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Coupling Shape and Topological Derivatives for a Thermoelasticity Problem: Application To Mixed Boundary Conditions

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Abstract. In this work, we study shape and topology optimization problems for thermoelasticity. A model based on linear thermoelasticity problem and a weak solution existence result with a Neumann and Dirichlet constraints are first proposed, before presenting models for which a mathematical analysis will be provided. Afterwards, we establish an optimal shape existence result using the uniform cone property. And finally, we establish shape and topological derivative results for the functional under consideration using the minmax method with constrained partial differential equations of the model in the stationary case.

1 Introduction

In this paper, we deal with thermoelasticity problem by using shape and topological optimization problem and their application in solar energy.

In the topological derivative section, we can consider the following assumptions: we recall the notions of Minkowski content, d-rectifiability, and positive reach that are used to define the topological derivative of an objective function with respect to perturbations by a d-dimensional closed subset $\overline{\omega}$ of $\mathbb{R}^N$. Given an open domain Ω with boundary $\partial\Omega$. Let us associate with ε, $0 < \varepsilon \le r$, the perturbed domain $\Omega_\epsilon = \Omega \setminus \overline{\omega_\epsilon}$ is obtained by removing the ϵ-dilatation $\overline{\omega_\epsilon}$ of the set $\overline{\omega}$. The boundary of Ω_ϵ is made up of two disjoint parts $\partial\Omega_\epsilon = \partial\Omega \cup \partial\overline{\omega_\epsilon}$. The notion of positive reach for a closed nowhere dense set $\overline{\omega}$ ensures that the boundary $\partial\overline{\omega_\epsilon}$ of the hole is of class $C^{1,1}$ and that $\overline{\Omega \setminus \overline{\omega}} = \overline{\Omega}$.

Before tacking the theorem of the topological derivative, we also need the following hypotheses. For the conception of these hypotheses, we relied on the works of of Novotny [1, 2, 3], and Delfour [4, 5, 6, 7, 8].

Let Ω_ϵ^m denote the component of Ω_ε for which $\partial\Omega$ is part of its boundary. Let Ω_ϵ^0 denote the blind component of Ω_ε whose boundary has no intersection with $\partial\Omega$. The function u_{Ω_ϵ} is divided between the two components Ω_ϵ^0 and Ω_ϵ^m as follows: $u_{\Omega_\epsilon} = u_{\Omega_0}$ within Ω_ϵ^0, and $\frac{\partial u_\epsilon}{\partial n} = 0$ on $\partial\Omega_\epsilon^m \cap \partial\overline{w}_\epsilon$ as $\partial\overline{w}_\epsilon$ consists of the two disjoint boundaries Ω_ϵ^0 and $\partial\Omega_\epsilon^0$ and $\partial\Omega_\epsilon^m \cap \partial\overline{w}_\epsilon$. We can construct an extension of Ω by defining the solution as follows: $u^0_{\Omega_\epsilon} = u_{\Omega_\epsilon}$ on $\partial\Omega_\epsilon^m \cap \partial\overline{w}_\epsilon$, $u^0_{\Omega_\epsilon} = u_{\Omega_\epsilon}$ on $\partial\Omega_\epsilon^0$. For more information, the reader can refer to [4].

The main objective of this article is to determine the shape and topological derivative of the functional $J(\Omega_\epsilon) = J(\Omega_\epsilon, u_{\Omega_\epsilon})$, where the perturbed domain Ω_ϵ of Ω is defined by $\Omega_\epsilon = T_\epsilon(\Omega)$ or $\Omega_\epsilon = \Omega \setminus \overline{w}_\epsilon$ depending on the derivative to be calculated.

Therefore, it is important to study one of many efficiency criterion of the material: we want to point out the geometrical and topological properties of the materials. The performance of the used material is linked to its physical properties. Furthermore, there is a relationship between physical and mechanical aspects of the material. For studied solid materials, it is interesting to get information on their deformation field.

In fact, the topology of the material may play a principal role for a selected efficiency criteria. For instance the homogenization theory allows us to get good physical and topological properties for a given material. For thermal or photovoltaic solar energy, one of the efficiency criteria is to know which deformation for the used material in order to get the best performance in the outpout sense.

Topological optimization gives an opportunity to get important informations on the topology of the considered domain in order to optimize at least a criteria. For our study it allows us to know the optimal deformation in the geometrical and topological point of view in order to have the best of the temperature distribution in the domain. Then, the expectation is to improve the output of thermal and photo voltaic solar systems.

The geometry and topology of solid materials can be studied using the tools of partial differential equation (PDE) and topological optimization. And this can lead to good material selection in physics and industrial applications.

In fact, domain optimization is used today in many industrial environments, for example in Airbus to reduce structures, improve vibration resistance and in many other areas of physics [9, 10, 11].

In [12], the authors address a shape optimization problem for a thermoelasticity model with uncertainties in the Robin boundary condition. The problem was formulated as the minimization of the body's volume under an inequality constraint on the expectation of the combination of the time-averaged L^2-norm of the von Mises stress and taken at the final time . They derived analytical expressions of the shape functional to obtain the shape derivative via second order correlations. An efficient numerical method based on the low rank approximation was proposed. The solution of the optimization problem was implemented numerically via the level method. In [13], the authors have taken thermal residual stresses into account in the topological optimization of structures built by additive manufacturing.

In [14] the authors have studied: Few results on fixed point theorems in two-dimensional multiplicative metric spaces. But in our work, the method of minimizing a functional allows us to show the existence of weak solutions.

The isogeometric approach has been adopted in research areas where sophisticated geometric representations are demanding, such as shell analysis [15], fluid-structure interaction [16], robust mesh [17]and shape design optimization [18, 19]. With respect with thermoelastic behavior, the thermomechanical contact of the mortar problem [20] and material distribution of functionally graded structures [21, 22] were studied using the isogeometric approach. For more information see [23].

In 1995, Rodrigues and Fernandes [24] attempted for the first time to solve the problem of optimizing the topology of the thermoelastic structure. By Focusing on the problem of minimum conformity, they used the homogenization method and the augmented Lagrange method. Li et al. [25] applied a scalable thickness design to the displacement minimization problem. Cho and Choi [26] developed a design sensitivity analysis method for weakly coupled thermoelastic elasticity problems and applied this method to solve the minimum conformance topology problem. In the same year, a similar method was used by Zuo et al. [27] and was applied to the topological design of thermally actuated folding micro-mechanisms. Xia and Wang [28] applied the method of defining levels with the augmented Lagrange multiplier method to the problem of minimal and comparative conformity results with other methods. Sun and Zhang [29] used independent interpolation models in mechanics and thermal fields for better results in 2009. For more informations see [30].

Chung et al. [31] studied the topological optimization of structures subjected to large deformations due to thermal and mechanical loads, which demonstrated how temperature changes affected the optimized design of large deformation structures. Considering the effect of temperature changes, Deng et al. [32] and Yan et al. [33] use simultaneous multi-scale formulations to optimize macro-scale topology and micro-scale material configurations. Li et al. [34] studied multi-scale optimization based on the level set approach in thermomechanics. Environment indicated that the porous material proves systematically favored for a coupled multi-physics problem. Zhu et al. [35] proposed a temperature-constrained topology optimization for coupled thermomechanical problems and revealed that temperature constraints play an important role in

relevant issues. For more reviews on thermoelastic design optimization, readers can refer to Wu et al. [36] and [37].
The document is organized as follows: In the first section, we present the introduction. In the second section, we give a model of the linear thermoelastic problem and recall an existence result for a solution of this problem. Section 3 is devoted to the existence of the optimal form of the minimization problem. Section 4 deals with the shape and in section 5 we study the topological derivatives. These sections gives theoretical results in asymptotic analysis. These results provide insights and information on the topological variation of the domain. Finaly in Section 6, we give the conclusion and some extensions.

2 Modeling and Existence of solution to (2.6)

2.1 Modeling

The model is essentially based on the principle of conservation of the mass and the momentum and the general law of thermoelasticity.
Let Ω be a domain included in big ball $\mathcal{D}_0$ at time t_0 which is in movement and becomes at time t, $\Omega_t \subset \mathcal{D}_t$. Let $a = (a_1, a_2, a_3)$ be a point of Ω which is $x = (x_1, x_2, x_3)$ in Ω_t. Let $\vec{u} = (u_1, u_2, u_3)$ the displacement vector (the deformation).
Using conservation of mass and the momentum we have

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = div\bar{\bar{\sigma}} + \vec{f} \tag{2.1}$$

and

$$div\vec{u} \ = \ \frac{\rho_0 - \rho}{\rho} \text{ in } \mathcal{D}_0 \tag{2.2}$$

where ρ_0 (resp. ρ) the volume mass of $\mathcal{D}_0$ (resp $\mathcal{D}_t$).
The general law of thermoelasticity, combining with law conservation gives directly in the permanent case

$$-\mu\Delta\vec{u} - (\lambda+\mu)grad(div(\vec{u})) + 3k\alpha grad\theta = \quad \vec{f} \quad in \quad \Omega. \tag{2.3}$$

The equation (2.3) relates a thermoelasticity problem. We will add some boundary conditions. The temperature θ is solution of a boundary value problem (in our case it is solution of the Neumann Laplacian problem) see the following section.

In the following section we will study these partial differential equations with boundary conditions. In what follows, we consider the following optimization problem

$$\min\left\{J(\Omega), \Omega \subset \mathbb{R}^N\right\} \tag{2.4}$$

where J is the functional defined by

$$J(\Omega) = \alpha \int_\Omega |u_\Omega - u_0|^2 dx + \beta \int_\Omega |\nabla u_\Omega|^2 dx \tag{2.5}$$

with $u_\Omega = (u_1, u_2, u_3)$ is solution to:

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u_\Omega}) + 3k\alpha\nabla\theta_\Omega = \overrightarrow{f} \text{ in } \Omega \\ -\Delta\theta_\Omega = g \text{ in } \Omega \\ \frac{\partial\theta_\Omega}{\partial n} = h \text{ on } \partial\Omega \\ \mathcal{B}u_\Omega = U \text{ on } \partial\Omega. \end{cases} \tag{2.6}$$

$\overrightarrow{f} \in L^1(\Omega, \mathbb{R}^N)$, $h \in L^2(\partial\Omega)$ and $g \in L^2(\Omega)$. $\mathcal{B}u_\Omega$ is a boundary operator $\partial\Omega$ denoting either a Dirichlet condition $\mathcal{B}u_\Omega = u_\Omega$ on $\partial\Omega$ or a Neumann condition $\mathcal{B}u_\Omega = \frac{\partial u_\Omega}{\partial n}$ on $\partial\Omega$.

