigma_1^{n-r+1} + \sigma_2^{n-r+1})(\sigma_1^{n+r+1} + \sigma_2^{n+r+1}) - (\sigma_1^{n+1} + \sigma_2^{n+1})^2 \\ &= \sigma_1^{n-r+1}\sigma_2^{n+r+1} + \sigma_1^{n+r+1}\sigma_2^{n-r+1} - 2\sigma_1^{n+1}\sigma_2^{n+1} \\ &= (\sigma_1\sigma_2)^{n-r+1}[\sigma_1^{2r} + \sigma_2^{2r} - 2(\sigma_1\sigma_2)^r] \\ &= (\sigma_1\sigma_2)^{n-r+1}[(\sigma_1^r + \sigma_2^r)^2 - 4(\sigma_1\sigma_2)^r] \\ &= (-k)^{n-r+1}[X_{k,r-1}^2 - 4(-k)^r] \\ &= (-k)^{n-r+1}X_{k,r-1}^2 - 4(-k)^{n+1} \end{aligned}$$

□

Remark 3.5. In particular case,

(i) If $r = 1$, we have $X_{k,n-1}X_{k,n+1} - X_{k,n}^2 = (1 + 4k)(-k)^n$,

(ii) If $r = n$, we have $X_{k,2n} = X_{k,n}^2 - kX_{k,n-1}^2 - 4(-k)^{n+1}$.

Theorem 3.6. *For any positive integer n and positive real number k.*

If $A = \begin{bmatrix} 1+2k & k\sqrt{1+4k} \\ -\sqrt{1+4k} & -2k \end{bmatrix}$, *then* $A^n = \begin{bmatrix} X_{k,n} & \frac{k(2X_{k,n}-X_{k,n-1})}{\sqrt{1+4k}} \\ \frac{-(2X_{k,n}-X_{k,n-1})}{\sqrt{1+4k}} & -X_{k,n} + X_{k,n-1} \end{bmatrix}$.

Proof. We shall demonstrate that the statement holds true for all $n \geq 1$ by applying mathematical induction.
For $n = 1$, we can see that

$$\begin{bmatrix} 1+2k & k\sqrt{1+4k} \\ -\sqrt{1+4k} & -2k \end{bmatrix} = \begin{bmatrix} X_{k,1} & \frac{k(2X_{k,1}-X_{k,0})}{\sqrt{1+4k}} \\ \frac{-(2X_{k,1}-X_{k,0})}{\sqrt{1+4k}} & -X_{k,1} + X_{k,0} \end{bmatrix}.$$

Thus the statement holds for $n = 1$.
Now, suppose that $n = m$ is true. That is

$$A^m = \begin{bmatrix} X_{k,m} & \frac{k(2X_{k,m}-X_{k,m-1})}{\sqrt{1+4k}} \\ \frac{-(2X_{k,m}-X_{k,m-1})}{\sqrt{1+4k}} & -X_{k,m} + X_{k,m-1} \end{bmatrix}.$$

We will use this assumption to prove that $n = m + 1$ is true.

$$\begin{aligned} A^{m+1} &= A^m A \\ &= \begin{bmatrix} X_{k,m} + kX_{k,m-1} & \frac{kX_{k,m}+2k^2X_{k,m-1}}{\sqrt{1+4k}} \\ \frac{-(X_{k,m}+2kX_{k,m-1})}{\sqrt{1+4k}} & -kX_{k,m-1} \end{bmatrix} \\ &= \begin{bmatrix} X_{k,m+1} & \frac{kX_{k,m}+2k(X_{k,m+1}-X_{k,m})}{\sqrt{1+4k}} \\ \frac{-(X_{k,m}+2(X_{k,m+1}-X_{k,m}))}{\sqrt{1+4k}} & -X_{k,m+1} + X_{k,m} \end{bmatrix} \\ &= \begin{bmatrix} X_{k,m+1} & \frac{k(2X_{k,m+1}-X_{k,m})}{\sqrt{1+4k}} \\ \frac{-(2X_{k,m+1}-X_{k,m})}{\sqrt{1+4k}} & -X_{k,m+1} + X_{k,m} \end{bmatrix} \end{aligned}$$

The proof of this theorem is now completed by the method of mathematical induction. □

Note. Based on the properties of the determinant of a matrix, it follows that

$$X_{k,n-1}X_{k,n} + kX_{k,n-1}^2 - X_{k,n}^2 = (1+4k)(-k)^n.$$

From the properties of matrix products and Theorem 3.6, we have

Theorem 3.7. *For any positive integer n, m and positive real number k, we get*

$$(1+4k)X_{k,m+n} = (1+4k)X_{k,m}X_{k,n} - k(X_{k,m-1} - 2X_{k,m})(X_{k,n-1} - 2X_{k,n}) \tag{3.5}$$

