

ON DETOUR EDGE PEBBLING NUMBER FOR CERTAIN CLASSES OF GRAPHS

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Abstract Assume G is a connected graph with the distribution of pebbles over its edges. An edge pebbling move on a graph G is defined to be the removal of two pebbles from one edge and one pebble will be added to its adjacent edge, while the other pebble will be discarded from the play. In this paper, we have working on the concept of detour edge pebbling number and find out the detour edge pebbling number for certain classes of graphs. We have carried out the edge pebbling move in the concept of detour pebbling to arrive a new graph invariant called the detour edge pebbling number. The detour edge pebbling number of an edge e of a graph G is the minimum number of pebbles such that these pebbles are placed on the edges of G , we can move a pebble to the edge e by making a sequence of pebble moves regardless of the initial configuration using the edge detour path. The detour edge pebbling number of a graph G , $f_e^*(G)$, is the maximum $f_e^*(G, e)$ over all the edges of G . Also, we give an example that offer grounds for the results.

1 Introduction

Graph pebbling is a mathematical game and which can be applied to any connected graph. The concept of pebbling in graphs was first suggested by Lagarias and Saks for to give an alternative proof of a theorem of Kleitman and Lemke in number theory and the concept was first mentioned in 1989 by Chung [2]. According to Hurlbert and Kenter [4], Graph pebbling is a blend of graph theory, number theory, and optimization and they have provided a clear view of graph pebbling.

A series of pebbling moves in graphs is what makes the game. To pebble a graph, we need to choose a vertex with at least two pebbles, get rid of two pebbles from it, and add one to its adjacent vertex, and the second pebble that is removed will be taken out of play. To have a better understanding of pebbling, readers can refer to [7, 13]. The possibility exists for graph pebbling is to serve as a model for distributing and transporting consumable resources.

Some notable invariants in pebbling are optimal pebbling number, t -pebbling number, cover pebbling number, monophonic pebbling number [1, 5, 10, 14]. In 2020, Paul [12] extended the concept of pebbling by finding the edge pebbling number of some standard graphs. Chartrand *et al.* [3] presented and investigated the fundamental concepts of detour distance in graphs. In 2023, Zhan [18] worked on the minimum number of detours in graphs and his paper paves the elegant ways of finding detours in graphs. Lourdusamy *et al.* [8, 9, 11, 6] worked on detour pebbling number. Vincylin *et al.* [15] found the detour edge pebbling number. In this paper, we compute the detour edge pebbling number for certain classes of graphs.

2 Preliminaries

Definition 2.1. [12] An edge pebbling move on a graph G is defined to be the removal of two pebbles from one edge and one pebble will be added to an adjacent edge, while the other will be

discarded from the play.

Theorem 2.1. [12] The edge pebbling number of a n -path $P_E(P_n) = 2^{n-2}$, $n > 1$

Theorem 2.2. [12] The edge pebbling number of a complete graph $P_E(K_n) = \frac{n(n-1)}{2}$

Theorem 2.3. [12] The edge pebbling number of a n -star graph $P_E(S_n) = n$

Theorem 2.4. [8] For the path graph P_n with n vertices, the detour pebbling number is $f^*(P_n) = 2^{n-1}$

Theorem 2.5. [8] For the complete graph K_n with n vertices, the detour pebbling number is $f^*(K_n) = 2^{n-1}$

Theorem 2.6. [8] For the wheel graph W_n with $n + 1$ vertices, the detour pebbling number is $f^*(W_n) = 2^n$

Theorem 2.7. [8] Let $K_{1,n}$ be an n -star. The detour pebbling number for the n -star graph is $f^*(K_{1,n}) = n+2$

3 Detour Edge pebbling Number

Definition 3.1. [15] The detour edge pebbling number of an edge e of a graph G is the minimum number of pebbles such that these pebbles are placed on the edges of G , we can move a pebble to the edge e by making a sequence of pebble moves regardless of the initial configuration using the edge detour path. The detour edge pebbling number of a graph G , $f_e^*(G)$, is the maximum $f_e^*(G, e)$ over all the edges of G .

