Palestine Journal of Mathematics

Vol 15(1)(2026) , 894–906

SOME IDENTITIES ARISING HOMODEREVATIONS IN PRIME RINGS AND BANACH ALGEBRAS

A. Ouazza and A. Boua

Communicated by: Francesco Rania

MSC 2010 Classifications: 46J10, 47B47, 16W25, 46J45, 16N60.

Keywords and phrases: Banach algebra, continuity, closed subspace, derivation, open subspace.

The authors would like to thank the editor and reviewers for their valuable feedback, which improved the quality of the article.

Abstract Let $\mathcal{A}$ be a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ with the center $Z(\mathcal{A})$. In this article, we show that if a homoderivation of $\mathcal{A}$ satisfies some local differential identities, then $\mathcal{A}$ must be commutative. Several examples and applications are provided to illustrate the main results under various conditions, and we conclude by showing that certain assumptions are indeed necessary for these results to hold.

1 Introduction

In all that follows, $\mathcal{R}$ will be an associative ring with center $Z(\mathcal{R})$, and usually $\mathcal{R}$ is 2-torsion free if whenever $2x = 0$ with $x \in \mathcal{R}$ forces $x = 0$. The ring $\mathcal{R}$ is said to be prime if for any $x, y \in \mathcal{R}$, $x\mathcal{R}y = \{0\}$ implies either $x = 0$ or $y = 0$; it is semi-prime if $x\mathcal{R}x = \{0\}$ implies $x = 0$. The Lie and Jordan products of $x, y \in \mathcal{R}$ are denoted $[x, y]$ and $x \circ y$ respectively, where $[x, y] = xy - yx$ and $x \circ y = xy + yx$. An additive mapping $h : \mathcal{R} \to \mathcal{R}$ is called a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ holds for all $x, y \in \mathcal{R}$. An example of such a mapping is $h(x) = u(x)-x$ for all $x \in \mathcal{R}$, where u is an endomorphism on $\mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$, and is called an inner derivation if there exists an element $a \in \mathcal{R}$ such that $d(x) = [a, x]$ for all $x \in \mathcal{R}$. Obviously, a homoderivation is also a derivation if $h(x)h(y) = 0$ for all $x, y \in \mathcal{R}$. Let $\mathcal{A}$ be a Banach algebra over the complex field. A linear map $f : \mathcal{A} \to \mathcal{A}$ is called a linear skew derivation if $f(xy) = f(x)y + g(x)f(x)$ for all $x, y \in \mathcal{A}$, where g is an automorphism of $\mathcal{A}$.

Determining structural conditions under which a ring must be commutative constitutes a fundamental direction of research in ring theory. Classical contributions, initiated by Herstein, demonstrate that assumptions that may initially appear rather weak already suffice to enforce commutativity. We now present several theorems due to Herstein [7, p 412], which state that a ring $\mathcal{R}$ is commutative if for each x and y in $\mathcal{R}$ there exists a positive integer $n > 1$ such that $x^n - x$ permutes with y.

The relationship between ring commutativity and the existence of specific types of derivations has attracted sustained attention in recent years. The central objective is to reduce whether elements of a ring commute to the verification of certain differential identities, thereby rigorously establishing the commutative structure of the ring. An important result was established by Posner in 1957, which states that if a prime ring has a non-zero derivation centralized on the entire ring, then the ring must be commutative. Motivated by this result, a similar result for automorphisms was obtained by J. Mayne [13, Theorem 1]; this work has also been extended in several directions. Later, in [2] Bell and Daif initiated the study of a class of maps that preserve commutativity in the following sense: "Let $\mathcal{S}$ be a subset of a ring $\mathcal{R}$. A map $f : \mathcal{S} \to \mathcal{R}$ is said to be strongly commutativity preserving (SCP) on $\mathcal{S}$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{S}$." More precisely, they proved that $\mathcal{R}$ must be commutative if $\mathcal{R}$ is a prime ring and $\mathcal{R}$ admits a derivation or non-identity endomorphism that is SCP on the right ideal of $\mathcal{R}$.

In Banach algebras, Yood [19] proved that if a semi-prime Banach algebra $\mathcal{A}$ with two non-void open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ holds for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ there are strictly positive integers n, m such that $[x^n, y^m] \in Z(\mathcal{A})$, then $\mathcal{A}$ is commutative. Inspired by the work of Ali and Khan [1] and Yood [17, 18, 19], Rehman [16] explored strong commutativity-preserving skew derivations on Banach algebras. He proved in [16, Theorem 3] that if $\mathcal{A}$ is a prime Banach algebra and f is a continuous linear skew-derivation, then under the assumption that there exist non-empty open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{A}$ such that $[f(x^m), f(y^n)] - [x^m, y^n] \in Z(\mathcal{A})$ for each $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, then the algebra $\mathcal{A}$ must be commutative. In [12, Theorem 1], M. Moumen, L. Taoufiq, and L. Oukhtite showed that if $\mathcal{A}$ is a prime Banach algebra and $\mathcal{H}_1$ and $\mathcal{H}_2$ are non-empty open subsets of $\mathcal{A}$, then if $\mathcal{A}$ admits a continuous non-injective derivation d such that for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ there are positive integers p, q such that $d(x^p y^q) + [x^p, y^q] \in Z(\mathcal{A})$, it follows that $\mathcal{A}$ must be commutative. Afterward, M. Moumen, L. Taoufiq, and A. Boua [11] proved that if $\mathcal{A}$ is a prime Banach algebra and $\mathcal{H}_1$ and $\mathcal{H}_2$ are non-empty open subsets of $\mathcal{A}$, and if d is a continuous non-injective derivation of $\mathcal{A}$ satisfying one of the following conditions:

(i) for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, there exists $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $d(x^p y^q) \pm x^p \circ y^q \in Z(\mathcal{A})$.

(ii) for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, there exists $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $d(x^p \circ y^q) \pm [x^p, y^q] \in Z(\mathcal{A})$,

then $\mathcal{A}$ is commutative. There has been a lot of research in recent years on the connection between a prime Banach algebra $\mathcal{A}$ commutativity and the existence of specific derivations of $\mathcal{A}$ or projections from $\mathcal{A}$ to its center $Z(\mathcal{A})$ (see [4], [5]).

Motivated by these results, we explore an alternative approach for addressing the commutativity of Banach algebras using homoderivation. In this work, we aim to prove results that yield analogous conclusions, but under the framework of distinct identities with homoderivations. In particular, we have proved that if a prime Banach algebra $\mathcal{A}$ has continuous linear homoderivation and there are two non-void open subsets $\mathcal{K}_1$, $\mathcal{K}_2$ of $\mathcal{A}$ which satisfy for all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there are strictly positive integers p, q such that $h(x^p y^q) + x^p \circ y^q \in Z(\mathcal{A})$, then $\mathcal{A}$ must be commutative. Additional results of a similar nature have been obtained in this context.