2.2 Existence of solution to (2.6)

We consider

$$\begin{cases} -\Delta\theta_\Omega = g \text{ in } \Omega \\ \frac{\partial\theta_\Omega}{\partial n} = h \text{ on } \partial\Omega. \end{cases} \tag{2.7}$$

Therefore, this will therefore lead us to study two boundary value problems separately. Consequently, We will seek a solution θ_Ω of problem (2.7) that we will inject into the following systems of equations to study the solution of the problem

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u_\Omega}) + 3k\alpha\nabla\theta_\Omega = \overrightarrow{f} \text{ in } \Omega \\ \frac{\partial u_\Omega}{\partial n} = U_2 \text{ on } \partial\Omega. \end{cases} \tag{2.8}$$

and the problem

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u_\Omega}) + 3k\alpha\nabla\theta_\Omega = \overrightarrow{f} \text{ in } \Omega \\ u_\Omega = U_1 \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

We seek θ belonging to a Hilbert space solution of (2.7). We give the definition of what we mean by classical solution of this problem.

Definition 2.1. Given a bounded open set Ω of class $\mathcal{C}^2$ of $\mathbb{R}^N$ of boundary $\partial\Omega$ and $h \in \mathcal{C}^0(\partial\Omega)$. We call a classical solution to problem (2.7) all function u of class $\mathcal{C}^2(\bar{\Omega})$ verifying:

$$\begin{cases} -\Delta u = g \text{ in } \Omega \\ \frac{\partial u(z)}{\partial n} = h, \ \forall\ z \in \partial\Omega. \end{cases} \tag{2.10}$$

It is therefore clear that there is no uniqueness in the solution to this Neumann problem. in fact, if u is a solution to this problem for any connected component Ω_0 of Ω and any $c \in \mathbb{R}$ then $u + c\chi_{\Omega_0}$ is still a classical solution.
We also see that there is no solution for any function $h \in \mathcal{C}^0(\partial\Omega)$ and $g \in \mathcal{C}^0(\Omega)$.
If the equation admits a classical solution then for any connected component Ω_0 of Ω, we have

$$\int_{\partial\Omega_0} h\,d\sigma + \int_{\Omega_0} g\,dx = 0. \tag{2.11}$$

We have the following theorem:

Theorem 2.2. *Let Ω be a bounded open set of $\mathbb{R}^N$ and $\partial\Omega$ its boundary of class $\mathcal{C}^2$. $h \in \mathcal{C}^2(\partial\Omega)$ and $g \in L^2(\Omega)$.*
Neumann's problem (2.7) admits a solution if and only if,

$$\int_{\partial\Omega} h\,d\sigma + \int_{\Omega} g\,dx = 0. \tag{2.12}$$

Proof. See [38, 39]. □

Theorem 2.3. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^1$. Then the following partial differential equation has a solution*

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u_\Omega}) + 3k\alpha\nabla\theta_\Omega = \overrightarrow{f} \textit{ in } \Omega \\ \frac{\partial u_\Omega}{\partial n} = U_2 \textit{ on } \partial\Omega. \end{cases} \tag{2.13}$$

This solution is unique up to an additive constant.

Proof. See [38, 39].

□

Theorem 2.4. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^1$. Then the following partial differential equation has a unique solution*

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u_\Omega}) + 3k\alpha\nabla\theta_\Omega = \overrightarrow{f} \text{ in } \Omega \\ u_\Omega = U_1 \text{ on } \partial\Omega. \end{cases} \tag{2.14}$$

Proof. See [38, 39].

□

3 Optimal shape results

Definition 3.1. $\mathcal{O}_\epsilon$ We say that an open set satisfies the cone property if for all $x \in \partial\Omega$ there exists a unit vector $\xi(x)$ such that $y \in B(x,\epsilon)\cap\overline{\Omega}$, $\mathcal{C}(y,\xi(x),\epsilon)$. We denote by $\mathcal{O}_\epsilon$ the following set:

$$\mathcal{O}_\epsilon = \{\Omega \text{ open},\ \Omega \subset D,\ vol(\Omega) = c,\ \Omega \text{ check the } \epsilon - \text{cône property}\}. \tag{3.1}$$

Theorem 3.2. *: Let Ω_n be an open sequence in the class $\mathcal{O}_\epsilon$. Then there exists an open $\Omega \in \mathcal{O}_\epsilon$ and an sub-sequence Ω_{n_k} which converges towards Ω both in the sense of Hausdorff, in the sense of the characteristic functions and in the sense of compact. Additionally, $\overline{\Omega}_{n_k}$ and $\partial\Omega_{n_k}$ converge in the Hausdorff sense to $\overline{\Omega}$ and $\partial\Omega$.*

Proof. See [40]. □

Theorem 3.3. *: Let K be a compact and B a bounded open of $\mathbb{R}^N$. Let Ω_n be a sequence of open with $\overline{\Omega}_n \subset K \subset B$, verifying the ownership of the $\epsilon-$ cône.*
Then there is an open Ω verifying the ownership of the $\epsilon-$ cône and an extracted sequence Ω_{n_k} such as

$$\Omega_{n_k} \xrightarrow{\ H\ } \Omega, \quad \chi_{\Omega_{n_k}} \xrightarrow{\ L^1 p.p\ } \chi_\Omega,$$

$$\overline{\Omega}_{n_k} \xrightarrow{\ H\ } \overline{\Omega}, \quad \partial\Omega_{n_k} \xrightarrow{\ H\ } \partial\Omega.$$

It is a result which will allow us to characterize the existence of solution.

Proof. See [40]

□

In the next section, we deal with the existence of optimal forms using the Neumann edge condition. We could also use the Dirichlet edge condition to show the optimal form result.

Theorem 3.4. *Let $\mathcal{O}_{ad} \subset \mathcal{O}_\epsilon$ be a set open bounded domain of $\mathbb{R}^n$. Then there exists on open set $\Omega \in \mathcal{O}_{ad}$ satisfying*

$$J(\Omega) = \min_{\Omega\in\mathcal{O}_{ad}} J(\Omega).$$

Proof. At first, we will find a lower bound for the functional $J\Omega)$, $\Omega \in \mathcal{O}_{ad}$. Because of the fact that $u_0 \in L^2(\Omega)$ and u_Ω is solution to the problem (2.6), there exists a constant M such that

$$0 \le J(\Omega) \le M.$$

Then, the functional J is bounded and there exists a minimizing sequence $\Omega_n \in \mathcal{O}_{ad}$ such that

$$J(\Omega_n) \longrightarrow m = \inf J(\Omega).$$

As $\Omega_n \in \mathcal{O}_{ad}$ and $\mathcal{O}_{ad}$ is closed to $\mathcal{O}_\epsilon$, then according to a compactness Theorem 3.3, there exists an open set $\Omega \in \mathcal{O}_{ad}$ and a subsequence Ω_{n_k} of Ω_n such that the following convergence

holds: $\Omega_{n_k} \xrightarrow{H} \Omega$, $\chi_{\Omega_{n_k}} \xrightarrow{L^1 p.p} \chi_\Omega$ and $\overline{\Omega}_{n_k} \xrightarrow{H} \overline{\Omega}$, $\partial\Omega_{n_k} \xrightarrow{H} \partial\Omega$.
Now, from the variational formula, we get

$$\mu \int_\Omega \nabla u_\Omega . \nabla v dx - (\lambda + \mu) \int_\Omega \nabla(\nabla \cdot u_\Omega) v dx - 3k\alpha \int_\Omega \nabla\theta_\Omega v dx = \int_\Omega f v dx. \tag{3.2}$$

Hence in Ω_{n_k} we get :

$$\mu \int_{\Omega_{n_k}} \nabla u_{\Omega_{n_k}} . \nabla v dx - (\lambda + \mu) \int_{\Omega_{n_k}} \nabla(\nabla \cdot u_{\Omega_{n_k}}) v dx - 3k\alpha \int_{\Omega_{n_k}} \nabla\theta_{\Omega_{n_k}} v = \int_{\Omega_{n_k}} f v dx. \tag{3.3}$$

Taking $v = u_{\Omega_{n_k}}$ in (3.3), we get

$$\mu \int_{\Omega_{n_k}} |\nabla u_{\Omega_{n_k}}|^2 dx - (\lambda + \mu) \int_{\Omega_{n_k}} |\nabla \cdot u_{\Omega_{n_k}}|^2 dx - 3k\alpha \int_{\Omega_{n_k}} \nabla\theta_{\Omega_{n_k}} u_{\Omega_{n_k}} = \int_{\Omega_{n_k}} f u_{\Omega_{n_k}} dx. \tag{3.4}$$

From (3.4), as Ω_{n_k} is bounded in D, we have

$$\mu \int_D \chi_{\Omega_{n_k}} |\nabla u_{\Omega_{n_k}}|^2 dx + \mu \int_D \chi_{\Omega_{n_k}} |\nabla \cdot u_{\Omega_{n_k}}|^2 dx \leq \int_D \chi_{\Omega_{n_k}} f u_{\Omega_{n_k}} dx + 3k\alpha \int_D \chi_{\Omega_{n_k}} \nabla\theta_{\Omega_{n_k}} u_{\Omega_{n_k}}$$
$$\leq ||f||^2 ||u_{\Omega_{n_k}}||^2 + 3k\alpha ||\nabla\theta_{\Omega_{n_k}}||^2 ||u_{\Omega_{n_k}}||^2. \tag{3.5}$$

The solution $\theta_{\Omega_{n_k}}$ of the Laplacian operator is also bounded in $H^1(D)/\mathbb{R}$, then the term $||\nabla\theta_{\Omega_{n_k}}||^2$ is also finite, so there exists a constant $\overline{M}$ depending on f, k, α and μ such that

$$||u_{\Omega_{n_k}}||_{H_{div}(D)} \leq \overline{M}.$$

The sequence $u_{\Omega_{n_k}}$ is bounded in $H_{div}(D)$, which is a reflexive space. Then there exists an extracted sequence of $(u_{\Omega_{n_k}})$ still denoted by $(u_{\Omega_{n_k}})_{k\geq 1}$ and u_Ω such that: if $k \longrightarrow \infty$,

$$(u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup u^*_\Omega \in L^2(\Omega_{n_k}),\ (\nabla u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup \nabla u^*_\Omega \in L^2(\Omega),\ (div\, u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup div\, u^*_\Omega \in L^2(\Omega) \tag{3.6}$$

In the same way, as $||\nabla\theta_{\Omega_{n_k}}||^2$ is also bounded in $H^1(\Omega)/\mathbb{R}$, there exists also a subsequence $(\theta_{\Omega_{n_k}})$ such that the following convergence holds

$$(\nabla\theta_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup \nabla\theta_\Omega \in L^2(\Omega). \tag{3.7}$$

From (3.3), we get

$$\mu \int_D \chi_{\Omega_{n_k}} \nabla u_{\Omega_{n_k}} . \nabla\varphi dx - (\lambda + \mu) \int_D \chi_{\Omega_{n_k}} \nabla(\nabla \cdot u_{\Omega_{n_k}}) \varphi dx - 3k\alpha \int_D \chi_{\Omega_{n_k}} \nabla\theta_{\Omega_{n_k}} \varphi = \int_D \chi_{\Omega_{n_k}} f \varphi dx. \tag{3.8}$$