Definition 3.2. [15] An edge detour path between e_x and e_y is a sequence of edges $P_e^* = \{e_x, e_1, e_2, \dots, e_k, e_y\}$ such that $e_i \cap e_{i+1} \neq \emptyset$, no edge appears more than once in the sequence and the path allows repeated vertices. Therefore, the edge detour distance between the edges e_x and e_y , $d^*(e_x, e_y) = |P_e^*| - 1$ where $|P_e^*|$ is the total number of edges in the edge detour path.

Example 3.1

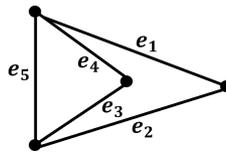


Figure 3.1. A connected graph G with 5 edges.

Consider the Figure 3.1. Let e_1 be the destination edge.

Now we can reach the destination edge e_1 from e_3 using the shortest path e_3, e_4, e_1 but the edge detour path from e_3 to e_1 is e_3, e_4, e_5, e_2, e_1 .

The edge detour path for e_1 to e_2 is e_1, e_4, e_3, e_2 . By placing 8 pebbles on e_4 , we can reach e_1 either by the way e_4, e_3, e_2, e_1 or e_4, e_5, e_2, e_1 but if 8 pebbles on e_5 , we cannot reach e_1 with the edge detour path e_5, e_4, e_3, e_2, e_1 . By placing 15 pebbles on e_5 , we cannot reach e_1 with the edge detour path e_5, e_4, e_3, e_2, e_1 . And if 16 pebbles are placed on the edge e_5 , we can reach e_1 by the edge detour path e_5, e_4, e_3, e_2, e_1 . Also, 16 pebbles are placed on the edges of G , the edge detour path from e_1 to any other edges of G has edge detour distance of 4, we can reach e_1 .

So, the detour edge pebbling number for e_1 is $f_e^*(G, e_1) = 16$.

Similarly, the detour edge pebbling number for e_2 is $f_e^*(G, e_2) = 16$, for e_3 is $f_e^*(G, e_3) = 16$, for e_4 is $f_e^*(G, e_4) = 16$ and $f_e^*(G, e_5) = 16$.

Thus, the detour edge pebbling number for G is $f_e^*(G) = 16$.

Theorem 3.1. [15] The detour edge pebbling number for the path P_n graph is $f_e^*(P_n) = 2^{n-2}$; $n \geq 2$.

Proof. Let P_n be the path graph with n vertices. Consider the edge set of P_n be $E(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$.

Take $2^{n-2} - 1$ pebbles for the initial distribution. Consider the target edge to be e_{n-1} . Placing $2^{n-2} - 1$ pebbles on e_1 , the edge e_{n-1} cannot be reached. Therefore, $f_e^*(P_n) \geq 2^{n-2}$.

For proving the sufficient part, let the target edge be either e_1 or e_{n-1} . The edge detour distance from e_k to e_1 is at most $k-1$ where $1 < k < n-1$. By using 2^{k-1} pebbles, reaching the destination edge either e_1 or e_{n-1} is possible. Now, consider e_k as the destination edge. The edge detour

distance from e_k to e_j is at most $n-2$ where $k < j \leq n-1$. Hence, the edge set $\{e_k, e_{k+1}, \dots, e_{n-1}\}$ contains at least 2^{n-3} pebbles, we can reach the destination edge e_k . Therefore, 2^{n-2} pebbles are sufficient to reach the destination edge because the edge detour distance from e_k to e_i is at most $k-1$ where $1 \leq i < k$.

Therefore, $f_e^*(P_n) = 2^{n-2}$.

Theorem 3.2. *The detour edge pebbling number for the complete graph K_n with n vertices is*

$$f_e^*(K_n) = 2^{\frac{n^2-n-2}{2}}; \text{ when } n \text{ is odd, } n \geq 3 \text{ and } f_e^*(K_n) = 2^{\frac{n^2-2n}{2}}; \text{ when } n \text{ is even.}$$

Proof. Consider the edge set of K_n be $E(K_n) = \{e_1, e_2, \dots, e_{\frac{n(n-1)}{2}}\}$.

Case 1. n is odd.

Take $2^{\frac{n^2-n-2}{2}} - 1$ pebbles for distribution. Let e_1 be any edge of K_n . Consider the target edge to be e_1 and assume there are zero number of pebbles on it. Placing $2^{\frac{n^2-n-2}{2}} - 1$ pebbles on any one of the edges of K_n except the edge e_1 , using edge detour path we cannot move a pebble to e_1 .