We begin this paper by presenting several results that rely on well-known theorems, without explicitly citing them. Then we consider the case of a ring with one type of identity and extend the result to the case of a Banach algebra. Subsequently, we explore additional identities that lead to commutativity in a Banach algebra under various conditions. In the last section of this paper, we provide examples showing that our conditions are not superfluous.

2 Main results

All that follows, $\mathcal{A}$ always denotes a real or complex Banach algebra with center $Z(\mathcal{A})$. The following lemma, due to Bonsall and Duncan [3], is crucial for developing the proof of the main results.

Lemma 2.1. *Let $\mathcal{R}$ be a prime ring, then the following assertions hold:*

(i) *If $x \in Z(\mathcal{R})$ and $xy \in Z(\mathcal{R})$, then $x = 0$ or $y \in Z(\mathcal{R})$.*

(ii) *The nonzero element from $Z(\mathcal{R})$ are not zero divisors.*

(iii) *If d is a derivation of $\mathcal{R}$ such that $[d(x), x] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ (in particular if $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$), then $\mathcal{R}$ is commutative or d is zero.*

Lemma 2.2. *Let $\mathcal{A}$ be a real or complex Banach algebra and $P(t) = \sum_{i=0}^{n} a_i t^i$ a polynomial in the real variable t with coefficients in $\mathcal{A}$. If for an infinite set of real values of t, $P(t) \in \mathcal{M}$, where $\mathcal{M}$ is a closed linear subset of $\mathcal{A}$, then every a_i is in $\mathcal{M}$.*

We begin our contribution by showing the main result in a prime ring. More precisely speaking, we will prove that

Theorem 2.3. *Let $\mathcal{R}$ be a prime ring with characteristic different from two, and let h be a homoderivation of $\mathcal{R}$ such that for all $(x, y) \in \mathcal{R}^2$, there are fixed positive integers m, n such that*

$$[h(x^m), h(y^n)] - [x^m, y^n] \in Z(\mathcal{R}),$$

then $\mathcal{R}$ is commutative.

Proof. If $h = 0$, then our hypothesis reduces to $[x^n, y^m] \in Z(\mathcal{R})$ for all $(x, y) \in \mathcal{R}^2$. For $m = n = 1$, $\mathcal{R}$ is obviously commutative. If $m > 1$ and $n = 1$ or $m = 1$ and $n > 1$, then by [15, Lemma 1], $\mathcal{R}$ has no nonzero nilpotent element, and hence [9, Theorem 1] assures that $\mathcal{R}$ is commutative. Hence, we may suppose that $h \neq 0$.
Let $\mathcal{A}_1 = \langle x^m \rangle$ and $\mathcal{A}_2 = \langle x^n \rangle$. It is easy to show that

$$[h(x^m), h(y^n)] - [x^m, y^n] \in Z(\mathcal{R}) \text{ for all } x \in \mathcal{A}_1,\ y \in \mathcal{A}_2. \tag{2.1}$$

In light of the main theorem in [6], either $\mathcal{A}_1$ has a non-central Lie ideal $\mathcal{L}_1$ or $x^m \in \mathcal{R}$ for all $x \in \mathcal{R}$. In the latter case, $\mathcal{R}$ is commutative. Similarly, suppose there exists a non-central Lie ideal $\mathcal{L}_2$ of $\mathcal{R}$ contained in $\mathcal{A}_2$. Moreover, by [[8] page 4-5], there exist I_1 and I_2 ideals of $\mathcal{R}$ such that $0 \neq [I_1, \mathcal{R}] \subseteq \mathcal{L}_1$ and $0 \neq [I_2, \mathcal{R}] \subseteq \mathcal{L}_2$. So we have

$$[h(x^m), h(y^n)] - [x^m, y^n] \in Z(\mathcal{R}) \text{ for each } x \in [I_1, I_2], \text{ and } y \in [I_1, I_2]. \tag{2.2}$$

Since I_1 and I_2 satisfy the same differential identities [10, Theorem 3], we have

$$[h(x^m), h(y^n)] - [x^m, y^n] \in Z(\mathcal{R}) \text{ for all } x, y \in [\mathcal{R}, \mathcal{R}].$$

Therefore, using Lemma 2.2, we find that $[\mathcal{R}, \mathcal{R}] \subseteq Z(\mathcal{R})$. It follows immediately that $\mathcal{R}$ is commutative. □

Now that the previous results have been established, we move on to study their implications in a more focused and structured context, specifically in the context of a prime Banach algebra, where the algebraic and topological properties interact in a way that enables a more thorough examination of the assumed identities and homoderivations.

Theorem 2.4. *Let $\mathcal{A}$ be a prime Banach algebra, and let $\mathcal{K}_1$, $\mathcal{K}_2$ be non-void open subsets of $\mathcal{A}$. If h is a continuous linear homoderivation of $\mathcal{A}$ such that*

$$\forall (x, y) \in \mathcal{K}_1 \times \mathcal{K}_2, \exists (p, q) \in \mathbb{N}^* \times \mathbb{N}^*,\ [h(x^p), h(y^q)] - [x^p, y^q] \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Proof. If $h = 0$, then our hypothesis reduces to $[x^p, y^q] \in Z(\mathcal{A})$ for all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$, so the required result follows from [19, Theorem 2]. Hence, we may suppose that $h \neq 0$.
For all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, let us define the following sets:

$$O_{p,q} = \{(x, y) \in \mathcal{A} \mid\ [h(x^p), h(y^q)] - [x^p, y^q] \notin Z(\mathcal{A})\} \text{ and}$$

$$F_{p,q} = \{(x, y) \in \mathcal{A} \mid\ [h(x^p), h(y^q)] - [x^p, y^q] \in Z(\mathcal{A})\}.$$

Obviously $(\cap O_{p,q}) \cap (\mathcal{K}_1 \times \mathcal{K}_2) = \emptyset$, otherwise there exists $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ such that $[h(x^p), h(y^q)] - [x^p, y^q] \notin Z(\mathcal{A})$; $\forall (p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, a contradiction.
Now we claim that each $O_{p,q}$ is open in $\mathcal{A} \times \mathcal{A}$ or, equivalently, its complement $F_{p,q}$ is closed. For this, we consider a sequence $((x_k, y_k))_{k \in \mathbb{N}} \subset F_{p,q}$, therefore,

$$[h(x_k^p), h(y_k^q)) + [x_k^p, y_k^q] \in Z(\mathcal{A})\ \ \forall k \in \mathbb{N}.$$