Passing to the limit as $k \longrightarrow \infty$ and using weak convergence (3.7) and (3.8), we get the following formulation

$$\mu \int_D \chi_\Omega \nabla u^*_\Omega . \nabla\varphi dx - (\lambda + \mu) \int_D \chi_\Omega \nabla(\nabla \cdot u^*_\Omega) \varphi dx - 3k\alpha \int_D \chi_\Omega \nabla\theta_\Omega \varphi = \int_D \chi_\Omega f \varphi dx \tag{3.9}$$

giving the following weak formulation

$$\mu \int_\Omega \nabla u^*_\Omega . \nabla\varphi dx - (\lambda + \mu) \int_\Omega \nabla(\nabla \cdot u^*_\Omega) \varphi dx - 3k\alpha \int_\Omega \nabla\theta_\Omega \varphi = \int_\Omega \chi_\Omega f \varphi dx. \tag{3.10}$$

Using Green formula in the first term of (3.10), we get

$$\int_\Omega -\mu\Delta u^*_\Omega dx - (\lambda + \mu) \int_\Omega \nabla(\nabla \cdot u^*_\Omega) \varphi dx - 3k\alpha \int_\Omega \nabla\theta_\Omega \varphi = \int_\Omega f \varphi dx \quad \forall\varphi \in H_{div}(\Omega).$$

since we have:

$$\begin{cases} -\mu\Delta u^*_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u^*_\Omega}) + 3k\alpha\nabla\theta_\Omega = \overrightarrow{f} \text{ in } \Omega \\ \frac{\partial u^*_\Omega}{\partial n} = U_2 \text{ on } \partial\Omega\,. \end{cases}$$

Finally by taking $\varphi = u_{\Omega_{n_k}}$ in (3.8), and $\varphi = u_\Omega$ in (3.9) we have :

$$\begin{aligned} &\lim\left(\mu\int_{\Omega_{n_k}}|\nabla u_{\Omega_{n_k}}|^2dx - (\lambda+\mu)\int_{\Omega_{n_k}}\nabla(\nabla\cdot u_{\Omega_{n_k}})u_{\Omega_{n_k}}dx - 3k\alpha\int_{\Omega_{n_k}}\nabla\theta_{\Omega_{n_k}}u_{\Omega_{n_k}}\right) \\ &= \lim\int_{\Omega_{n_k}} f u_{\Omega_{n_k}}dx = \int_\Omega f(x)u^*_\Omega \\ &= \mu\int_\Omega|\nabla u^*_\Omega|^2dx - (\lambda+\mu)\int_\Omega\nabla(\nabla\cdot u^*_\Omega)u^*_\Omega dx - 3k\alpha\int_\Omega\nabla\theta_\Omega u^*_\Omega dx \end{aligned}$$

$$\int_{\Omega_{n_k}}|\nabla u_{\Omega_{n_k}} - \nabla u_\Omega|^2dx = \int_{\Omega_{n_k}}|\nabla u_{\Omega_{n_k}}|^2dx - 2\int_{\Omega_{n_k}}\nabla u_{\Omega_{n_k}}\nabla u_\Omega + \int_{\Omega_{n_k}}|\nabla u_\Omega|^2dx.$$

Then taking the limit in the right hand side after equality, as $k \longrightarrow \infty$

$$\int_{\Omega_{n_k}}|\nabla u_{\Omega_{n_k}} - \nabla u_\Omega|^2dx = \int_{\Omega_{n_k}}|\nabla u_{\Omega_{n_k}}|^2dx - 2\int_{\Omega_{n_k}}\nabla u_{\Omega_{n_k}}\nabla u_\Omega + \int_{\Omega_{n_k}}|\nabla u_\Omega|^2dx = 0.$$

Since

$$\int_{\Omega_{n_k}}|\nabla u_{\Omega_{n_k}} - \nabla u_\Omega|^2dx = 0.$$

And we show

$$\int_{\Omega_{n_k}}|\nabla(\nabla\cdot u_{\Omega_{n_k}})u_{\Omega_{n_k}} - \nabla(\nabla\cdot u_\Omega)u_\Omega|^2dx = 0$$

$$\int_{\Omega_{n_k}} f\left(u_{\Omega_{n_k}} - u_\Omega\right) = 0 \text{ and } \int_{\Omega_{n_k}} u_{\Omega_{n_k}}\left(\nabla\theta_{\Omega_{n_k}} - \nabla\theta_\Omega\right) = 0.$$

We have :

$$u_{\Omega_{n_k}}(x)\xrightarrow{L^2}u_\Omega(x)$$

$$\nabla(\nabla\cdot u_{\Omega_{n_k}})u_{\Omega_{n_k}}\xrightarrow{L^2}\nabla(\nabla\cdot u_\Omega)u_\Omega$$

$$\nabla\theta_{\Omega_{n_k}}\xrightarrow{L^2}\nabla\theta_\Omega.$$

Since :

$$J(\Omega_{n_k}) \longrightarrow J(\Omega).$$

We prove that Ω^* is a minimiser of J. □

4 Shape Derivative

In this part, we give the main results of the paper. We first start with the theorem giving the shape derivative. For this purpose, we rely on the work of the pioneers [4, 5, 6, 7, 8, 41] on the minmax method.

4.1 Preliminaries

We will need some notations. The reader can refer to [4, 40, 42, 43, 44] for further information.
Notations : For $V \in C^0([0,\tau] : \mathcal{C}_1^0(\mathbb{R}^N, \mathbb{R}^N))$ and the diffeomorphism $T_\epsilon(V) = T_\epsilon$

$$\frac{d}{d\epsilon}T_\epsilon(X) = V(\epsilon, T_\epsilon(X)), \qquad T_0(X) = X, \qquad \frac{dT_\epsilon}{d\epsilon} = V(\epsilon) \circ T_\epsilon, \qquad T_0 = I,$$

where $V(\epsilon)(X) = V(\epsilon, X)$ and I is the identity matrix on $\mathbb{R}^N$. Moreover

$$\frac{d}{d\epsilon}DT_\epsilon = DV(\epsilon) \circ T_\epsilon DT_\epsilon, \qquad DT_0 = I, \quad \frac{d}{d\epsilon}J_\epsilon = divV(\epsilon) \circ T_\epsilon J_\epsilon, \quad J_0 = 1,$$

where $DV(\epsilon)$ and DT_ϵ are the Jacobian matrix of $V(\epsilon)$ and T_ϵ. For $k \geq 1$, $\mathcal{C}_0^k(\mathbb{R}^N, \mathbb{R}^N)$ is the space of k times continuously differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}^N$ going to zero at infinity; for $k = 0$, $\mathcal{C}_0^k(\mathbb{R}^N, \mathbb{R}^N)$ is the space of continuous functions from $\mathbb{R}^N$ to $\mathbb{R}^N$ going to zero at infinity. We shall also use the notation

$$V(\epsilon) := V(\epsilon) \circ T_\epsilon, \quad V\epsilon(X) = V(\epsilon, T_\epsilon(X)), \quad g(\epsilon) := T_\epsilon - I, \quad g(\epsilon)(X) = T_\epsilon(X)) - X.$$

Lemma 4.1. *Assume that $V \in C^0$, then $f_1 \in C^1$.*
For $\tau > 0$ sufficiently small, $J_\epsilon = \det DT_\epsilon = |\det DT_\epsilon| = |J_\epsilon|$, $0 \leq \epsilon \leq \tau$, and there exist constants $0 < \alpha < \beta$ such that

$forall \quad \xi \in \mathbb{R}^N, \alpha|\xi|^2 \leq B(\epsilon)\xi \cdot \xi \leq \beta|\xi|^2$ et $\alpha \leq J_\epsilon \leq \beta$. Moreover, one has :

(i) When ϵ goes to zero, $V_\epsilon \to V(0)$ in $C_0^1(\mathbb{R}^N, \mathbb{R}^N)$, $DT_\epsilon \to I$ in $C_0^0(\mathbb{R}^N, \mathbb{R}^N)$,

$J_\epsilon \to 1$ in $C_0^0(\mathbb{R}^N)$,

$\frac{DT_\epsilon - I}{\epsilon}$ is bounded in $C_0^0(\mathbb{R}^N, \mathbb{R}^N)$, $\frac{J_\epsilon - 1}{\epsilon}$ is bounded in $C_0^0(\mathbb{R}^N)$.

(ii) When ϵ goes to zero,

$$B(\epsilon) \to I \quad in \quad C_0^0(\mathbb{R}^N, \mathbb{R}^N), \qquad \frac{B(\epsilon) - I}{\epsilon} \quad is\ bounded\ in \quad C_0^0(\mathbb{R}^N, \mathbb{R}^N),$$

$$B'(v) = div\ V_\epsilon I - DV_\epsilon - (DV_\epsilon)^* \to B'(0) = div\ V(0) - DV(0) - (DV(0))^* \quad in \quad C_0^0(\mathbb{R}^N, \mathbb{R}^N).$$

where DV_ϵ is the jacobian matrix of V_ϵ and DV_ϵ^* is the transpose of DV_ϵ.

(iii) Given $h \in H^1(\mathbb{R}^N)$, as ϵ goes to zero,

$$h \circ T_\epsilon \to h \quad in \quad L^2(\Omega), \qquad \frac{h \circ T_\epsilon - h}{\epsilon} \quad is\ bounded\ in \quad L^2(\Omega), \qquad \nabla h \cdot V_\epsilon \to \nabla h \cdot V(0) \quad in \quad L^2(\Omega).$$

Proof. See [5, 6].

□

In this section, we recall the framework used in [44] and extended in [7] for the multivalued case.