Therefore, $f_e^*(K_n) \geq 2^{\frac{n^2-n-2}{2}}$.

Now we prove the sufficient part by distributing $2^{\frac{n^2-n-2}{2}}$ pebbles on the edges of K_n .

Let the target edge be any edge of K_n . Here, the edge detour path of K_n consists of all the edges $\{e_1, e_2, \dots, e_{\frac{n(n-1)}{2}}\}$ has length $\frac{n(n-1)}{2}$. Therefore, if $2^{\frac{n^2-n-2}{2}}$ pebbles are placed on any one of the edges of K_n , target edge can be reached. Also, if we alter the configurations of pebbles on the edges, by Theorem 3.1, shifting a pebble to the destination edge is possible.

Thus, $f_e^*(K_n) = 2^{\frac{n^2-n-2}{2}}$, when n is odd.

Case 2. n is even.

For k_2 , the result is obviously true. For $n \geq 4$, the edge detour path of K_n does not consist of all the edges i.e., for $n = 4$, the edge detour path lacks one edge from the total number of edges of k_4 since it has only 5 edges and for $n = 6$, the edge detour path does not have 2 edges from the total number of edges of k_6 . So the lacking of edges in the edge detour path keeps on increasing by one edge for $n \geq 4$. Therefore, $\frac{n^2-2n}{2} + 1$ is the length of the edge detour path of K_n when n is even.

Take $2^{\frac{n^2-2n}{2}} - 1$ pebbles for distribution. Let e_1 be any edge of K_n . Consider the target edge to be e_1 . Now, $2^{\frac{n^2-2n}{2}} - 1$ pebbles are insufficient to reach the target edge using the edge detour path by considering the pebble allotment in an unusual way of placing all the pebbles on any one of the edges of K_n except the edge e_1 . Therefore, $f_e^*(K_n) \geq 2^{\frac{n^2-2n}{2}}$.

Let's now show the sufficiency by distributing $2^{\frac{n^2-2n}{2}}$ pebbles on the edges of K_n .

Let the target edge be any edge of K_n . Here, the edge detour path of K_n does not consist of all the edges. Therefore, if $2^{\frac{n^2-2n}{2}}$ pebbles are placed on any one of the edges of K_n , target edge can be reached. Also, if we alter the configurations of pebbles on the edges, by Theorem 3.1, shifting a pebble to the destination edge is possible.

Thus, $f_e^*(K_n) = 2^{\frac{n^2-2n}{2}}$, when n is even.

Theorem 3.3. *The detour edge pebbling number for the n -star graph is $f_e^*(S_n) = n$.*

Proof. The n -star graph is a complete bipartite graph $K_{1,n}$. Let the edge set of S_n be $E(S_n) = \{e_1, e_2, \dots, e_n\}$.

For $n = 1$, the result is obviously true. For $n \geq 2$, Let e_1 be any edge of S_n . Consider the target edge to be e_1 and assume there are zero number of pebbles on it. Placing $n-1$ pebbles on the edges of n -star graph by keeping all the pendant edges have one pebble each except the target edge e_1 , the target edge cannot be reached. Therefore $f_e^*(S_n) \geq n$.

Now we prove the sufficient part by distributing n pebbles on the edges of S_n .

Let the target edge be any edge of S_n . Without loss of generality, let it be e_1 . Here, the edge detour path of S_n does not consist of all the edges. Thus, there would only be one path in every pebble move, and that path would undoubtedly be the edge detour path i.e., e_i to e_1 for some i , is a edge detour path. Consider the pebble allotment in an unusual way of placing 2 pebbles on anyone of the edges of n -star graph except the target edge e_1 , reaching the target edge is possible. And by placing n pebbles on the edges of S_n with the distribution of $n-2$ edges have one pebble each and 2 pebbles on a remaining edge of S_n , reaching the target edge is possible. Also, if

we alter the configurations of pebbles on the edges, by Theorem 3.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the edges of the graph S_n . Therefore, $f_e^*(S_n) = n$.