Exploiting the continuity of h, we observe that the sequence $[h(x_k^p), h(y_k^q)] + [x_k^p, y_k^q])_{k \in \mathbb{N}}$ converges to $[h(x^p), h(y^q)] + [x^p, y^q]$, and the fact that $Z(\mathcal{A})$ is closed implies $[h(x^p), h(y^q)] + [x^p, y^q] \in Z(\mathcal{A})$. So $(x, y) \in F_{p,q}$, so $F_{p,q}$ is closed, $O_{p,q}$ is open.
By Baire's category theorem, if $O_{p,q}$ is dense, then its intersection is also dense, which contradicts the existence of $\mathcal{K}_1$ and $\mathcal{K}_2$. Accordingly, there exist positive integers $m,\ n$ such that $O_{p,q}$ is not dense in $\mathcal{A}$, which forces the existence of a non-void open subset $O \times O'$ in $F_{p,q}$ such that

$$[h(x^n), h(y^m)] + [x^n + y^m] \in Z(\mathcal{A})\ \ \forall x \in O;\ \forall y \in O'.$$

Fix $y \in O'$, let $x \in O$ and $z \in \mathcal{A}$, then $x + tz \in O$ for all sufficiently small real t. Therefore,

$$P(t) = [h((x + tz)^n), h(y^m)] + [(x + tz)^n, y^m] \in Z(\mathcal{A}).$$

Since $P(t)$ can equivalently be written as

$$P(t) = A_{n,0}(x,z,y) + A_{n-1,1}(x,z,y)t + A_{n-2,2}(x,z,y)t^2 + \cdots + A_{0,n}(x,z,y)t^n,$$

where the last term in this polynomial is $A_{0,n}(x,z,y) = [h(z^n), h(y^m)] + [z^n, y^m]$, then Lemma 2.2 yields

$$A_{0,n}(x,z,y) = [h(z^n), h(y^m)] + [z^n, y^m] \in Z(\mathcal{A}).$$

Following from the previous results, for $y \in O'$ we get

$$[h(x^n), h(y^m)] + [x^m, y^n] \in Z(\mathcal{A}) \text{ for all } x \in \mathcal{A}.$$

By proceeding in a manner analogous to the previous argument, we conclude

$$[h(x^n), h(y^m)] + [x^m, y^n] \in Z(\mathcal{A}) \text{ for all } (x,y) \in \mathcal{A}^2.$$

According to Theorem 2.3, $\mathcal{A}$ is commutative. □

From Theorem 2.4, we readily deduce the following corollary:

Corollary 2.5. *Let $\mathcal{A}$ be a prime Banach algebra and $\mathcal{K}_1$, $\mathcal{K}_2$ non-void open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous linear homoderivation h such that*

$$\forall (x,y) \in \mathcal{K}_1 \times \mathcal{K}_2, \exists (p,q) \in \mathbb{N}^* \times \mathbb{N}^*,\ [h(x^p), h(y^q)] + [x^p, y^q] \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Example 2.6. Let $\mathbb{C}$ be the field of complex numbers, let

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C} \right\} = \mathcal{M}_2(\mathbb{C})$$

be a non-commutative unital prime algebra of all 2×2 matrices over $\mathbb{C}$ with the usual matrix addition, and define matrix multiplication as follows: $A \times_l B = lA \times B$ for all $A, B \in \mathcal{A}$ where $l = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ and $|\alpha| > 1$. For $A = (a_{ij})_{1 \le i,j \le 2} \in \mathcal{A}$, define $||A|| = \sum_{1 \le i,j \le 2}^{2} |a_{ij}|$. Then $\mathcal{A}$ is a normed linear space. Now, define the map

$$h\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -2b \\ -2c & 0 \end{pmatrix},$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}$. Since $\mathcal{A}$ is finite-dimensional, it is easy to verify that h is a non-zero homoderivation. Observe that

$$\mathcal{K}_1 = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\} \text{ and } \mathcal{K}_2 = \left\{ \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

are open subsets of $\mathcal{A}$. For all $(X,Y) \in \mathcal{K}_1 \times \mathcal{K}_2$ and for all $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$, we have $[h(X^p), h(Y^p)] \in Z(\mathcal{A})$ and $[X^p, Y^p] \in Z(\mathcal{A})$, then $[h(X^p), h(Y^p)] + [X^p, Y^p] \in Z(\mathcal{A})$. By Theorem 2.4, we conclude that $\mathcal{A}$ is not a Banach algebra under the defined norm.

The result of Theorem 2.3 remains valid if the homoderivation h is substituted by a skew derivation φ associated with an automorphism of $\mathcal{A}$. Based on this observation, we derive the following statement:

Corollary 2.7. [16, Theorem 3] *Let $\mathcal{A}$ be a prime Banach algebra, and let $\mathcal{K}_1$, $\mathcal{K}_2$ be non-void open subsets of $\mathcal{A}$. If φ is a continuous linear skew-derivation of $\mathcal{A}$ such that*

$$\forall (x,y) \in \mathcal{K}_1 \times \mathcal{K}_2, \exists (p,q) \in \mathbb{N}^* \times \mathbb{N}^*,\ [\varphi(x^p), \varphi(y^q)] - [x^p, y^q] \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Hereafter, we provide a proof of our result through two identities involving both products: the first based on the Jordan product and the conventional product, and the second obtained by interchanging their roles.

Theorem 2.8. *Let $\mathcal{A}$ be a prime Banach algebra and let $\mathcal{K}_1$, $\mathcal{K}_2$ be non-empty open subsets of $\mathcal{A}$. If h is a continuous linear homoderivation of $\mathcal{A}$ such that for all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there exist strictly positive integers p, q such that*

$$h(x^p \circ y^q) + x^p y^q \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Proof. If $f = 0$, then our hypothesis reduces to $x^p y^q \in Z(\mathcal{A})$ for all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$, so the required result follows from [19]. Hence, we may suppose that $f \neq 0$.
For all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, let us define the following sets:

$$O_{p,q} = \{(x, y) \in \mathcal{A} \mid\ h(x^p \circ y^q) + x^p y^q \notin Z(\mathcal{A})\} \text{ and}$$
$$F_{p,q} = \{(x, y) \in \mathcal{A} \mid\ h(x^p \circ y^q) + x^p y^q \in Z(\mathcal{A})\}.$$

Using reasoning analogous to that employed in the proof of Theorem 2.4, we conclude that there exist positive integers m and n such that

$$h(x^n \circ y^m) + x^n y^m \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}. \tag{2.3}$$

Replacing x by x^m and y by y^n in (2.3), we find that

$$h(x^{nm} \circ y^{mn}) + x^{nm} y^{mn} \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}. \tag{2.4}$$