Definition 4.2. A Lagrangian is a function of the form:

$$(\epsilon, x, y) \mapsto L(\epsilon, x, y) : [0, \tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0, \tag{4.1}$$

where Y is a vector space, X is a non-empty subset of a vector space, and $y \mapsto L(\epsilon, x, y)$ is affine. Associate with the parameter ϵ the parametrized minimax:

$$\begin{aligned} l(\epsilon) &= \inf_{x \in X} \sup_{y \in Y} L(\epsilon, x, y) : [0, \tau] \to \mathbb{R}, \\ Dl(0) &= \lim_{\epsilon \to 0^+} \frac{l(\epsilon) - l(0)}{\epsilon}. \end{aligned} \tag{4.2}$$

When the limits exist, we shall use the following compact notation:

$$d_\epsilon L(0,x,y) = \lim_{\epsilon\to 0^+} \frac{L(\epsilon,x,y) - L(0,x,y)}{\epsilon}, \tag{4.3}$$

$$d_x L(\epsilon,x,y;\varphi) = \lim_{\theta\to 0^+} \frac{L(\epsilon,x+\theta\varphi,y) - L(\epsilon,x,y)}{\theta}, \quad \varphi \in X, \tag{4.4}$$

$$d_y L(\epsilon,x,y;\psi) = \lim_{\theta\to 0^+} \frac{L(\epsilon,x,y+\theta\psi) - L(\epsilon,x,y)}{\theta}, \quad \psi \in Y. \tag{4.5}$$

The notation $\epsilon \to 0^+$ and $\theta \to 0^+$ means that ϵ and θ go to 0 through strictly positive values. Since $L(\epsilon,x,y)$ is affine in y, for all $(\epsilon,x) \in [0,\tau]\times X$,

$$\forall y,\psi \in Y, \quad d_y L(\epsilon,x,y;\psi) = L(\epsilon,x,\psi) - L(\epsilon,x,0) = d_y L(\epsilon,x,0;\psi). \tag{4.6}$$

We define the state equation as follows : for all $\epsilon \geq 0$

$$\text{Find } x^\epsilon \in X \text{ such that for all } \psi \in Y, d_y L(\epsilon,x^\epsilon,0;\psi) = 0. \tag{4.7}$$

The set of solutions (states) x^ϵ at $\epsilon \geq 0$ is denoted:

$$E(\epsilon) = \{x^\epsilon \in X : \forall\psi \in Y, dy\, L(\epsilon,x^\epsilon,0;\psi) = 0\}. \tag{4.8}$$

The adjoint state equation is defined as follows : for all$\epsilon \geq 0$ is :

$$\text{Find } p^\epsilon \in Y \text{ such that for all } \varphi \in X, \quad d_x L(\epsilon,x^\epsilon,p^\epsilon;\varphi) = 0, \tag{4.9}$$

and the set of solutions is denoted $Y(\epsilon,x^\epsilon)$. Finally, the set of minimizers for the minimax is given by:

$$X(\epsilon) = \{x^\epsilon \in X : \; l(\epsilon) = \inf_{x\in X}\sup_{y\in Y} L(\epsilon,x,y) = \sup_{y\in Y} L(\epsilon,x^\epsilon,y)\}. \tag{4.10}$$

We need the following assumption for everything that follows:

Hypothesis (H0)

Let X be the vector space.

(i) : For all $\epsilon \in [0,\tau]$, $x^0 \in X(0)$, $x^\epsilon \in X(\epsilon)$ and $y \in Y$, the function $\theta \mapsto L(\epsilon, x^0 + \theta(x^\epsilon - x^0), y) : [0,1] \to \mathbb{R}$ is continuous. This implies that for almost all θ the derivative exists and is equal to $d_x L(\epsilon, x^0 + \theta(x^\epsilon - x^0), y; x^\epsilon - x^0)$ and it is the integral of its derivative. In particular

$$L(\epsilon,x^s,y) = L(\epsilon,x^0,y) + \int_0^1 d_x L(\epsilon, x^0+\theta(x^\epsilon - x^0), y; x^\epsilon - x^0)\, d\theta.$$

ii) : For all $\epsilon \in [0,\tau]$, $x^0 \in X(0)$, $x^\epsilon \in X(s)$ and $y \in Y$, $\phi \in X$ and for almost all $\theta \in [0,1]$, $d_x L(\epsilon, x^0 + \theta(x^\epsilon - x^0), y; \phi)$ exists and the functions $\theta \mapsto d_x L(\epsilon, x^0 + \theta(x^\epsilon - x^0), y; \phi)$ belong to $L^1[0,1]$.

Theorem 4.3. *Consider the Lagrangian functional*

$$L(\epsilon,x,y) : [0,\tau]\times X \times Y \to \mathbb{R}, \quad \tau > 0,$$

where X and Y are vector spaces and the function $y \to L(\epsilon,x,y)$ is affine. Let $(H0)$ and the following hypotheses be satisfied:

- $(H1)$*: For all $\epsilon \in [0,\tau]$, $l(\epsilon)$ is finite, $X(\epsilon) = \{x^\epsilon\}$, and $Y(0,x^0) = \{p^0\}$ are singletons.*
- $(H2'')$*: $d_\epsilon L(0,x^0,y^0)$ exists.*
- $(H3'')$*: The following limit exists:*

$$R(x^0,p^0) = \lim_{\epsilon\to 0^+} \int_0^1 d_x L\left(\epsilon, x^0, \theta(x^\epsilon - x^0), p^0, \frac{x^\epsilon - x^0}{\epsilon}\right) d\theta.$$

Then, Dl(0) exists and $Dl(0) = d_\epsilon L(0,x^0,p^0) + R(x^0,p^0)$.

Proof. See [7, 44].

□

4.2 Shape derivative for the functional

In what follows, we will apply the previous shape and topological derivative results to our model problem and calculate the derivative associated with the functional (2.5). To do this, we've established the existence of solutions to the model given by the partial differential equation (2.6), and the differentiability of the Lagrangian associated with the form functional (2.5) is also guaranteed. The following theorem gives the main result for the form derivative of the functional.

Theorem 4.4. *The shape derivative exists if and only if* $\lim_{\epsilon\longrightarrow 0} R(\epsilon)$ *exists with*

$$\begin{aligned} R(\epsilon) &= \int_\Omega 2\beta\left[\frac{B(\epsilon)-I}{\epsilon}\right]\nabla\left(\frac{u_\Omega^\epsilon+u_\Omega^0}{2}\right)\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)+\int_\Omega \mu\left[\frac{B(\epsilon)-I}{\epsilon}\right]\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)\nabla p^0 dx \\ &- \int_\Omega \beta\left|\nabla\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right)\right|^2 dx+\int_\Omega 2\alpha\left[\frac{J(\epsilon)-1}{\epsilon}\right]\left[\frac{u_\Omega^\epsilon+u_\Omega^0}{2}-u_0\circ T_\epsilon\right]\left(u_\Omega^\epsilon-u_\Omega^0\right)\,dx \\ &+ \int_\Omega 2\alpha\left[u_0\circ T_\epsilon-u_0\right]\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)dx-\alpha\int_\Omega\left|\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right|^2 dx. \end{aligned}$$

If $R(\epsilon)\longrightarrow 0$ *the shape derivative is given by the expression:*

$$\begin{aligned} DJ(\Omega,V(0)) &= \int_\Omega\left[\beta|\nabla u_\Omega^0|^2 divV(0)+\alpha\left|u_\Omega^0-u_0\right|^2 divV(0)-2(u_\Omega^0-u_0)\nabla u_0\,V(0)\right]dx \\ &+ \int_\Omega \mu B'(0)\nabla u_\Omega^0\nabla p^0 dx-\int_\Omega\left[fp^0+3k\alpha\nabla\theta_\Omega p^0\right]divV(0)\,dx. \end{aligned}$$

Proof. The shape functional associated the perturbed domain is given by

$$j(\chi_\epsilon(x_0))=J(\Omega)=\alpha\int_{\Omega_\epsilon}|u_{\Omega_\epsilon}-u_0|^2dx+\beta\int_{\Omega_\epsilon}|\nabla u_{\Omega_\epsilon}|^2dx \tag{4.11}$$

where u_{Ω_ϵ} is solution of the variational problem

$$\mu\int_{\Omega_\epsilon}\nabla u_{\Omega_\epsilon}.\nabla v dx-(\lambda+\mu)\int_{\Omega_\epsilon}\nabla(\nabla\cdot u_{\Omega_\epsilon})v dx-3k\alpha\int_{\Omega_\epsilon}\nabla\theta_{\Omega_\epsilon}v=\int_{\Omega_\epsilon}fv dx. \tag{4.12}$$

and

$$\int_{\Omega_\epsilon}\nabla\theta_{\Omega_\epsilon}\nabla\varphi dx=\int_{\Omega_\epsilon}g\varphi dx \tag{4.13}$$

for all $v\in H_{div(\Omega,\mathbb{R}^3)}$ strongly and $\varphi\in H_{div(\Omega,\mathbb{R}^3)}$ for some given functions $\overrightarrow{f}\in L^1(\Omega,\mathbb{R}^N)$ and $g\in L^2(\Omega)$

In the case where $div\, u_\Omega$=0 in (4.12) is reduced to the following variational problem

$$\mu\int_{\Omega_\epsilon}\nabla u_{\Omega_\epsilon}.\nabla v\;dx=\int_{\Omega_\epsilon}fv\;dx+3k\alpha\int_{\Omega_\epsilon}\nabla\theta_{\Omega_\epsilon}v\;dx \tag{4.14}$$

and θ_Ω solution to (4.13). Before starting the formal derivative, will first need to define the following set.

$$H_{div(\Omega_\epsilon,\mathbb{R}^3)}=\{\phi\circ T_\epsilon^{-1},\;\phi\in H_{div(\Omega,\mathbb{R}^3)}\}.$$

We also introduce

$$u_\Omega^\epsilon=u_{\Omega_\epsilon}\circ T_\epsilon\qquad v_0=v\circ T_\epsilon.$$

So, we aim to find $u_\Omega^\epsilon\in H_{div(\Omega,\mathbb{R}^3)}$ thus that

$$\int_{\Omega_\epsilon}\mu\nabla(u_\Omega^\epsilon\circ T_\epsilon^{-1}).\nabla(v_0\circ T_\epsilon^{-1})-f(v_0\circ T_\epsilon^{-1})-3k\alpha\nabla\theta_\Omega(v_0\circ T_\epsilon^{-1})dx=0\quad\forall\quad v_0\in H_{div(\Omega,\mathbb{R}^3)}. \tag{4.15}$$

By applying the change of variable formula, we have

$$\int_{\Omega}\left[\mu\nabla(u_{\Omega}^{\epsilon}\circ T_{\epsilon}^{-1}).\nabla(v_0\circ T_{\epsilon}^{-1})-f(v_0\circ T_{\epsilon}^{-1})-3k\alpha\nabla\theta_{\Omega}(v_0\circ T_{\epsilon}^{-1})\right]\circ T_{\epsilon}J_{\epsilon}dx=0.$$

And on the other hand, we also have

$$\int_{\Omega}\left[f(v_0\circ T_{\epsilon}^{-1})+3k\alpha\nabla\theta_{\Omega}(v_0\circ T_{\epsilon}^{-1})\right]\circ T_{\epsilon}J_{\epsilon}dx=\int_{\Omega}\left[f(v_0\circ T_{\epsilon}^{-1}\circ T_{\epsilon})+3k\alpha\nabla\theta_{\Omega}(v_0\circ T_{\epsilon}^{-1}\circ T_{\epsilon})\right]J_{\epsilon}dx.$$

And thus, we obtain

$$\int_{\Omega}\left[f(v_0\circ T_{\epsilon}^{-1})+3k\alpha\nabla\theta_{\Omega}(v_0\circ T_{\epsilon}^{-1})\right]\circ T_{\epsilon}J_{\epsilon}dx=\int_{\Omega}\left[fv_0+3k\alpha\nabla\theta_{\Omega}v_0\right]J_{\epsilon}dx.$$