Theorem 3.4. *The detour edge pebbling number for the ladder graph is $f_e^*(L_n) = 2^{2^n} + n - 3$; $n \geq 3$.*

Proof. Consider the edge set of L_n be $E(L_n) = \{e_1, e_2, \dots, e_{3n-2}\}$. For $n = 3$, the edge detour path consists of all the edges of L_3 . Therefore, reaching any target edge is possible since the length of the edge detour path is $3n-2$ for L_3 . For $n \geq 4$, the edge detour path does not consist of all the edges of L_n i.e., for $n = 4$, the edge detour path lacks of one edge from the total number of edges of L_4 since the edge detour path has only 9 edges. And the lacking of edges in the edge detour path keeps on increasing for $n \geq 4$. Take $2^{2^n} + n - 4$ pebbles for distribution. Consider the target edge to be e_1 and assume there are zero number of pebbles on it. Place $2^{2^n} - 1$ pebbles on the edge e^* of L_n and make sure that the edge detour distance between the edges e_1 and e^* is maximum and keep the edges with one pebble that are not in the edge detour path. Now, the target edge cannot be reached using the edge detour path since the edge detour distance is at most $2n$. Thus, $f_e^*(L_n) \geq 2^{2^n} + n - 3$. Let's now show the sufficiency by distributing $2^{2^n} + n - 3$ pebbles on the edges of L_n . Consider the target edge to be e_1 and assume there are zero number of pebbles on it. Here, the edge detour path of L_n does not consist of all the edges since the length of the edge detour path is $2n+1$. Therefore, if we consider the pebble allotment in an unusual way of placing all the 2^{2^n} pebbles on the edge e^* and $n - 3$ pebbles with the distribution such that the edges that are not in the edge detour path will have one pebble each, target edge can be reached. Also, if we alter the configurations of pebbles on the edges, by Theorem 3.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the edges of the graph L_n . Thus, $f_e^*(L_n) = 2^{2^n} + n - 3$; $n \geq 3$.

Remark 3.1. The detour edge pebbling number for cycle C_n with n vertices is same as the detour pebbling number of C_n since the number of vertices in C_n equals the number the edges and every vertex has exactly two edges incident in it. Hence $f_e^*(C_n) = 2^{n-1}$

4 Detour edge pebbling number on some notable graph structures

Definition 4.1. [16] The Durer graph is a well-covered graph with 12 vertices and 18 edges. It is named after Albrecht Durer, the German mathematician. And its construction as the generalized Petersen graph $G(6,2)$. Like any convex polyhedron's graph, the Durer graph is a 3-vertex-connected, simple, and planar.

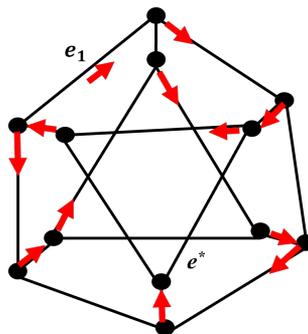


Figure 4.1. Edge detour path from e_1 to e^* in Durer Graph.

Theorem 4.1. *The detour edge pebbling number for the Durer graph with m edges is $f_e^*(DG) = 2^{m-6}$.*

Proof. Consider the edge set of DG be $E(DG) = \{e_1, e_2, \dots, e_{18}\}$. Take $2^{m-6} - 1$ pebbles for distribution. Consider the target edge to be any edge of Durer graph and let it be e_1 and assume there are zero number of pebbles on it. Place $2^{m-6} - 1$ pebbles on the edge e^* of Durer graph and make sure that the length of the edge detour path connecting the

edges e_1 and e^* is $n+1$. Now, the target edge cannot be reached using the edge detour path. Thus, $f_e^*(DG) \geq 2^{m-6}$.

Now we prove the sufficient part by distributing 2^{m-6} pebbles on the edges of DG .

Consider the target edge to be e_1 and assume there are zero number of pebbles on it. The edge detour distance from e_1 to any other edges of DG is n . Therefore, if we consider the pebble allotment in an unusual way of placing all the 2^{m-6} pebbles on the edge e^* , target edge can be reached. Also, if we alter the configurations of pebbles on the edges, by Theorem 3.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the edges of the graph DG .

Thus, $f_e^*(DG) = 2^{m-6}$.

Definition 4.2. [17] The Moser spindle graph is a unit-distance and planar graph with 7 vertices and 11 edges. It is named after Leo Moser and William Moser, brothers and Austrian-Canadian mathematicians. And its construction as the complement graph of the utility graph $K_{3,3}$ with one subdivided edge, was used by Horvat et al. to demonstrate the NP -hardness of determining whether a graph has a $2D$ unit-distance representation.