Proof. From $A^{m+n} = A^m A^n$, we obtain

$$A^{m+n} = \begin{bmatrix} X_{k,m+n} & \frac{k(2X_{k,m+n}-X_{k,m+n-1})}{\sqrt{1+4k}} \\ \frac{-(2X_{k,m+n}-X_{k,m+n-1})}{\sqrt{1+4k}} & -X_{k,m+n} + X_{k,m+n-1} \end{bmatrix}$$

and

$$A^m A^n = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ where}$$

$a_{11} = \frac{-k(X_{k,m-1}-2X_{k,m})(X_{k,n-1}-2X_{k,n})}{1+4k} + X_{k,m}X_{k,n}$,
$a_{12} = \frac{k(X_{k,m}X_{k,n-1}-X_{k,m-1}(X_{k,n-1}-X_{k,n}))}{\sqrt{1+4k}}$,
$a_{21} = \frac{-X_{k,m}X_{k,n-1}+X_{k,m-1}(X_{k,n-1}-X_{k,n})}{\sqrt{1+4k}}$,
$a_{22} = \frac{X_{k,m}(-(1+2k)X_{k,n-1}+X_{k,n})+X_{k,m-1}((1+3k)X_{k,n-1}-(1+2k)X_{k,n})}{1+4k}$.
Consider the element in the first row and the first column, we have

$$(1+4k)X_{k,m+n} = (1+4k)X_{k,m}X_{k,n} - k(X_{k,m-1} - 2X_{k,m})(X_{k,n-1} - 2X_{k,n}).$$

□

Remark 3.8. From (3.5), if $m = n$, we have

$$(1+4k)X_{k,2n} = (1+4k)X_{k,n}^2 - k(X_{k,n-1} - 2X_{k,n})^2$$

or

$$(1+4k)X_{k,2n} = X_{k,n}^2 - kX_{k,n-1}^2 + 4kX_{k,n-1}X_{k,n}.$$

Note. From Remark 3.5 and Remark 3.8 we have the following identity:

$$X_{k,n-1}X_{k,n} - X_{k,2n} = (-k)^n.$$

3.2 Determinants and Permanents of Hessenberg Matrices with the Sequence $X_{k,n}$

We define a new $n \times n$ lower Hessenberg matrices and present the determinants and the permanents of their matrices which are the sequence $X_{k,n}$.

Theorem 3.9. *Let $A_n = [a_{i,j}]$ be an $n \times n$ lower Hessenberg matrix, is defined by*

$$a_{i,j} = \begin{cases} 1+2k & \text{if } i=j=1; \\ 1+k & \text{if } i=j \text{ for } i,j \geq 2; \\ 1 & \text{if } i=j-1; \\ (-k)^{i-j} & \text{if } i \geq j+1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Then

$$\det A_n = X_{k,2n-1}, \tag{3.7}$$

for any positive integer n and positive real number k.

Proof. We prove by mathematical induction on n. By hypothesis, the result holds for all $n \leq 2$. Then, we suppose that the result is true for all positive integer m such that $m \geq 3$. We will prove it for $m+1$. Firstly, we use elementary row operations on the matrix A_{m+1}. We multiply in m^{th} row by k then add to $(m+1)^{\text{th}}$ row. So, we get the $(m+1)^{\text{th}}$ row as $[\underbrace{0\ \ 0\ \ \cdots\ \ 0}_{(m-1)^{\text{th}}}\ \ k^2\ \ 1+2k]$.

That is

$$A_{m+1} = \begin{bmatrix} 1+2k & 1 & 0 & \cdots & & \cdots & & 0 \\ -k & 1+k & 1 & 0 & & \cdots & & 0 \\ k^2 & -k & 1+k & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ (-k)^{m-1} & & & & & & 1+k & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & k^2 & 1+k \end{bmatrix}.$$