So, we have

$$\int_{\Omega_{\epsilon}}\left[\mu\nabla(u_{\Omega}^{\epsilon}\circ T_{\epsilon}^{-1}).\nabla(v_0\circ T_{\epsilon}^{-1})\right]\circ T_{\epsilon}J_{\epsilon}dx=\int_{\Omega}\mu DT_{\epsilon}^{-1}(DT_{\epsilon}^{-1})^{*}\nabla u_{\Omega}^{\epsilon}\nabla v_0 J_{\epsilon}dx,$$

$$\int_{\Omega_{\epsilon}}\left[\mu\nabla(u_{\Omega}^{\epsilon}\circ T_{\epsilon}^{-1}).\nabla(v_0\circ T_{\epsilon}^{-1})\right]\circ T_{\epsilon}J_{\epsilon}dx=\int_{\Omega}\mu B(\epsilon)\nabla u_{\Omega}^{\epsilon}\nabla v_0 dx$$

$$\text{with}\qquad B(\epsilon)=DT_{\epsilon}^{-1}(DT_{\epsilon}^{-1})^{*}J_{\epsilon},\qquad J_{\epsilon}=det\,DT_{\epsilon}$$

DT_{ϵ} denotes the Jacobian matrix of T_{ϵ}. And we have

$$\int_{\Omega}\mu B(\epsilon)\nabla u_{\Omega}^{\epsilon}\nabla v_0 dx-\int_{\Omega}\left[fv_0+3k\alpha\nabla\theta_{\Omega}v_0\right]J_{\epsilon}dx. \tag{4.16}$$

Let us recall that the shape functional is given by

$$J(\boldsymbol{\Omega}_{\epsilon})=\beta\int_{\Omega_{\epsilon}}|\nabla u_{\Omega_{\epsilon}}|^2dx+\alpha\int_{\Omega_{\epsilon}}|u_{\Omega_{\epsilon}}-u_0|^2dx$$

$$J(\boldsymbol{\Omega}_{\epsilon})=\beta\int_{\Omega_{\epsilon}}|\nabla u_{\Omega_{\epsilon}}|^2J_{\epsilon}dx+\alpha\int_{\Omega_{\epsilon}}|u_{\Omega_{\epsilon}}\circ T_{\epsilon}-u_0\circ T_{\epsilon}|^2J_{\epsilon}dx.$$

And on the other hand, we also have

$$J(\boldsymbol{\Omega}_{\epsilon})=\int_{\Omega}\beta|\nabla u_{\Omega}^{\epsilon}|^2J_{\epsilon}dx+\int_{\Omega}\alpha|u_{\Omega}^{\epsilon}-u_0\circ T_{\epsilon}|^2J_{\epsilon}dx.$$

We now define the Lagrangian in terms of ϵ.

$$L(\epsilon,\phi,\boldsymbol{\Phi})=\int_{\Omega}\left[\beta|\nabla\phi|^2J_{\epsilon}+\alpha|\phi-u_0\circ T_{\epsilon}|^2J_{\epsilon}\right]dx+\int_{\Omega}\mu B(\epsilon)\nabla\phi\nabla\boldsymbol{\Phi}dx-\int_{\Omega}\left[f\boldsymbol{\Phi}+3k\alpha\nabla\theta_{\Omega}\boldsymbol{\Phi}\right]J_{\epsilon}dx.$$

$$l(\epsilon)=\inf_{\phi\in H_{div}}\ \sup_{\Phi\in H_{div}}L(\epsilon,\phi,\boldsymbol{\Phi}),\quad Dl(0)=\lim_{\epsilon\longrightarrow 0}\frac{l(\epsilon)-l(0)}{\epsilon}=DJ(\boldsymbol{\Omega},V(0)).$$

Let evaluate the derivative of the Lagrangian by first setting

$$V_{\epsilon}=V(\epsilon)\circ T_{\epsilon},\quad V_{\epsilon}(X)=V(\epsilon,T_{\epsilon}(X)),\quad g(\epsilon)=T_{\epsilon}-I.$$

$$\begin{aligned}
d_{\epsilon}L(\epsilon,\phi,\boldsymbol{\Phi}) &= \int_{\Omega}\left[\beta|\nabla\phi|^2 divV(\epsilon)\circ T_{\epsilon}\,J_{\epsilon}\ +\alpha|\phi-u_0\circ T_{\epsilon}|^2 divV(\epsilon)\circ T_{\epsilon}J_{\epsilon}\right]dx\\
&- \int_{\Omega}\left[2(\phi-u_0\circ T_{\epsilon})\nabla u_0\,V(\epsilon)\circ T_{\epsilon}\,J_{\epsilon}\right]dx\\
&+ \int_{\Omega}\mu B'(\epsilon)\nabla\phi\nabla\boldsymbol{\Phi}dx-\int_{\Omega}\left[f\boldsymbol{\Phi}+3k\alpha\nabla\theta_{\Omega}\boldsymbol{\Phi}\right]divV(\epsilon)\circ T_{\epsilon}\,J_{\epsilon}\,dx.
\end{aligned}$$

By taking $\phi = u_\Omega^0$ and $\Phi = p^0$ we have

$$\begin{aligned} d_\epsilon L(\epsilon, u_\Omega^0, p^0) &= \int_\Omega \Big[\beta|\nabla u_\Omega^0|^2 divV(\epsilon)\circ T_\epsilon\, J_\epsilon \,+ \alpha \left|u_\Omega^0 - u_0\circ T_\epsilon\right|^2 divV(\epsilon)\circ T_\epsilon J_\epsilon\Big]\, dx \\ &- \int_\Omega \left[2(u_\Omega^0 - u_0\circ T_\epsilon)\nabla u_0\, V(\epsilon)\circ T_\epsilon\, J_\epsilon\right] dx \\ &+ \int_\Omega \mu B'(\epsilon)\nabla u_\Omega^0 \nabla p^0 dx - \int_\Omega \left[fp^0 + 3k\alpha\nabla\theta_\Omega p^0\right] divV(\epsilon)\circ T_\epsilon\, J_\epsilon\, dx. \end{aligned}$$

So, we obtain

$$\begin{aligned} d_\epsilon L(0, u_\Omega^0, p^0) = \int_\Omega \Big[\beta|\nabla u_\Omega^0|^2 divV_0 + \alpha\left|u_\Omega^0 - u_0\right|^2 divV(0) - 2(u_\Omega^0 - u_0)\nabla u_0\, V(0)\Big]\, dx \\ + \int_\Omega \mu B'(0)\nabla u_\Omega^0 \nabla p^0 dx - \int_\Omega \left[fp^0 + 3k\alpha\nabla\theta_\Omega p^0\right] divV(0)\, dx. \end{aligned} \tag{4.17}$$

For the derivative of the Lagrangian with respect to ϕ we have

$$d_\phi L(\epsilon, \phi, \Phi, \phi') = \int_\Omega [2\beta B(\epsilon)\nabla\phi\nabla\phi' + 2\alpha(\phi - u_0\circ T_\epsilon)\phi' J_\epsilon]\, dx + \int_\Omega \mu B(\epsilon)\nabla\phi'\nabla\Phi dx. \tag{4.18}$$

And on the other hand, the derivative of the Lagrangian with respect to Φ is given by

$$d_\Phi L(\epsilon, \phi, \Phi, \Phi') = \int_\Omega \mu B(\epsilon)\nabla\phi\nabla\Phi'\, dx - \int_\Omega [f\Phi' + 3k\alpha\nabla\theta_\Omega\Phi']\, J_\epsilon\, dx. \tag{4.19}$$

So, we have the state equation for $\epsilon \geq 0$ is given by

$$u_\Omega^\epsilon \in H_{div}, \forall \Phi' \in H_{div}, \quad \int_\Omega [\mu B(\epsilon)\nabla u_\Omega^\epsilon \nabla\Phi' - (f\Phi' + 3k\alpha\nabla\theta_\Omega\Phi')\, J_\epsilon]\, dx = 0. \tag{4.20}$$

And the adjoint state equation for $\epsilon = 0$ is also given by

$$p^0 \in H_{div}, \forall \phi' \in H_{div}, \quad \int_\Omega \left[2\beta\nabla u_\Omega^0 \nabla\phi' + 2\alpha(u_\Omega^0 - u_0)\phi' + \mu\nabla\phi'\nabla p^0\right] dx = 0. \tag{4.21}$$

$$\begin{aligned} R(\epsilon) &= \int_0^1 d_\phi L\left(\epsilon, u_{\Omega_0} + \Psi\left(u_\Omega^\epsilon - u_\Omega^0\right), p_0, \left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right)\right)\, d\Psi. \\ R(\epsilon) &= \int_\Omega 2\beta B(\epsilon)\nabla\left(\frac{u_\Omega^\epsilon + u_\Omega^0}{2}\right)\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right) + 2\alpha\left[\left(\frac{u_\Omega^\epsilon + u_\Omega^0}{2}\right) - u_0\circ T_\epsilon\right]\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right) J_\epsilon dx \\ &+ \int_\Omega \mu B(\epsilon)\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right)\nabla p^0\, dx. \end{aligned}$$

By substituting $\phi' = \frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}$ into the adjoint equation for p^0, we obtain:

$$\int_\Omega 2\beta\nabla u_\Omega^0\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right) + 2\alpha\left(u_\Omega^0 - u_0\right)\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right) + \mu\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right)\nabla p^0 dx = 0.$$

So, we have

$$\begin{aligned} \int_\Omega 2\beta\left[\nabla\left(\frac{u_\Omega^\epsilon + u_\Omega^0}{2}\right) - \nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{2}\right)\right]\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right) + \mu\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right)\nabla p^0 dx \\ + \int_\Omega 2\alpha\left[\frac{u_\Omega^\epsilon + u_\Omega^0}{2} - \frac{u_\Omega^\epsilon - u_\Omega^0}{2} - u_0\circ T_\epsilon + u_0\circ T_\epsilon - u_0\right]\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right) = 0. \end{aligned}$$

As a result, we have

$$\int_\Omega 2\beta\nabla\left(\frac{u_\Omega^\epsilon+u_\Omega^0}{2}\right)\nabla\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)+\int_\Omega \mu\nabla\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)\nabla p^0 dx-\alpha\int_\Omega\left|\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right|^2$$
$$-\int_\Omega\beta\left|\nabla\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right)\right|^2+\int_\Omega 2\alpha\left[\frac{u_\Omega^\epsilon+u_\Omega^0}{2}-u_0\circ T_\epsilon\right]\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)$$
$$+\int_\Omega 2\alpha\left[u_0\circ T_\epsilon-u_0\right]\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)=0.$$

We can now rewrite the expression of $R(\epsilon)$ as follows.