Substituting x for y and y for x in (2.4), we arrive at

$$h(y^{nm} \circ x^{mn}) + y^{nm} x^{mn} \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}, \tag{2.5}$$

and subtracting (2.5) from (2.4), which leads to

$$[x^{nm}, y^{mn}] \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}. \tag{2.6}$$

Fix $x \in \mathcal{A}$ and put $a = x^{nm}$. For $y \in \mathcal{A}$ and $t \in \mathbb{R}$, (2.6) yields that

$$P(t) = [a, (a + ty)^{nm}] \in Z(\mathcal{A}),$$

it follows from the fact that $P(t)$ can be expressed as $P(t) = \sum_{k=0}^{nm} [a, u_{nm-k,k}(a, y)] t^k$, where $u_{nm-k,k}(a, y)$ denotes the sum of all terms in which a appears exactly $nm - k$ times and y appears exactly k times, then Lemma 2.2 assures that

$$[a, u_{nm-k,k}(a, y)] \in Z(\mathcal{A}) \quad \text{for all } k \leq nm.$$

The coefficient of t in this polynomial is $[a, u_{nm-1,1}(a, y)]$, where

$$u_{nm-1,1}(a, y) = \sum_{k=0}^{nm-1} a^{nm-1-k} y a^k,$$

therefore, $\sum_{k=0}^{nm-1} [a, a^{nm-1-k} y a^k] \in Z(\mathcal{A})$. Since

$$\begin{aligned}
&\sum_{k=0}^{nm-1} [a, a^{nm-1-k} y a^k] \\
&= [a, a^{nm-1} y] + [a, a^{nm-2} y a^a] + [a, a^{nm-3} y a^2] + \cdots + [a, y a^{nm-1}] \\
&= (a a^{nm-1} y - a^{nm-1} y a) + (a a^{nm-2} y a - a^{nm-2} y a^2) + \cdots + (a y a^{nm-1} - y a^{nm}) \\
&= (a^{nm} y - a^{nm-1} y a) + (a^{nm-1} y a - a^{nm-2} y a^2) + \cdots + (a y a^{nm-1} - y a^{nm}) \\
&= a^{nm} y - y a^{nm} \\
&= [a^{nm}, y],
\end{aligned}$$

then we conclude that $[a^{nm}, y] \in Z(\mathcal{A})$ in such a way that

$$[x^{n^2m^2}, y] = 0 \quad \text{for all } x, y \in \mathcal{A}.$$

Let $y = x^{n^2m^2}y$, then

$$[x^{n^2m^2}, y] = [x^{n^2m^2}, x^{n^2m^2}y] = x^{n^2m^2}[x^{n^2m^2}, y],$$

in view of the preceding argument $x^{n^2m^2} \in Z(\mathcal{A})$ or $[x^{n^2m^2}, y] = 0$ for all $x \in \mathcal{A}$ and $y \in \mathcal{A}$ implies that $[x^{n^2m^2}, z] = 0$ or $[x^{n^2m^2}, y] = 0$ for all $x \in \mathcal{A}$, $y \in \mathcal{A}$, and $z \in \mathcal{A}$. For $z = y$, we conclude that $[x^{n^2m^2}, y] = 0$ for all $x \in \mathcal{A}$ and $y \in \mathcal{A}$, which forces $x^{n^2m^2}$ to be a non-zero element of $Z(\mathcal{A})$. The primeness of $\mathcal{A}$ assures that $y \in Z(\mathcal{A})$ and hence $\mathcal{A}$ is commutative. □

Example 2.9. Let $\mathbb{Z}$ be the set of integers. Let $\mathcal{M}_3(\mathbb{Z})$ be endowed with the usual operations on matrices and the norm $||.||_1$ defined by $||M||_1 = \sum_{1\leq i,j\leq 3} |m_{ij}|$ for all $M = (m_{ij})_{1\leq i,j\leq 3} \in M_3(\mathbb{Z})$. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} |x, y, z \in \mathbb{Z} \right\} \subset \mathcal{M}_3(\mathbb{Z}),$$

it is easy to verify that $\mathcal{A}$ is a non commutative real Banach algebra. Let h defined by

$$h \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a continuous homoderivation of $\mathcal{A}$ satisfying

$$h(M^p \circ N^p) + M^pN^p = 0 \;\text{ for all } M, N \in \mathcal{A} \text{ and } p > 2.$$

According to Theorem 2.8, we conclude that $\mathcal{A}$ is not prime.

Theorem 2.10. *Let $\mathcal{A}$ be a prime Banach algebra and $\mathcal{K}_1$, $\mathcal{K}_2$ non-void open subsets of $\mathcal{A}$. If h is a continuous linear non-zero homoderivation of $\mathcal{A}$ satisfying for all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there are strictly positive integers p, q such that*

$$h(x^py^q) + x^p \circ y^p \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Proof. For all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, let us define the following sets:

$$O_{p,q} = \{(x, y) \in \mathcal{A}^2 \mid \; h(x^py^q) + x^p \circ y^q \notin Z(\mathcal{A})\} \text{ and}$$

$$F_{p,q} = \{(x, y) \in \mathcal{A}^2 \mid \; h(x^py^q) + x^p \circ y^q \in Z(\mathcal{A})\}.$$

Obviously $(\cap O_{p,q}) \cap (\mathcal{K}_1 \times \mathcal{K}_2) = \emptyset$, otherwise there exists $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ such that $h(x^py^q) + x^p \circ y^q \notin Z(\mathcal{A})$; $\forall (p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, which is absurd with the hypothesis of the theorem.
Now we claim that every $O_{p,q}$ is open in $\mathcal{A} \times \mathcal{A}$. This means that we have to show that $F_{p,q}$, the complement of $O_{p,q}$, is closed. For this, we consider a sequence $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{p,q}$, so,

$$h((x_k)^p(y_k)^q) + (x_k)^p \circ (y_k)^q \in Z(\mathcal{A}) \text{ for all } k \in \mathbb{N}.$$

Using the continuity of h, the sequence $(h((x_k)^p(y_k)^q) + (x_k)^p \circ (y_k)^q)_{k\in\mathbb{N}}$ converges to $h(x^py^q) + x^p \circ y^q$, and the fact that $Z(\mathcal{A})$ is closed implies $h(x^py^q) + x^p \circ y^q \in Z(\mathcal{A})$. Therefore, $(x, y) \in F_{p,q}$ so that $F_{p,q}$ is closed implies that $O_{p,q}$ is open.
If every $O_{p,q}$ is dense, we know that their intersection is also dense by the Baire category theorem, which contradicts $(\bigcap O_{p,q}) \bigcap (\mathcal{K}_1 \times \mathcal{K}_2) = \emptyset$. Hence, there are positive integers m, n such

that $O_{p,q}$ is not dense in $\mathcal{A}$, which forces the existence of a non-void open subset $O \times O'$ in $F_{p,q}$ such that

$$h(x^n y^m) + x^n \circ y^m \in Z(\mathcal{A}) \ \ \forall (x, y) \in O \times O'.$$