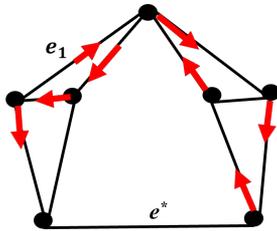


Figure 4.2. Edge detour path from e_1 to e^* in Moser Spindle Graph.

Theorem 4.2. The detour edge pebbling number for the Moser spindle graph with m edges is $f_e^*(MsG) = 2^{m-3} + 2$.

Proof. Consider the edge set of MsG be $E(MsG) = \{e_1, e_2, \dots, e_{11}\}$.

Take $2^{m-3} + 1$ pebbles for distribution. Consider the target edge to be any edge of Moser spindle graph and let it be e_1 and assume there are zero number of pebbles on it. Place $2^{m-3} - 1$ pebbles on the edge e^* of Moser spindle graph and make sure that the length of the edge detour path connecting the edges e_1 and e^* is $n+2$ and place the remaining two pebbles on the edges that are not in the edge detour path connecting the edges e_1 and e^* . Now, the target edge cannot be reached using the edge detour path. Thus, $f_e^*(MsG) \geq 2^{m-3} + 2$.

Let's now show the sufficiency by distributing $2^{m-3} + 2$ pebbles on the edges of MsG .

Consider the target edge to be e_1 and assume there are zero number of pebbles on it. The edge detour distance from e_1 to any other edges of MsG is at most $n+1$. Therefore, if we consider the pebble allotment in an unusual way of placing 2^{m-3} pebbles on the edge e^* and two pebbles on the edges that are not in the edge detour path connecting the edges e_1 and e^* , then the target edge can be reached. Also, if we alter the configurations of pebbles on the edges, by Theorem 3.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the edges of the graph MsG .

Thus, $f_e^*(MsG) = 2^{m-3} + 2$.

5 Detour edge pebbling number on book-type graphs

Theorem 5.1. The detour edge pebbling number for the stacked book graph is $f_e^*(B_{m,3}) = 2^{4m+1} + m$; $m \geq 4$.

Proof. In $B_{m,3}$, there are two spines s_1 and s_2 . Let the vertex incident to both s_1 and s_2 can be called as mid-spine vertex. Let the outer vertices incident to s_1 and s_2 can be called as end-spine vertices. In $B_{m,3}$, the edge detour path from s_1 to any one of the edges adjacent to s_2 has all the edges of $B_{m,3}$ but except m edges which are adjacent to both s_1 and s_2 . Thus, the edge detour distance from s_1 to any one of the edges adjacent to s_2 is $4m+1$.

Let e^* be any one of the edges adjacent to s_2 and let it be the destination edge. Take $2^{4m+1} + m - 1$ pebbles for distribution. Place $2^{4m+1} - 1$ pebbles on the edge s_1 and place 1 pebble each on

the edges which are adjacent to both s_1 and s_2 . Now, using the edge detour path, reaching the destination edge e^* is not achievable since $2^{4m+1}+m-1$ pebbles are sufficient only to reach the edges which are at a distance of at most $4m$ from s_1 .

So, $f_e^*(B_{m,3}) \geq 2^{4m+1}+m$.

Now we prove the sufficient part by using $2^{4m+1}+m$ pebbles on the edges of $B_{m,3}$.

Case 1. Let e_l be the destination edge where $l \in s_1$ and s_1 .

Without loss of generality, let s_2 be the destination edge. The edge detour distance from s_2 to e_m is at most $4m+1$ where $m \in$ edges incident with end-spine vertices and mid-spine vertex. Now, reaching the destination edge s_2 using the edge detour path is achievable by distributing at most $2^{4m+1}+m$ pebbles.

The edge detour distance from s_2 to e_n is at most $4m$ where $n \in$ off-spine edges. Now, reaching the destination edge s_2 using the edge detour path is achievable by distributing at most $2^{4m}+m+1$ pebbles.

The edge detour distance from s_2 to another spine of $B_{m,3}$ is at most $4m+1$. Now, reaching the destination edge s_2 using the edge detour path is achievable by distributing at most $2^{4m+1}+m$ pebbles. By symmetry, we can prove for s_1 .

Case 2. Let e_m be the destination edge where $m \in$ edges incident with end-spine vertices and mid-spine vertex.