$$\begin{aligned}R(\epsilon) &= \int_\Omega 2\beta\left[\frac{B(\epsilon)-I}{\epsilon}\right]\nabla\left(\frac{u_\Omega^\epsilon+u_\Omega^0}{2}\right)\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)+\int_\Omega\mu\left[\frac{B(\epsilon)-I}{\epsilon}\right]\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)\nabla p^0dx\\ &- \alpha\int_\Omega\left|\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right|^2\\ &- \int_\Omega\beta\left|\nabla\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right)\right|^2+\int_\Omega 2\alpha\left[\frac{J(\epsilon)-1}{\epsilon}\right]\left[\frac{u_\Omega^\epsilon+u_\Omega^0}{2}-u_0\circ T_\epsilon\right]\left(u_\Omega^\epsilon-u_\Omega^0\right)dx\\ &+ \int_\Omega 2\alpha\left[u_0\circ T_\epsilon-u_0\right]\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)dx.\end{aligned}$$

From this, we have the following estimate

$$\begin{aligned}|R(\epsilon)| &\leq \int_\Omega 2|\beta|\left|\left|\frac{B(\epsilon)-I}{\epsilon}\right|\right|\left|\left|\nabla\left(\frac{u_\Omega^\epsilon+u_\Omega^0}{2}\right)\right|\right|\left|\left|\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)\right|\right|\\ &+ \int_\Omega|\mu|\left|\left|\frac{B(\epsilon)-I}{\epsilon}\right|\right|\left|\left|\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)\right|\right|\left|\left|\nabla p^0\right|\right|dx\\ &+ |\alpha|\int_\Omega\left|\left|\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right|\right|^2+\int_\Omega|\beta|\left|\left|\nabla\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\sqrt{\epsilon}}\right)\right|\right|^2\\ &+ \int_\Omega 2|\alpha|\left|\left|\frac{J(\epsilon)-1}{\epsilon}\right|\right|\left|\left|\frac{u_\Omega^\epsilon+u_\Omega^0}{2}-u_0\circ T_\epsilon\right|\right|\left|\left|\left(u_\Omega^\epsilon-u_\Omega^0\right)\right|\right|\\ &+ \int_\Omega 2|\alpha|\left|\left|\left[u_0\circ T_\epsilon-u_0\right]\right|\right|\left|\left|\left(\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}\right)\right|\right|dx.\end{aligned}$$

By Lemma 4.1 the terms $\frac{B(\epsilon)-I}{\epsilon}$ and $\frac{J(\epsilon)-1}{\epsilon}$ are uniformy bounded. To conclude that the limit of $R(\epsilon)$ exists and is zero, it remains to show that $u_\Omega^\epsilon \longrightarrow u_\Omega^0$ in H_{div}-strong and that the L^2 norm of $\frac{u_\Omega^\epsilon-u_\Omega^0}{\epsilon}$ is bounded. From the state equation (4.20) of u_Ω^ϵ and u_Ω^0, for $\Phi'\in H_{div}$:

$$\int_\Omega\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)\nabla\Phi'=\int_\Omega 3k\alpha\left(J(\epsilon)-1\right)\nabla\theta_\Omega\Phi'dx+\int_\Omega f\Phi'dx-\int_\Omega\mu\left(B(\epsilon)-I\right)\nabla u_\Omega^\epsilon\nabla\Phi'dx.$$

Substitute $\Phi'=u_\Omega^\epsilon-u_\Omega^0$ to obtain the following estimate

$$\begin{aligned}&\left|\left|\nabla\left(u_\Omega^\epsilon-u_\Omega^0\right)\right|\right|^2\\ &\leq 3|k\alpha|\left|\left|\left(J(\epsilon)-1\right)\right|\right|\left|\left|\nabla\theta_\Omega\right|\right|\left|\left|\left(u_\Omega^\epsilon-u_\Omega^0\right)\right|\right|+\left|\left|f\right|\right|\left|\left|\left(u_\Omega^\epsilon-u_\Omega^0\right)\right|\right|\\ &+ |\mu|\left|\left|\left(B(\epsilon)-I\right)\right|\right|\left|\left|u_\Omega^\epsilon-u_\Omega^0\right|\right|\left|\left|\nabla\left(u_\Omega^\epsilon\right)\right|\right|.\end{aligned}$$

So Ω is a bounded open lipschitzian domain, there exists a constant such that

$$||(u_\Omega^\epsilon - u_\Omega^0)|| \leq C(\Omega)\, ||\nabla\, (u_\Omega^\epsilon - u_\Omega^0)||$$

and we have

$$||\nabla\, (u_\Omega^\epsilon - u_\Omega^0)|| \leq 3|k\alpha|\, ||(J(\epsilon)-1)||\, ||\nabla\theta_\Omega||\, C(\Omega) + ||f||\, C(\Omega) + |\mu|\, ||(B(\epsilon)-I)||\, C(\Omega)\, ||\nabla\, (u_\Omega^\epsilon)||$$

but the right-hand side of this enequality goes to zero as ϵ goes to zero. Therefore $u_\Omega^\epsilon \longrightarrow u_\Omega^0$ in H_{div}. Finally, going back to the inaquality and dividing by $\epsilon > 0$ and we have

$$\begin{aligned} & \left|\left|\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right)\right|\right| \\ \leq\ & 3|k\alpha| \left|\left|\left(\frac{J(\epsilon)-1}{\epsilon}\right)\right|\right| ||\nabla\theta_\Omega||\, C(\Omega) + \left|\left|\frac{f}{\epsilon}\right|\right| C(\Omega) \\ +\ & |\mu| \left|\left|\left(\frac{B(\epsilon)-I}{\epsilon}\right)\right|\right| C(\Omega)\, ||\nabla\, (u_\Omega^\epsilon)||\,. \end{aligned}$$

Since the right hand of the above inequality is bounded, $\nabla\left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}\right)$ is bounded. This means that $\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}$ is bounded in H_{div} and hence $\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon}$ is bounded in L^2. And the term $R(\epsilon)$ is zero.

The shape derivative is given by

$$\begin{aligned} DJ(\Omega, V(0)) &= \int_\Omega \left[\beta|\nabla u_\Omega^0|^2 div V_0 + \alpha \left|u_\Omega^0 - u_0\right|^2 div V(0) - 2(u_\Omega^0 - u_0)\nabla u_0\, V(0)\right] dx \\ &+ \int_\Omega \mu B'(0)\nabla u_\Omega^0 \nabla p^0 dx - \int_\Omega \left[f p^0 + 3k\alpha\nabla\theta_\Omega p^0\right] div V(0)\, dx. \end{aligned}$$

□

5 Topological derivative

In this part we give the fundamental results of the topological derivative of the functional. Before tackling this main result, we will need the following preliminary results.. For more informations, the reader can refer to Delfour papers [3, 4, 7, 44, 45, 46, 47].

5.1 Preliminaries

Definition 5.1. Given d, $0 \leq d \leq N$, the d-dimensional upper and lower Minkowski contents of a set E are defined through an $r-$dilatation of the set E as follows:

$$M^{*d}(E) = \limsup_{r \searrow 0} \frac{\mathrm{m}_N(E_r)}{\alpha_{N-d} r^{N-d}}, \quad M_*^d(E) = \liminf_{r \searrow 0} \frac{\mathrm{m}_N(E_r)}{\alpha_{n-d} r^{N-d}}.$$

where m_N is the Lebesgue measure in $\mathbb{R}^N$ and α_{N-d} is the volume of the ball of radius one in $\mathbb{R}^{N-d}$. When the two limits exist and are equal, we say that E admits a d-dimensional Minkowski content, and their common value will be denoted $M^d(E)$.

Definition 5.2. Let E be a subset of a metric space X. We say that E is d-rectifiable if it is the image of a compact subset K of $\mathbb{R}^d$ by a Lipschitz continuous function $f : \mathbb{R}^d \to X$.

Definition 5.3. Let $E \subset \mathbb{R}^N$ be H^d-measurable. We say that E is
1) countably d-rectifiable if there exist countably many Lipschitzian functions $f_i : \mathbb{R}^d \to \mathbb{R}^N$ such that $E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)$.

2) countably H^d-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^d \to \mathbb{R}^N$ such that $E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)$ is H^d-negligible : $H^d\left(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)\right) = 0$.

3) H^d-rectifiable if it is countably H^d-rectifiable and $H^d(E) < \infty$.

Definition 5.4. Let E be a closed subset of $\mathbb{R}^N$.

1) The set of points of $\mathbb{R}^N$ which have a unique projection on E.

$$\mathrm{Unp}(E) = \{y \in \mathbb{R}^N : \exists \text{ an unique } x \in E \text{ such that } d_E(y) = \|y - x\|\},$$

where $d_E(y)$ is the distance function from a point y to E.

2) The reach of a point $x \in E$ and the reach of E are respectively defined as follows :‘

$$\mathrm{reach}(E, x) = \sup\{r > 0 : B(x, r) \subset \mathrm{Unp}(E)\}, \quad \mathrm{reach}(E) = \inf_{x \in E} \mathrm{reach}(E, x).$$

Remark 5.5. We say that E is a positive reach if $\mathrm{reach}(E) > 0$.
The definition of $\mathrm{Unp}(E)$ implies the existence of a projection on E, a function $p_E : \mathrm{Unp}(E) \to E$ which associates $x \in \mathrm{Unp}(E)$ such that $d_E(x) = \|x - p_E(x)\|$.

Definition 5.6. Let E be a closed subset of $\mathbb{R}^N$ such that $E_r \subset \Omega$ for some $r > 0$, and f be a continuous function on a bounded open subset Ω of $\mathbb{R}^N$. The $d - dimensional$ topological derivative of the volume functional

$$V(\chi_\Omega) = \int_{\mathbb{R}^N} f\chi_\Omega \, dx = \int_\Omega f \, dx,$$

with respect to E is defined as follows :

$$dV(\chi_\Omega; \delta E) = \lim_{r \searrow 0} \frac{V(\chi_\Omega \setminus E_r) - V(\chi_\Omega)}{\alpha_{N-d} r^{N-d}},$$

whenever the limit exists.

Theorem 5.7. *Let E be a compact subset of $\mathbb{R}^N$ and $0 \le d < N$ an integer.*
Let α_k denote the volume of the unit ball in $\mathbb{R}^k$. We assume that the following properties hold:
1) *E is a d-rectifiable subset of $\mathbb{R}^N$ such that $\partial E = E$ and $0 < H^d(E) < \infty$;*
2) *E has positive reach; that is, there exists $R > 0$ such that $d_E^2 \in C^{1,1}(E_R)$;*
3) *f is continuous in E_R.*

$$\int_E f \, dH^d = \lim_{r \searrow 0} \frac{1}{(N-d)\alpha_{N-d} r^{N-d}} \int_{E_r} f \, d\psi = \lim_{r \searrow 0} \frac{1}{(N-d)\alpha_{N-d} r^{N-d}} \int_{E_r} f \circ p_E \, d\psi.$$

$$\int_E f \, dH^d = \lim_{r \searrow 0} \frac{1}{(N-d)\alpha_{N-d} r^{N-d-1}} \int_{\partial E_r} f \, dH^{N-1}.$$

In the following theorem $N \ge 2$ is a natural number, d the dimension of a subset E of $\mathbb{R}^N$, r the radius of the ball and α_{N-d} is the volume of the unit ball in $\mathbb{R}^{N-d}$.