Now, using Lemma 2.2, we have

$$\begin{aligned} \det A_{m+1} &= (1+2k)\det A_m + \sum_{t=1}^{m}\left((-1)^{m+1-t} a_{m+1,t}\left[\prod_{j=t}^{m} a_{j,j+1}\right]\det A_{t-1}\right) \\ &= (1+2k)\det A_m + \sum_{t=1}^{m-1}\left((-1)^{m+1-t} a_{m+1,t}\left[\prod_{j=t}^{m} a_{j,j+1}\right]\det A_{t-1}\right) \\ &\quad + \sum_{t=m}\left((-1)^{m+1-t} a_{m+1,t}\left[\prod_{j=t}^{m} a_{j,j+1}\right]\det A_{t-1}\right) \\ &= (1+2k)\det A_m + (-1)a_{m+1,m}a_{m,m+1}\det A_{m-1} \\ &= (1+2k)\det A_m - k^2\det A_{m-1} \\ &= (1+2k)X_{k,2m-1} - k^2 X_{k,2(m-1)-1} \\ &= X_{k,2m-1} + 2kX_{k,2m-1} - k\left(X_{k,2m-1} - X_{k,2m-2}\right) \\ &= X_{k,2m} + kX_{k,2m-1} \\ &= X_{k,2m+1} \\ &= X_{k,2(m+1)-1} \end{aligned}$$

Then, $\det A_n = X_{k,2n-1}$ for all $n \geq 1$. □

Theorem 3.10. *Let $B_n = [b_{i,j}]$ be an $n \times n$ lower Hessenberg matrix, is defined by*

$$b_{i,j} = \begin{cases} 1+3k & \text{if } i=j=1; \\ 1+2k & \text{if } i=j \text{ for } i,j \geq 2; \\ 1 & \text{if } i=j-1; \\ k^2 & \text{if } i=j+1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Then

$$\det B_n = X_{k,2n}, \tag{3.9}$$

for any positive integer n and positive real number k.

Proof. We prove by mathematical induction on n. By hypothesis, the result holds for all $n \leq 2$. Then, we suppose that the result is true for all positive integer m such that $m \geq 3$. We will prove it for $m+1$. So, we get the matrix B_{m+1}, that is

$$B_{m+1} = \begin{bmatrix} 1+3k & 1 & 0 & \cdots & \cdots & & 0 \\ k^2 & 1+2k & 1 & 0 & \cdots & & 0 \\ 0 & k^2 & 1+2k & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & & 0 & 1+2k & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & k^2 & 1+2k \end{bmatrix}.$$

Now, using Lemma 2.2, we have

$$\begin{aligned} \det B_{m+1} &= (1+2k)\det B_m + \sum_{t=1}^{m}\left((-1)^{m+1-t}\, b_{m+1,t}\left[\prod_{j=t}^{m} b_{j,j+1}\right]\det B_{t-1}\right) \\ &= (1+2k)\det B_m + \sum_{t=1}^{m-1}\left((-1)^{m+1-t}\, b_{m+1,t}\left[\prod_{j=t}^{m} b_{j,j+1}\right]\det B_{t-1}\right) \\ &\quad + \sum_{t=m}\left((-1)^{m+1-t}\, b_{m+1,t}\left[\prod_{j=t}^{m} b_{j,j+1}\right]\det B_{t-1}\right) \\ &= (1+2k)\det B_m + (-1) b_{m+1,m} b_{m,m+1}\det B_{m-1} \\ &= (1+2k)\det B_m - k^2\det B_{m-1} \\ &= (1+2k)X_{k,2m} - k^2 X_{k,2(m-1)} \\ &= X_{k,2m} + 2kX_{k,2m} - k\left(X_{k,2m} - X_{k,2m-1}\right) \\ &= X_{k,2m+1} + kX_{k,2m} \\ &= X_{k,2m+2} \\ &= X_{k,2(m+1)} \end{aligned}$$

Then, $\det B_n = X_{k,2n}$ for all $n \geq 1$. □

Now, we will present the permanents of matrices that are sequence $X_{k,n}$. Kenan, [18] gave the relationship the determinant and the permanent of a Hessenberg matrix by using Lemmas 2.2 and 2.4.

Then, let A_n be an $n \times n$ lower Hessenberg matrix $A = [a_{r,s}]$ is given in (2.1) and also let E_n be an $n \times n$ lower Hessenberg matrix which is defined by $e_{r,r+1} = -a_{r,r+1}$ for all r, $e_{r,s} = a_{r,s}$ for $r \geq s$ and 0 otherwise. So, we have det E_n = perA_n and det A_n = perE_n. Then, we have the following Corollary without proof.