The edge detour distance from e_m to e_n is $4m$ where $n \in$ off-spine edges. Now, reaching the destination edge e_m using the edge detour path is achievable by distributing at most $2^{4m}+m+1$ pebbles.

The edge detour distance from e_m to e_l is at most $4m+1$ where $l \in s_1$ and s_2 . Now, reaching the destination edge e_m using the edge detour path is achievable by distributing at most $2^{4m+1}+m$ pebbles.

The edge detour distance from e_m to any other edges incident with end-spine vertices and mid-spine vertex is at most $4m+1$. Now, reaching the destination edge e_m using the edge detour path is achievable by distributing at most $2^{4m+1}+m$ pebbles.

Case 3. Let e_n be the destination edge where $n \in$ off-spine edges.

The edge detour distance from e_n to e_m is at most $4m$ where $m \in$ edges incident with end-spine vertices and mid-spine vertex. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{4m}+m+1$ pebbles.

The edge detour distance from e_n to e_l is at most $4m$ where $l \in s_1$ and s_2 . Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{4m}+m+1$ pebbles.

The edge detour distance from e_n to any other off-spine edges is at most $4m+1$. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{4m+1}+m$ pebbles.

Thus, the detour edge pebbling number for the stacked book graph $B_{m,3}$ graph is $f_e^*(B_{m,3}) = 2^{4m+1}+m; m \geq 4$.

Example 5.1. we can check the result on $B_{4,3}$. Here, the edge detour path from one of the spine-adjacent edges to the spine s_2 is mentioned and the edges that are not in the edge detour path also highlighted.

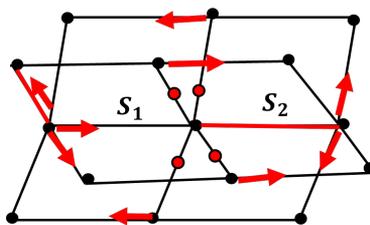


Figure 5.1. The stacked book graph $B_{4,3}$.

Theorem 5.2. The detour edge pebbling number for the stacked book graph is $f_e^*(B_{m,4}) = 2^{6m+2}+m; m \geq 4$.

Proof. In $B_{m,4}$, there are three spines s_1, s_2 and s_3 . The edge detour distance from s_1 to any one of the edges adjacent to s_3 is at most $6m+2$.

Let e^* be any one of the edges adjacent to s_3 and let it be the destination edge. Take $2^{6m+2}+m-1$ pebbles for distribution. Place $2^{6m+2}-1$ pebbles on the edge s_1 and place 1 pebble each on the off-spine edges on which both of its adjacent edges are adjacent to s_2 . Now, using the edge detour path, reaching the destination edge e^* is not achievable since $2^{6m+2}+m-1$ pebbles are sufficient only to reach the edges which are at a distance of at most $6m+1$ from s_1 .

So, $f_e^*(B_{m,4}) \geq 2^{6m+2}+m$.

Let's now show the sufficiency by using $2^{6m+2}+m$ pebbles on the edges of $B_{m,4}$.

Case 1. Let e_l be the destination edge where $l \in s_1, s_2$ and s_3 .

Without loss of generality, let s_1 be the destination edge. The edge detour distance from s_1 to e_m is at most $6m+2$ where $m \in$ edges adjacent to s_1, s_2 and s_3 . Now, reaching the destination edge s_1 using the edge detour path is achievable by distributing at most $2^{6m+2}+m$ pebbles.

The edge detour distance from s_1 to e_n is at most $6m+1$ where $n \in$ off-spine edges. Now, reaching the destination edge s_1 using the edge detour path is achievable by distributing at most $2^{6m+1}+m+1$ pebbles.

The edge detour distance from s_1 to any other spine of $B_{m,4}$ is at most $6m+2$. Now, reaching the destination edge s_1 using the edge detour path is achievable by distributing at most $2^{6m+2}+m$ pebbles. By symmetry, we can prove for s_2 and s_3 .

Case 2. Let e_m be the destination edge where $m \in$ edges adjacent to s_1, s_2 and s_3 .

The edge detour distance from e_m to e_n is at most $6m+1$ where $n \in$ off-spine edges. Now, reaching the destination edge e_m using the edge detour path is achievable by distributing at most $2^{6m+1}+m+1$ pebbles.