Fix $y \in O'$, let $x \in O$ and $z \in \mathcal{A}$, then $x + tz \in O$ for all sufficiently small real t. Therefore,

$$P(t) = h((x + tz)^n y^m) + (x + tz)^n \circ y^m \in Z(\mathcal{A}).$$

We can write

$$P(t) = A_{n,0}(x, z, y) + A_{n-1,1}(x, z, y)t + A_{n-2,2}(x, z, y)t^2 + \cdots + A_{0,n}(x, z, y)t^n,$$

where the first term in this polynomial is $A_{0,n}(x, z, y) = h(z^n y^m)] + z^n \circ y^m$, who belongs to $Z(\mathcal{A})$, by Lemma 2.2 we conclude that

$$A_{0,n}(x, z, y) = h(z^n y^m) + z^n \circ y^m \in Z(\mathcal{A}).$$

Consequently, given $y \in O'$, we get

$$h(x^n y^m) + x^m \circ y^n \in Z(\mathcal{A}) \ \text{ for all } x \in \mathcal{A}.$$

Reversing the roles of O and O' in the above setting, with x fixed in $\mathcal{A}$, we obtain

$$h(x^n y^m) + x^m \circ y^n \in Z(\mathcal{A}) \ \text{ for all } (x, y) \in \mathcal{A}^2.$$

Replacing x by x^m and y by y^n, it follows that

$$h(x^{nm} y^{mn}) + x^{nm} \circ y^{mn} \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}. \tag{2.7}$$

Taking $x = y$ in (2.7), we immediately obtain

$$h(y^{nm} x^{mn}) + y^{nm} \circ x^{mn} \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}, \tag{2.8}$$

and subtracting (2.8) from (2.7). Since $y^{nm} \circ x^{mn} = x^{nm} \circ y^{mn}$ for all $(x, y) \in \mathcal{A}^2$, we get

$$h([x^{nm}, y^{mn}]) \in Z(\mathcal{A}) \ \text{ for all } \ x, y \in \mathcal{A}. \tag{2.9}$$

Fixing $x \in \mathcal{A}$ and setting $a = x^{nm}$. For $y \in \mathcal{A}$ and $t \in \mathbb{R}$, it follows from (2.9) that

$$P(t) = h([a, (a + ty)^{nm}]) = \sum_{mn}^{k=0} h([a, A_{nm-k,k}])t^k,$$

where $A_{nm-k,k}$ denotes the sum of all terms in which y appears exactly $nm - k$ times and a appears exactly k times. The first term in this polynomial is $h([a, A_{nm,0}]) \in Z(\mathcal{A})$, then Lemma 2.2 assures that

$$h([a, A_{nm_k,k}(x, y)]) \in Z(\mathcal{A}) \quad \text{for all } 0 \leq k \leq nm.$$

The coefficient of t in this polynomial is $h([a, A_{nm-1,1}])$, and

$$A_{nm-1,1}(a, y) = \sum_{k=0}^{nm-1} a^{nm-1-k} y a^k.$$

Since

$$\begin{aligned}
&\sum_{k=0}^{nm-1} [a, a^{nm-1-k} y a^k] \\
&= [a, a^{nm-1} y] + [a, a^{nm-2} y a^a] + [a, a^{nm-3} y a^2] + \cdots + [a, y a^{nm-1}] \\
&= (a a^{nm-1} y - a^{nm-1} y a) + (a a^{nm-2} y a - a^{nm-2} y a^2) + \cdots + (a y a^{nm-1} - y a^{nm-1} a) \\
&= (a^{nm} y - a^{nm-1} y a) + (a^{nm-1} y a - a^{nm-2} y a^2) + \cdots + (a y a^{nm-1} - y a^{nm}) \\
&= a^{nm} y - y a^{nm} \\
&= [a^{nm}, y],
\end{aligned}$$

then we deduce that

$$h([a, A_{nm-1,1}(x,y)]) = \sum_{k=0}^{nm-1} h([a, x^{nm-1}yx^k]) = h([a^{nm}, y]) \in Z(\mathcal{A}).$$

Thus, we obtain the following statement

$$h([x^{n^2m^2}, y]) \in Z(\mathcal{A}) \text{ for all } (x,y) \in \mathcal{A}^2.$$

If $x^{n^2m^2} \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$.
Fix $y \in \mathcal{A}$ and let a be a non-zero element of $Z(\mathcal{A})$. From

$$(y+ta)^{n^2m^2} = \sum_{k=0}^{n^2m^2} \binom{k}{n^2m^2} a^{n^2m^2-k} y^k t^k \in Z(\mathcal{A}),$$

once again, by Lemma 2.2, we obtain $\binom{k}{n^2m^2} a^{n^2m^2-k} y^k t^k \in Z(\mathcal{A})$ for all $0 \le k \le n^2m^2$. In particular, $n^2m^2a^{n^2m^2-1}y \in Z(\mathcal{A})$, which leads to $a^{n^2m^2-1}y \in Z(\mathcal{A})$. As $a^{n^2m^2-1}$ is a non-zero element in $Z(\mathcal{A})$, the primeness of $\mathcal{A}$ assures that $y \in Z(\mathcal{A})$; hence $\mathcal{A}$ is commutative.
If there exists an element $x \in \mathcal{A}$ such that $x^k \notin Z(\mathcal{A})$, let a be a non-zero element in $Z(\mathcal{A})$ such that $h(a^k) \neq 0$ for all $0 \le k \le n^2m^2$, then

$$\begin{aligned}
[(x+ta)^{n^2m^2}, y] &= \left[\sum_{k=0}^{n^2m^2} \binom{k}{n^2m^2} x^{n^2m^2-k} a^k t^k, y\right] \\
&= \sum_{k=0}^{n^2m^2} \binom{k}{n^2m^2} x^{n^2m^2-k} a^k t^k y - \sum_{k=0}^{n^2m^2} \binom{n^2m^2}{k} y x^{n^2m^2-k} a^k t^k \\
&= \sum_{k=0}^{n^2m^2} \binom{k}{n^2m^2} a^k [x^{n^2m^2-k}, y] t^k
\end{aligned}$$

and

$$h([(x+ta)^{n^2m^2}, y]) = \sum_{k=0}^{n^2m^2} \binom{k}{n^2m^2} h(a^k [x^{n^2m^2-k}, y]) t^k,$$

denote $A_k(x,a) = \binom{k}{n^2m^2} h(a^k[x^{n^2m^2-k}, y])$. As before, by Lemma 2.2, we conclude that $A_k(x,a) \in Z(\mathcal{A})$ for all $0 \le k \le n^2m^2$. In the special case where $k = n^2m^2 - 1$, we have