5.2 Topological derivative for the functional

In the following, we establish the main result of the topological derivative. To perform this, we first consider the PDE with a Neumann edge condition, then with a Dirichlet edge condition. It is considered that there will be no major change in the two results obtained for the two edge conditions. Except that the spaces we're working with will change from one condition to the other.

Neumann condition

Theorem 5.8. *Let $0 \leq d < N$, $E = \overline{w}$ and $s = \alpha_{N-d} r^{N-d}$. The topological derivative exists if and only if the following limit :*

$$L = \lim_{\epsilon \longrightarrow 0} \left(L_0(\epsilon) + L_1(\epsilon) \right),$$

exists with

$$L_0(\epsilon) = \int_{\Omega_\epsilon} \beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2$$

and

$$L_1(\epsilon) = \frac{1}{s} \left[\int_{\partial \Omega_\epsilon^m \cap \partial \overline{w}_\epsilon} \nabla u_{\Omega_0} . \nabla d\overline{\omega}\, p_0 \, dH^{N-1} \right] dx - \frac{1}{s} \int_{\Omega_\epsilon} \mu \nabla \left(u_{\Omega_\epsilon} - u_{\Omega_0} \right) . \nabla p_\epsilon \, dx.$$

Moreover, the topological derivative of the function is given by the expression :

$$\begin{aligned} DJ(\Omega) &= \lim_{\epsilon \longrightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\alpha_{N-d} r^{N-d}} \\ &= L - \left[\int_{\overline{\omega}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu \nabla u_{\Omega_0} . \nabla p_0 - f p_0 - 3k\alpha \nabla \theta_\Omega p_0 \right] dH^d. \end{aligned}$$

where p_0, u_{Ω_0} are solutions of systems

$$\int_\Omega 2\beta \nabla u_{\Omega_0} . \nabla \phi' dx + \int_\Omega 2\alpha (u_{\Omega_0} - u_0) \phi' dx + \int_\Omega \mu \nabla \phi' . \nabla p_0 \, dx = 0.$$

Proof. Let us associate with ϵ, $0 < \epsilon \leq r$, the perturbed domain $\Omega_\epsilon = \Omega \backslash \overline{\omega_\epsilon}$, where by assumption,
$\partial \Omega_\epsilon = \partial \Omega \cup \partial \overline{\omega_\epsilon}$, and $\partial \Omega \cap \partial \overline{\omega_\epsilon} = \emptyset$ and $\partial \overline{\omega_\epsilon} \in \mathcal{C}^{1,1}$.
In the following, consider also the functional defined in Ω_ϵ, by

$$J(\Omega_\epsilon) = \beta \int_{\Omega_\epsilon} \mid \nabla u_{\Omega_\epsilon} \mid^2 dx + \alpha \int_{\Omega_\epsilon} \mid u_{\Omega_\epsilon} - u_0 \mid^2 dx \tag{5.1}$$

where u_{Ω_ϵ} is the solution to the following problem

$$\begin{cases} -\mu \Delta u_{\Omega_\epsilon} - (\lambda + \mu) \nabla (div \overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha \nabla \theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \Omega_\epsilon \\ u_{\Omega_\epsilon} = 0 \text{ in } \partial \Omega \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} =, \text{ on } \partial \overline{\omega_\epsilon}. \end{cases} \tag{5.2}$$

We can be extended to Ω by introducing the solution $\overline{\omega}_\epsilon^0 \longrightarrow \mathbb{R}$ of the problem

$$-\mu \Delta u_{\Omega_\epsilon} - (\lambda + \mu) \nabla (div \overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha \nabla \theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \overline{\omega}_\epsilon^0 \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial \overline{\omega}_\epsilon. \tag{5.3}$$

We suppose that Ω_ϵ has two components Ω_ϵ^m and Ω_ϵ^0. Ω_ϵ^m is the component of Ω_ϵ for which $\partial \Omega$ is part of its boundary. Ω_ϵ^0 is the blind component of Ω_ϵ whose boundary has an empty intersection with $\partial \Omega$. The function u_{Ω_ϵ} is distributed between the two components Ω_ϵ^0 and Ω_ϵ^m as

$$-\mu \Delta u_{\Omega_\epsilon} - (\lambda + \mu) \nabla (div \overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha \nabla \theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \Omega_\epsilon^m, \tag{5.4}$$

$$\begin{cases} u_{\Omega_\epsilon} = 0 \text{ on } \partial \Omega, \\ u_{\Omega_\epsilon} = 0 \text{ on } \partial \Omega_\epsilon^m \cap \partial \overline{\omega}_\epsilon. \end{cases}$$

Since $\partial\overline{\omega}_\epsilon$ is made up of two disjoint boundary $\partial\Omega_\epsilon^0$ and $\partial\Omega_\epsilon^m \cap \partial\overline{\omega}_\epsilon$, we can construct an extension to Ω by defining the solution $\overline{\omega}_\epsilon^0 \longrightarrow \mathbb{R}$

$$-\mu\Delta u_{\Omega_\epsilon} - (\lambda+\mu)\nabla(div\overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha\nabla\theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \overline{\omega}_\epsilon^0 \text{ and } u^0_{\Omega_\epsilon} = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^m \cap \partial\overline{\omega}_\epsilon \; u^0_{\Omega_\epsilon} = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^0. \tag{5.5}$$

Considering a shape function J defined by (5.1), where $u_\Omega \in H_{div(\Omega,\mathbb{R}^3)}$ is solution to the variational problem

$$\mu\int_\Omega \nabla u_\Omega.\nabla v dx - (\lambda+\mu)\int_\Omega \nabla(\nabla\cdot u_\Omega)v dx - 3k\alpha\int_\Omega \nabla\theta_\Omega v dx = \int_\Omega f v dx \tag{5.6}$$

and

$$\int_\Omega \nabla\theta_\Omega\nabla\varphi dx = \int_\Omega g\varphi dx \tag{5.7}$$

for all $v \in H_{div(\Omega,\mathbb{R}^3)}$ and $\varphi \in H_{div(\Omega,\mathbb{R}^3)}$ for some given functions $f \in H^1(\mathbb{R}^3),\; g \in H^{-1}(\Omega)$. In the case where $div u_\Omega = 0$ in Ω, (5.6) is reduced to the following variational problem

$$\mu\int_\Omega \nabla u_\Omega.\nabla v dx = \int_\Omega f v dx + 3k\alpha\int_\Omega \nabla\theta_\Omega v dx \tag{5.8}$$

and θ_Ω solution to (5.7). The shape functional associated the perforated domain is given by

$$j(\chi_\epsilon(x_0)) = J(\Omega_\epsilon) = \alpha\int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 dx + \beta\int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 dx \tag{5.9}$$

where u_{Ω_ϵ} is solution of the variational problem

$$\mu\int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon}.\nabla v dx - (\lambda+\mu)\int_{\Omega_\epsilon} \nabla(\nabla\cdot u_{\Omega_\epsilon})v dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \tag{5.10}$$

We aim to compute the topological derivative of the functional $J(\Omega_\epsilon)$

$$DJ(\Omega) = \lim_{\epsilon\longrightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\alpha_{N-d} r^{N-d}}.$$

And for this purpose, we define the following set:

$$V_H = \left\{ u_{\Omega_\epsilon} \in (L^2(\Omega))^d, div(u_{\Omega_\epsilon}) \in L^2(\Omega), u_{\Omega_\epsilon} = 0 \text{ on } \partial\Omega, \frac{\partial u_{\Omega_\epsilon}}{\partial n} = \text{ on } \partial\overline{\omega_\epsilon} \right\}. \tag{5.11}$$

Our variational formulation (5.2) consists of finding $u_{\Omega_\epsilon} \in V_H$ such that

$$\mu\int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon}.\nabla v dx - (\lambda+\mu)\int_{\Omega_\epsilon} \nabla(\nabla\cdot u_{\Omega_\epsilon})v dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \tag{5.12}$$

In the case where $div(u_{\Omega_\epsilon}) = 0$ in Ω_ϵ we have

$$\mu\int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon}.\nabla v dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx,\; \forall\, v \in V_H. \tag{5.13}$$

Thus, the Lagrangian dependent on ϵ will be written in the form :

$$L(\epsilon,\phi,\Phi) = \beta\int_{\Omega_\epsilon} |\nabla\phi|^2 dx + \alpha\int_{\Omega_\epsilon} |\phi - u_0|^2 dx + \mu\int_{\Omega_\epsilon} \nabla\phi.\nabla\Phi dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon}\Phi dx - \int_{\Omega_\epsilon} f\Phi dx \tag{5.14}$$

$$J(\Omega_\epsilon) = \inf_{\phi\in V_H} \sup_{\Phi\in V_H} L(\epsilon,\phi,\Phi).$$

From this, we can now evaluate the derivative of the Lagrangian, dependent on ϵ, with respect to ϕ.

$$d_\phi L(\epsilon,\phi,\Phi,\phi') = \int_{\Omega_\epsilon} 2\beta\nabla\phi.\nabla\phi' dx + \int_{\Omega_\epsilon} 2\alpha(\phi-u_0)\phi' dx + \int_{\Omega_\epsilon} \mu\nabla\phi'.\nabla\Phi dx.$$

Subsequently, we obtain the variational formulation of the adjoint state equation given by $d_\phi L(0,u_{\Omega_0},p_0,\phi')=0$, where $u_{\Omega_0}=u_{\Omega_\epsilon}$ for $\epsilon=0$. Find $p_0\in H_0^1(\Omega)$ such that

$$\int_\Omega 2\beta\nabla u_{\Omega_0}.\nabla\phi' dx + \int_\Omega 2\alpha(u_{\Omega_0}-u_0)\phi' dx + \int_\Omega \mu\nabla\phi'.\nabla p_0\ dx = 0. \tag{5.15}$$

Next, we derive the Lagrangian with respect to Φ.

$$d_\Phi L(\epsilon,\phi,\Phi,\Phi') = \int_{\Omega_\epsilon} \mu\nabla\phi.\nabla\Phi'\, dx - \int_{\Omega_\epsilon} f\Phi' dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon}\Phi' dx.$$