Let H_n be $n \times n$ matrix, is defined by

$$H_n = \begin{bmatrix} 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \ddots & 1 \\ 1 & 1 & 1 & \ddots & 1 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{3.10}$$

Corollary 3.11. *Let V_n and W_n be $n \times n$ matrices and define $V_n = H_n \circ A_n$ and $W_n = H_n \circ B_n$ where $\circ$ denotes the operator of Hadamard product of matrix. Then,*

$$\text{per}V_n = X_{k,2n-1} \quad \textit{and} \quad \text{per}W_n = X_{k,2n}. \tag{3.11}$$

4 Conclusion

In this paper, we introduced a new sequence $X_{k,n}$, defined as a combination of terms from the k-Jacobsthal sequence, given by $X_{k,n} = 2kJ_{k,n} + J_{k,n+1}$. The various properties of the sequence, including a Binet formula and several algebraic identities, have been studied. Moreover, using matrix methods, we defined a matrix whose powers contain elements corresponding to the sequence $X_{k,n}$. Additionally, we introduced lower Hessenberg matrices whose determinants and permanents correspond to the odd and even terms of the sequence $X_{k,n}$. For future work, one may consider extending this construction by combining it with other sequences. Furthermore, the matrix methods associated with sequences could be applied to problems in cryptography and related areas.

References

[1] A. F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly, **34(1)**, 40–54, (1996).

[2] F. Koken and D. Bozkurt, *On the Jacobsthal-Lucas numbers by matrix methods*, Int. J. Contemp. Math. Sci., **3(33)**, 1629–1633, (2008).

[3] S. Vasanthi and B. Sivakumar, *Jacobsthal matrices and their properties*, Indian J. Sci. Technol., **15(5)**, 207–215, (2022).

[4] K. Adegoke, R. Frontczak and T. Goy, *Some special sums with squared Horadam numbers and generalized tribonacci numbers*, Palestine Journal of Mathematics, **11(1)**, 66–73, (2022).

[5] F. Sergio, *On the k-Jacobsthal Numbers*, American Review of Mathematics and Statistics., **2(1)**, 67—77, (2014).

[6] D. Bród and A. Michalski, *On generalized Jacobsthal and Jacobsthal-Lucas numbers*, In Annales Mathematicae Silesianae, **36(2)**, 115—128, (2022).

[7] D. Jhala, K. Sisodiya and G. P. S. Rathore, *On some identities for k-Jacobsthal numbers*, Int. J. Math. Anal.(Ruse), **7(12)**, 551—556, (2013).

[8] F. T. Aydın, *On generalizations of the Jacobsthal sequence*, Notes on number theory and discrete mathematics., **24(1)**, 120—135, (2018).

[9] S. Falcon, *On the k-Jacobsthal numbers*. American Review of Mathematics and Statistics, **2(1)**, 67–77, (2014).

[10] G. Morales, *New approach to third-order Jacobsthal sequence*, Palestine Journal of Mathematics, **14(1)**, (2025).

[11] S. F. Santana and J. L. Díaz-Barrero, *Some properties of sums involving Pell numbers*, Missouri Journal of Mathematical Sciences, **18(1)**, 33–40, (2006).

[12] G. Karadeniz Gözeri, *On Pell, Pell-Lucas, and balancing numbers*, Journal of Inequalities and Applications, **2018(1)**, 3, (2018).

[13] S. J. Petroudi, M. Pirouz and M. Shadab, *A Study of Pell-Narayana sequence: Properties and circulant matrix aspects*, Palestine Journal of Mathematics, **12(3)**, 103–114, (2023).

[14] S. Kapoor and P. Kumar, *On Narayana-Lucas Hybrinomial Sequence*, Palestine Journal of Mathematics, **13(4)**, (2024).

[15] E. Morteza, *More on the Fibonacci sequence and Hessenberg matrices*, Integers, **6**, A32, (2006).

[16] N. D. Cahill, J. R. D'Errico, D. A. Narayan and J. Y. Narayan, *Fibonacci determinants*, College Math. J., **33(3)**, 221–225, (2002).

[17] A. A. Ocal, N. Tuglu and E. Altinisik, *On the representation of k-generalized Fibonacci and Lucas Numbers*, Appl. Math. Comput., **170**, 584–596, (2005).

[18] K. Kaygısız and A. Sahin, *Determinant and permanent of Hessenberg matrix and generalized Lucas polynomials*, Bull. Iran. Math. Soc., **39**, 1065–1078, (2013).

Author information

Jirawat Kantalo, Department of Mathematics and Statistics, Faculty of Science and Technology, Sakon Nakhon Rajabhat University, Thailand.
E-mail: jirawat@snru.ac.th

Fongchan Ban A-Pai, Department of Mathematics and Statistics, Faculty of Science and Technology, Sakon Nakhon Rajabhat University, Thailand.
E-mail: fongchan@snru.ac.th

Supunnee Sompong, Department of Mathematics and Statistics, Faculty of Science and Technology, Sakon Nakhon Rajabhat University, Thailand.
E-mail: s_sompong@snru.ac.th

Received: 2025-05-27
Accepted: 2025-11-09