$$A_{n^2m^2-1}(x,a) = n^2m^2 h(a^{n^2m^2-1}[x,y]) \text{ for all } y \in \mathcal{A},$$

a simplification can be achieved by n^2m^2 (since $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$), which leads to $h(a^{n^2m^2-1}[x,y]) \in Z(\mathcal{A})$, hence

$$h(a^{n^2m^2-1}[x,y]) = h(a^{n^2m^2-1})h([x,y]) + h(a^{n^2m^2-1})[x,y] + a^{n^2m^2-1}h([x,y]) \text{ for all } y \in \mathcal{A},$$

or $h(a^{n^2m^2-1})h([x,y])$ and $a^{n^2m^2-1}h([x,y])$ are elements of $Z(\mathcal{A})$, then $h(a^{n^2m^2-1})[x,y] \in Z(\mathcal{A})$. As $h(a^{n^2m^2-1})$ is a non-zero element of $Z(\mathcal{A})$, by Remark 2.1, we conclude that $[x,y] = 0$ for all $y \in \mathcal{A}$.
Proceeding similarly, we arrive at $[x,y] = 0$ for all $y \in \mathcal{A}$ and $x \in \mathcal{A}$. By the primeness of $\mathcal{A}$ it follows that $y \in Z(\mathcal{A})$; therefore, $\mathcal{A}$ is commutative. □

By employing both the Jordan and Lie products simultaneously in an identity, we conclude this article with the final result:

Theorem 2.11. *Let $\mathcal{A}$ be a prime Banach algebra and $\mathcal{K}_1$, $\mathcal{K}_2$ non-void open subsets of $\mathcal{A}$. Let h be a continuous linear non-injective non-zero homoderivation of $\mathcal{A}$ satisfying the following condition.*
For all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$, there exists $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$h(x^p \circ y^q) + [x^p, y^p] \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Proof. Proceeding analogously to the proof Theorem 2.10, we demonstrate that there are strictly positive integers n, m such that

$$h(x^n \circ y^m) + [x^n, y^m] \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}.$$

If $y = x$, then $2h(x^{n+m}) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$. As $\mathcal{A}$ is $2-$torsion free holds, it follows that $h(x^{n+m}) \in Z(\mathcal{A})$.
Now, we assert that the restriction of h is non-injective.
In order to achieve this, we consider $u \in \mathcal{A}$ such that $h(u) = 0$ and v is a non-zero element of $Z(\mathcal{A})$; we have $h((u + tv)^{n+m}) \in Z(\mathcal{A})$ for all $t \in \mathbb{R}$. In view of $v^k \in Z(\mathcal{A})$ for all $k \in \mathbb{N}^*$, we may write

$$h((u + tv)^{n+m}) = \sum_{k=0}^{n+m} \binom{k}{n+m} h(u^{n+m-k} v^k) t^k = \sum_{k=0}^{n+m} C_k t^k,$$

where $C_k = \binom{k}{n+m} h(u^{n+m-k} v^k)$, $C_0 = h(u^{n+m}) \in Z(\mathcal{A})$ and $C_{n+m} = h(v^{n+m}) \in Z(\mathcal{A})$, form Lemma 2.2 we deduce that $C_k \in Z(\mathcal{A})$ for all $0 \leq k \leq n + m$. In particular, for $k = n + m - 1$ we have $(n + m)h(uv^{n+m-1}) \in Z(\mathcal{A})$, simplifying by $n + m$ we obtain $h(uv^{n+m-1}) \in Z(\mathcal{A})$, then

$$u\, h(v^{n+m-1}) \in Z(\mathcal{A}).$$

Assume that $h(v^{n+m-1}) = 0$, then it follows that $v^{n+m-1} = 0$ (because $v^{n+m-1} \in Z(\mathcal{A})$) and h is injective in $Z(\mathcal{A})$, then $v = 0$. This is a contradiction. Thus $h(v^{n+m-1}) \neq 0$ and in view of Remark 2.1 we deduce that $u \in Z(\mathcal{A})$. Since the restriction of h on $Z(\mathcal{A})$ is injective, it follows that $u = 0$ and that h is injective. Finally, we conclude that if the restriction of h on $Z(\mathcal{A})$ is non-injective, then there is a non-zero element u of $Z(\mathcal{A})$ such that $h(u) = 0$, we have $h((x + tu)^{n+m}) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$ and $t \in \mathcal{R}$. Given that $u^k \in Z(\mathcal{A})$ for all $k \in \mathbb{N}^*$, we conclude that

$$h((x + tu)^{n+m}) = \sum_{k=0}^{n+m} \binom{k}{n+m} h(x^{n+m-1} u^k) t^k.$$

The last term in this polynomial is $h(u^{n+m}) \in Z(\mathcal{A})$. Lemma 2.2 implies that

$$\binom{k}{n+m} h(x^{n+m-1} u^k) \in Z(\mathcal{A}) \text{ for all } 0 \leq k \leq n + m,$$

in particular, for $k = n+m-1$, there is $(n+m)h(xu^{n+m-1}) \in Z(\mathcal{A})$; therefore, $h(xu^{n+m-1}) \in Z(\mathcal{A})$ hence,

$$h(x)h(u^{n+m-1}) + h(x)u^{n+m-1} + xh(u^{n+m-1}) \in Z(\mathcal{A}),$$

then $h(x)u^{n+m-1} \in Z(\mathcal{A})$ (because $h(u^{n+m-1}) = 0$). Applying Remark 2.1, we have $h(x) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$; consequently, $h(x^2) = h(x)^2 + 2xh(x) \in Z(\mathcal{A})$ implies $2xh(x) \in Z(\mathcal{A})$, hence $xh(x) \in Z(\mathcal{A})$ (because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$). According to Remark 2.1, we get $x \in Z(\mathcal{A})$. So $\mathcal{A}$ is commutative. □

Application 2.12. Let $\mathbb{K}$ be the fields of complex or real numbers; let

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{K} \right\}$$

be a non-commutative unital prime algebra of all 2×2 upper triangular matrices over $\mathbb{K}$ with the usual operations on matrices and the norm $||.||_1$ defined by $|| \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} ||_1 = |a| + |b| + |c|$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{A}$, then $\mathcal{A}$ is a normed linear space. Next, define the map

$$h\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix},$$

for every $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{A}$. Given that $\mathcal{A}$ is finite-dimensional, it is straightforward to verify that h is a non-zero continuous homoderivation.
Let $\mathcal{K}$ be a non-empty subset of $\mathcal{A}$ included in $Z(\mathcal{A})$. For all M, N two elements of $\mathcal{A}$, and for all p, q strictly positive integers, we have $M^p \in Z(\mathcal{A})$ and $N^q \in Z(\mathcal{A})$, hence $h(N^pN^q)+M^p \circ N^q \in Z(\mathcal{A})$. By Theorem 2.10, we conclude that $\mathcal{A}$ is commutative. As a result, $\mathcal{K}$ is the empty set. We conclude that the only open subset included in $Z(\mathcal{A})$ is the empty set.