The edge detour distance from e_m to e_l is at most $6m+2$ where $l \in s_1, s_2$ and s_3 . Now, reaching the destination edge e_m using the edge detour path is achievable by distributing at most $2^{6m+2}+m$ pebbles.

The edge detour distance from e_m to any other edges adjacent to s_1, s_2 and s_3 is at most $6m+2$. Now, reaching the destination edge e_m using the edge detour path is achievable by distributing at most $2^{6m+2}+m$ pebbles.

Case 3. Let e_n be the destination edge where $n \in$ off-spine edges.

The edge detour distance from e_n to e_m is at most $6m+1$ where $m \in$ edges adjacent to s_1, s_2 and s_3 . Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{6m+1}+m+1$ pebbles.

The edge detour distance from e_n to e_l is at most $6m+1$ where $l \in s_1, s_2$ and s_3 . Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{6m+1}+m+1$ pebbles.

The edge detour distance from e_n to any other off-spine edges is at most $6m$. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{6m}+m+2$ pebbles.

Thus, the detour edge pebbling number for the stacked book graph $B_{m,4}$ graph is $f_e^*(B_{m,4}) = 2^{6m+2}+m; m \geq 4$.

Example 5.2. we can check the result on $B_{4,4}$. Here, the edge detour path from one of the spine-adjacent edges to the spine s_1 is mentioned and the edges that are not in the edge detour path also highlighted.

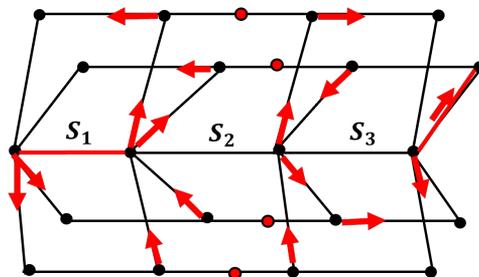


Figure 5.2. The stacked book graph $B_{4,4}$.

6 Comparative interpretation table

We have explored the limitations outlined below and the number of pebbles required is fewer using the edge pebbling number and detour pebbling number. The monophonic pebbling number is derived using the monophonic path. These parameters sparked our interest to find out the detour edge pebbling number using the edge detour path which supplies the maximum number of pebbles with extra moves. The various graph pebbling numbers are shown in the table 6.1 demonstrate that the detour edge pebbling number of a graph is unique.

Table 6.1: Several pebbling numbers for some standard graphs

Edge pebbling number [12]	Monophonic pebbling number [10]	Detour pebbling number [8]	Detour edge pebbling number
$P_E(P_n) = 2^{n-2}$	$\mu(P_n) = 2^{n-1}$	$f^*(P_n) = 2^{n-1}$	$f_e^*(P_n) = 2^{n-2}$
$P_E(K_n) = \frac{n(n-1)}{2}$	$\mu(K_n) = n$	$f^*(K_n) = 2^{n-1}$	$f_e^*(K_n) = 2^{\frac{n^2-n-2}{2}}$, when n is odd & $f_e^*(K_n) = 2^{\frac{n^2-2n}{2}}$, when n is even.
$P_E(S_n) = n$	$\mu(S_n) = n+2$	$f^*(S_n) = n+2$	$f_e^*(S_n) = n$
-	$\mu(W_n) = 2^{n-2}+2$	$f^*(W_n) = 2^n$	$f_e^*(W_n) = 2^{n+\lceil \frac{n}{3} \rceil}$, when n is odd & $f_e^*(W_n) = 2^{n+\frac{n}{2}}$, when n is even.

7 Concluding Remarks

In this paper, detour edge pebbling number for some standard graphs, notable graph structures and book-type graphs are determined. In section 2, pre-existing results are discussed. In section 3, we provided a detailed, step-by-step guide to illustrate the process of finding the detour edge pebbling number, using a concrete example. In section 3, we found the detour edge cover pebbling number for some standard graphs. In section 4, we analysed the Durer and Moser spindle graphs and found the detour edge pebbling for those graphs. In section 5, we discussed about the book-type graphs and found the detour edge pebbling for those graphs. In the previous section, pre-existing limitations are addressed, and the detour edge pebbling concept is compared to some existing theories using a comparative interpretation table, demonstrating its uniqueness. Therefore, the outcomes of this research are diverse, significant, and open avenues for future research.

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