Application 2.13. Let $\mathcal{A} = \mathcal{L}_c(E)$ be the space of continuous linear applications from E to E, where E is a normed space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$); endowed with usual application addition and composition and the norm defined by $||f|| = sup_{||x|| \leq 1}||f(x)||$ for all $x \in \mathcal{A}$, it is a normed algebra over $\mathbb{K}$.
Let $\mathcal{H}$ be a subspace of $\mathcal{A}$ defined by $\mathcal{H} = \{\alpha I_E \mid \alpha \in \mathbb{K}\}$, where I_E is the identity of E. We observe that $\mathcal{H} \subset Z(\mathcal{A})$, and in light of Application 2.12, we get the interior of $\mathcal{H}$ is empty because $\mathcal{A}$ is not commutative and $Int(\mathcal{H}) \subset Int(Z(\mathcal{A})) = \emptyset$.

Corollary 2.14. *Let $\mathcal{A}$ be a prime Banach algebra and $\mathcal{D}$ a dense part of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous non-zero homoderivation h such that*

$$\exists (p, q) \in \mathbb{N}^* \times \mathbb{N}^* : \quad h(x^p \circ y^q) + [x^p, y^q] \in Z(\mathcal{A}) \ \textit{for all } x, y \in \mathcal{D},$$

then $\mathcal{A}$ is commutative.

Proof. Let $a, b \in \mathcal{A}$; there exist two sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$ converging to a and b. Accordingly $(a_k)_{k\in\mathbb{N}} \subset \mathcal{D}$ and $(b_k)_{k\in\mathbb{N}} \subset \mathcal{D}$, which implies that

$$h((a_k)^p \circ (b_k)^q) + [(a_k)^p, (b_k)^q] \in Z(\mathcal{A}) \ \text{ for all } k \in \mathbb{N}.$$

As h is continuous, the sequence $(h((a_k)^p \circ (b_k)^q) + [(a_k)^p, (b_k)^q])_{k\in\mathbb{N}}$ converges to $h(a^p \circ b^q) + [a^p, b^q]$, or $Z(\mathcal{A})$ is closed, then $h(a^p \circ b^q) + [a^p, b^q] \in Z(\mathcal{A})$. This yields

$$\exists (p, q) \in \mathbb{N}^* \times \mathbb{N}^* : \quad h(a^p \circ b^q) + [a^p, b^q] \in Z(\mathcal{A}) \ \text{ for all } a, b \in \mathcal{A}.$$

By Theorem 2.11, we conclude that $\mathcal{A}$ is commutative. □

Remark 2.15. A straightforward corollary is that conclusions akin to those of the Theorem 2.10 remain valid if one of the open sets is replaced by a dense subset of $\mathcal{A}$.

The following examples serve to illustrate that the assumption that "$\mathcal{K}_1$ and $\mathcal{K}_2$ are open" in our theorems is crucial and that omitting this condition may lead to the failure of the stated conclusions.

Example 2.16. Consider the real prime Banach algebra $\mathcal{A} = \mathcal{M}_2(\mathbb{R})$ endowed with $||.||_1$ defined by $||M||_1 = \sum_{1 \leq i,j \leq 2} |a_{ij}|$ for all $M = (a_{ij})_{1 \leq i,j \leq 2}$. Let

$$\mathcal{H}_1 = \left\{ \begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\} \text{ and } \mathcal{H}_2 = \left\{ \begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix} \mid t \in \mathbb{R}^+ \right\}.$$

We assert that $\mathcal{H}_1$ is not open in $\mathcal{A}$ because its complement $\mathcal{H}_1^c$ is not closed. Indeed, the sequence

$$\left(\begin{pmatrix} 1+\frac{1}{n} & \frac{1}{n} \\ 1+\frac{1}{n} & \frac{1}{n} \end{pmatrix}\right)_{n\in\mathbb{N}^*}$$

in $\mathcal{H}_1^c$ converges to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin \mathcal{H}_1^c$. Let h be the homoderivation defined in Application 2.12 by

$$h\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.$$

One readily observes that h is continuous but fails to be injective. Furthermore, for

$$A = \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \in \mathcal{H}_1,\ B = \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix} \in \mathcal{H}_2.$$

Let $(p,q) \in \mathbb{N}^2$, we obtain

$$A^pB^q = \begin{pmatrix} a^pb^q & 0 \\ a^pb^q & 0 \end{pmatrix},\quad A^p \circ B^q = \begin{pmatrix} 2a^pb^q & 0 \\ 2a^pb^q & 0 \end{pmatrix},$$

$$[h(A^p), h(B^q)] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } [A^p, B^q] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

result of which

$$h(A^p \circ B^q) = h([A^p, B^q]) = h(A^pB^q) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Exploiting this fact

(i) $[h(A^p), h(B^q)] - [A^p, B^q] \in Z(\mathcal{A})$,

(ii) $[h(A^p), h(B^q)] + [A^p, B^q] \in Z(\mathcal{A})$,

(iii) $h(A^p \circ B^q) + A^pB^q \in Z(\mathcal{A})$,

(iv) $h(A^pB^q) + A^p \circ B^q \in Z(\mathcal{A})$,

(v) $h(A^p \circ B^q) + [A^p, B^q] \in Z(\mathcal{A})$.

It follows that h fulfills the assumptions of our theorems, although $\mathcal{A}$ is not a commutative algebra.

The following example shows that we cannot replace $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{Z}/5\mathbb{Z}$ in the hypothesis for Theorem 2.10 and Theorem 2.11.

Example 2.17. Let $\mathbb{F}_5$ be the field $\mathbb{Z}/5\mathbb{Z}$ and $\mathcal{A} = (\mathcal{M}_2(\mathbb{F}_5), +, \times, .)$. Clearly, $\mathcal{A}$ is a 2-torsion free prime Banach algebra over $\mathbb{F}_5$ with norm $||.||_1$ defined by $||A||_1 = \sum_{1\leq i,j\leq 2} |a_{ij}|$ for all $A = (a_{ij})_{1\leq i,j\leq 2} \in \mathcal{A}$, where $||.||$ is the norm defined on $\mathbb{F}_5$ by

$$||\bar{0}|| = 0,\ ||\bar{1}|| = 1,\ ||\bar{2}|| = 2,\ ||\bar{3}|| = 3,\ ||\bar{4}|| = 4.$$

Observe that

$$\mathcal{H} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{F}_5\right\}$$

is an open subset of $\mathcal{A}$. Indeed, for all $A \in \mathcal{H}$, the open ball $B(A,1) = \{X \in \mathcal{A} \mid ||A - X||_1 < 1\} = \{A\} \subset \mathcal{H}$. Let d be the inner derivation associated to $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $d(M) = [A, M]$ defined on $\mathcal{A}$ by

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix},$$

and h defined on $\mathcal{A}$ by $h(M) = d(M) - M$ for all $M \in \mathcal{A}$ is a continuous homoderivation. Moreover, for all $(p, q) \in \mathbb{N}^{*2}$ and for all $(M, N) \in \mathcal{H}^2$, we have

(i) $[h(M^p), h(N^q)] - [M^p, N^q] \in Z(\mathcal{A})$,

(ii) $[h(M^p), h(N^q)] + [M^p, N^q] \in Z(\mathcal{A})$,

(iii) $h(M^p \circ N^q) + M^p N^q \in Z(\mathcal{A})$,

(iv) $h(M^p N^q) + M^p \circ N^q \in Z(\mathcal{A})$,

(v) $h(M^p \circ N^q) + [M^p, N^q] \in Z(\mathcal{A})$.

This means that h satisfies the hypotheses of our theorems but $\mathcal{A}$ is not commutative.

3 Conclusion and discussions

In this paper, we have established new criteria based on differential identities to investigate the commutativity of prime rings and prime Banach algebras over the fields of real and complex numbers via homoderivations. In order to achieve the commutativity of this type of algebra in Theorem 2.4, Theorem 2.8, Theorem 2.10, and Theorem 2.11, our work is grounded in local identities that employ various product structures, including Lie, Jordan, and the conventional product. More precisely, this work aims to broaden the scope of the investigation by incorporating additional local identities involving homoderivations in order to achieve the same conclusions as those obtained in our paper [14], but utilizing a different type of derivation. It is evident that, given precise conditions, each of the previously studied local identities guarantees the commutativity of this class of algebras. However, can we find other more general identities with homoderivations and endomorphisms involving commutativity?

Open questions

We conclude this work by highlighting several intriguing open questions that invite further exploration and are expected to stimulate future research in this area.
Let $\mathcal{A}$ be a semi-prime Banach algebra and $\mathcal{K}_1$, $\mathcal{K}_2$ non-void open subsets of $\mathcal{A}$. Suppose $h : \mathcal{A} \to \mathcal{A}$ is a continuous linear homoderivation and $f : \mathcal{A} \to \mathcal{A}$ a continuous linear endomorphism of $\mathcal{A}$. In the following situations, what can be said about the form of h and f and the structure of $\mathcal{A}$?

- For all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there are strictly positive integers p, q such that
$$h(x^p y^q) + f(x^p \circ y^p) \in Z(\mathcal{A}).$$
- For all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there are strictly positive integers p, q such that
$$h(x^p \circ y^q) + f(x^p y^q) \in Z(\mathcal{A}).$$
- For all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there are strictly positive integers p, q such that $h(x^p y^q) \in Z(\mathcal{A})$.
- For all $(x, y) \in \mathcal{K}_1 \times \mathcal{K}_2$ there are strictly positive integers p, q such that $h(x^p \circ y^q) \in Z(\mathcal{A})$.

References

[1] S. Ali and A. N. Khan. "On commutativity of Banach algebras with derivations." Bulletin of the Australian Mathematical Society 91.3 (2015): 419-425.

[2] H. E. Bell and M. N. Daif. "On commutativity and strong commutativity-preserving maps." Canadian Mathematical Bulletin 37.4 (1994): 443-447.

[3] F. F. Bonsall and J. Duncan, Complete Normed Algebras (Springer-Verlag, New York, Heidelberg, 1973).

[4] A. Boua, A. Raji, and M. El Hamdaoui. "Projection and commutativity of prime Banach algebras." Palestine Journal of Mathematics 13.4 (2024).

[5] A. Boua, A. Raji, and M. El Hamdaoui. "Commutativity of prime rings and Banach algebras with generalized (β, α)-derivations." Palestine Journal of Mathematics 13.4 (2024).

[6] C-L. Chuang. "The additive subgroup generated by a polynomial." Israel Journal of Mathematics 59.1 (1987): 98-106.

[7] I. N. Herstein. "Two remarks on the commutativity of rings." Canadian Journal of Mathematics 7 (1955): 411-412.

[8] I.N. Herstein, Topics in Ring Theory, Univ. of Chicago Press, Chicago (1969).

[9] I. N. Herstein. "A commutativity theorem." Journal of Algebra 38.1 (1976): 112-118.

[10] T. K. Lee. "Semiprime rings with differential identities." Bull. Inst. Math. Acad. Sinica 20.1 (1992): 27-38.

[11] M. Moumen, L. Taoufiq, and A. Boua. "On prime Banach algebras with continuous derivations." Mathematica 65.88 (2023): 122-132.

[12] M. Moumen, L. Taoufiq, and L. Oukhtite. "Some differential identities on prime Banach algebras." Journal of Algebra and Its Applications 22.12 (2023): 2350258.

[13] J. Mayne. "Centralizing automorphisms of prime rings." Canadian Mathematical Bulletin 19.1 (1976): 113-115.

[14] A. Ouazza, A. Boua, and M. El Hamdaoui, "Commutativity with some differential identities on prime Banach algebras." Communications in Algebra (2025): 1-11.

[15] M. A. Quadri, M. A. Khan, and M. Ashraf. "Some elementary commutativity theorems for rings." Math. Student 56.1-4 (1988): 223-226.

[16] N. Rehman. "On rings and Banach algebras with skew derivations." Note Mat 40.1 (2020): 73-79.

[17] B. Yood. "Commutativity theorems for Banach algebras." Michigan Mathematical Journal 37.2 (1990): 203-210.

[18] B. Yood. "On commutativity of unital Banach algebras." Bulletin of the London Mathematical Society 23.3 (1991): 278-280.

[19] B. Yood. "Some commutativity theorems for Banach algebras." Publicationes Mathematicae Debrecen 45.1-2 (1994): 29-33.

Author information

A. Ouazza, Department of Mathematics, Polydisciplinary Faculty, LMSD, Taza, Sidi Mohammed Ben Abdellah University, Fez, Morocco.
E-mail: ayoub.ouazza@usmba.ac.ma

A. Boua, Department of Mathematics, Polydisciplinary Faculty, LMSD, Taza, Sidi Mohammed Ben Abdellah University, Fez, Morocco.
E-mail: abdelkarim.boua@usmba.ac.ma

Received: 2025-06-12
Accepted: 2025-09-23