The initial state $u_{\Omega_0}=u_\Omega$ is a solution of $d_\Phi L(0,u_{\Omega_0},0,\Phi')=0\ \forall\ \Phi'\in H_0^1$ and in this case, we have:

$$\int_\Omega \mu\nabla u_{\Omega_0}.\nabla\Phi'\, dx - \int_\Omega f\Phi' dx - 3k\alpha\int_\Omega \nabla\theta_\Omega\Phi' dx = 0.$$

Then, we have:

$$\int_\Omega \left[\mu\nabla u_{\Omega_0}.\nabla\Phi' - f\Phi' - 3k\alpha\nabla\theta\Phi'\right] dx = 0.$$

The state u_{Ω_ϵ} for all $\epsilon\geq 0$ satisfies

$$\int_{\Omega_\epsilon} \left[\mu\nabla u_{\Omega_\epsilon}.\nabla\Phi' - f\Phi' - 3k\alpha\nabla\theta\Phi'\right] dx = 0,\ \ \forall\Phi'\in V_H.$$

In the following, we aim to find the derivative of the Lagrangian, with respect to ϵ. To perform this, let us first compute the quotient

$$\frac{L(\epsilon,\phi,\Phi)-L(0,\phi,\Phi)}{s}.$$

$$\begin{aligned} L(\epsilon,\phi,\Phi)-L(0,\phi,\Phi) &= \beta\int_{\Omega_\epsilon}|\nabla\phi|^2dx + \alpha\int_{\Omega_\epsilon}|\phi-u_0|^2dx \\ &+ \mu\int_{\Omega_\epsilon}\nabla\phi.\nabla\Phi dx - 3k\alpha\int_{\Omega_\epsilon}\nabla\theta_{\Omega_\epsilon}\Phi dx - \int_{\Omega_\epsilon} f\Phi dx \\ &- \left[\beta\int_\Omega|\nabla\phi|^2dx + \alpha\int_\Omega|\phi-u_0|^2dx\right] \\ &- \left[\mu\int_\Omega\nabla\phi.\nabla\Phi dx - 3k\alpha\int_\Omega\nabla\theta_\Omega\Phi dx - \int_\Omega f\Phi dx\right] \\ L(\epsilon,\phi,\Phi)-L(0,\phi,\Phi) &= -\left[\int_{\overline{\omega}_\epsilon}\beta|\nabla\phi|^2 + \alpha|\phi-u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dx. \end{aligned}$$

For $d=0, \overline{\omega}=\{x_0\}, \overline{\omega}_\epsilon=\{x\in\mathbb{R}^N : |x-x_0|\leq\epsilon\}=\overline{B}(x_0,\epsilon)$.

$$\begin{aligned} d_sL(0,\phi,\Phi) &= \lim_{s\longrightarrow 0} -\frac{1}{|B(x_0,\epsilon)|}\left[\int_{B(x_0,\epsilon)}\beta|\nabla\phi|^2 + \alpha|\phi-u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dx \\ &= -\beta|\nabla\phi(x_0)|^2 - \alpha|\phi(x_0)-u_0(x_0)|^2 - \mu\nabla\phi(x_0).\nabla\Phi(x_0) \\ &+ f\Phi(x_0) + 3k\alpha\nabla\theta_\Omega\Phi(x_0). \end{aligned}$$

By evaluating the last equation at the point u_{Ω_0}, p_0, we obtain:

$$\begin{aligned} d_s L(0, u_{\Omega_0}, p_0) &= -\beta|\nabla u_{\Omega_0}(x_0)|^2 - \alpha|u_{\Omega_0}(x_0) - u_0(x_0)|^2 - \mu\nabla u_{\Omega_0}(x_0).\nabla p_0(x_0) \\ &+ f p_0(x_0) + 3k\alpha\nabla\theta_\Omega p_0(x_0). \end{aligned}$$

Hence, if $0 < d \leq N-1$, we have:

$$\begin{aligned} \frac{L(\epsilon,\phi,\Phi) - L(0,\phi,\Phi)}{s} &= -\frac{1}{|\overline{\omega}_\epsilon|}\left[\int_{\overline{\omega}_\epsilon} \beta|\nabla\phi|^2 + \alpha|\phi - u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dx \\ &= -\frac{1}{\alpha_{N-d} r^{N-d}}\left[\int_{\overline{\omega}_\epsilon} \beta|\nabla\phi|^2 + \alpha|\phi - u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dx \\ &\longrightarrow -\left[\int_{\overline{\omega}} \beta|\nabla\phi|^2 + \alpha|\phi - u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dH^d. \end{aligned}$$

Therefore, taking the last result at the point u_{Ω_0}, p_0 becomes:

$$d_s L(0, u_{\Omega_0}, p_0) = -\left[\int_{\overline{\omega}} \beta|\nabla u_{\Omega_0}|^2 + \alpha|u_{\Omega_0} - u_0|^2 + \mu\nabla u_{\Omega_0}.\nabla p_0 - f p_0 - 3k\alpha\nabla\theta_\Omega p_0\right] dH^d. \tag{5.16}$$

We will now define $R(\epsilon)$ by

$$R(\epsilon) = \int_0^1 d_x\, L\left(\epsilon, u_{\Omega_0} + \Psi\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right), p_0, \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s}\right)\right)\, d\Psi.$$

By substituting $\phi' = \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\epsilon}$ and $\Psi = \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{2}$ into the adjoint equation for p_0, we obtain:

$$\begin{aligned} R(\epsilon) &= \int_{\Omega_\epsilon} 2\beta\nabla\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2}\right).\nabla\left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s}\right) dx \\ &+ \int_{\Omega_\epsilon} 2\alpha\left[\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2}\right) - u_0\right]\left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s}\right) dx + \int_{\Omega_\epsilon} \mu\nabla\left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s}\right).\nabla p_0\, dx \\ &= \frac{1}{s}\left[\int_{\Omega_\epsilon} 2\beta\nabla\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2}\right).\nabla\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right) dx + 2\alpha\left[\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2}\right) - u_0\right]\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right)\right] dx \\ &+ \frac{1}{s}\left[\int_{\Omega_\epsilon} \mu\nabla\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right).\nabla p_0\right] dx \\ R(\epsilon) &= \frac{1}{s}\left[\int_{\Omega_\epsilon} 2\beta\left(\nabla\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2}\right) - \nabla u_{\Omega_0} + \nabla u_{\Omega_0}\right).\nabla\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right)\right] dx \\ &+ \frac{2\alpha}{s}\int_{\Omega_\epsilon}\left[\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2}\right) - u_{\Omega_0} + u_{\Omega_0} - u_0\right]\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right) dx \\ &+ \frac{1}{s}\left[\int_{\Omega_\epsilon} \mu\nabla\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right).\nabla p_0\right] dx \end{aligned}$$

$$\begin{aligned}
R(\epsilon) &= \frac{1}{s}\left[\int_{\Omega_\epsilon} 2\beta\left(\nabla\left(\frac{u_{\Omega_\epsilon}-u_{\Omega_0}}{2}\right)+\nabla u_{\Omega_0}\right).\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)\right]dx \\
&+ \frac{1}{s}\left[\int_{\Omega_\epsilon} 2\alpha\left[\left(\frac{u_{\Omega_\epsilon}-u_{\Omega_0}}{2}\right)+u_{\Omega_0}-u_0\right]\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)\right]dx \\
&+ \frac{1}{s}\left[\int_{\Omega_\epsilon} \mu\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right).\nabla p_0\right]dx \\
R(\epsilon) &= \int_{\Omega_\epsilon}\left(\beta\left|\nabla\left(\frac{u_{\Omega_\epsilon}-u_{\Omega_0}}{\sqrt{s}}\right)\right|^2+\alpha\left|\frac{u_{\Omega_\epsilon}-u_{\Omega_0}}{\sqrt{s}}\right|^2\right)dx \\
&+ \frac{1}{s}\left[\int_{\Omega_\epsilon} 2\beta\nabla u_{\Omega_0}\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)+2\alpha(u_{\Omega_\epsilon}-u_{\Omega_0})(u_{\Omega_0}-u_0)\right]dx \\
&+ \frac{1}{s}\left[\int_{\Omega_\epsilon} \mu\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right).\nabla p_0\right]dx.
\end{aligned}$$

Thus, for all $u_{\Omega_0}\in H_{div(\Omega)}$, equation (5.13) becomes:

$$\begin{aligned}
\int_{\Omega_\epsilon}\left(\mu\nabla u_{\Omega_\epsilon}.\nabla v-fv-3k\alpha\nabla\theta v\right)dx &= -\left[\int_{\overline{\omega}_\epsilon}\left(\mu\nabla u_{\Omega_\epsilon}.\nabla v-fv-3k\alpha\nabla\theta v\right)dx\right] \\
&= \int_{\partial\overline{\omega}_\epsilon}\frac{\partial u_{\Omega_0}}{\partial n}v\,dH^{N-1},\ \forall\ v\in H_{div(\Omega)}.
\end{aligned}$$

Now, considering the assumption about $\overline{\omega}$, $\partial\overline{\omega}_\epsilon\in\mathcal{C}^{1,1}$ and

$$\frac{\partial u_{\Omega_0}}{\partial n}=\nabla u_{\Omega_0}.n\Omega_\epsilon=-\nabla u_{\Omega_0}.\nabla d\overline{\omega}\ \text{ on }\ \overline{\omega}_\epsilon.$$

And in this case, we have:

$$\int_{\Omega_\epsilon}\left(\mu\nabla u_{\Omega_\epsilon}.\nabla v-fvdx-3k\alpha\nabla\theta v\right)dx=-\int_{\partial\overline{\omega}_\epsilon}\nabla u_{\Omega_0}.\nabla d\overline{\omega}\,v\,dH^{N-1}. \tag{5.17}$$

By taking the difference between equation (5.13) and equation (5.17), we obtain:

$$\int_{\Omega_\epsilon}\mu\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)\nabla v=\int_{\partial\overline{\omega}_\epsilon}\nabla u_{\Omega_0}.\nabla d\overline{\omega}\,v\,dH^{N-1}.$$

The adjoint equation for $\epsilon\geq 0$ yields:

$$\int_{\Omega_\epsilon}2\beta\nabla u_{\Omega_0}.\nabla\phi' dx+\int_{\Omega_\epsilon}2\alpha(u_{\Omega_0}-u_0)\phi' dx+\int_{\Omega_\epsilon}\mu\nabla\phi'.\nabla p_\epsilon\ dx=0. \tag{5.18}$$

By taking $\phi'=u_{\Omega_\epsilon}-u_{\Omega_0}$ in equation (5.18), we have:

$$\int_{\Omega_\epsilon}2\beta\nabla u_{\Omega_0}.\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)dx+\int_{\Omega_\epsilon}2\alpha(u_{\Omega_0}-u_0)\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)dx+\int_{\Omega_\epsilon}\mu\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right).\nabla p_\epsilon\ dx=0.$$

$$\int_{\Omega_\epsilon}2\beta\nabla u_{\Omega_0}.\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)+2\alpha(u_{\Omega_0}-u_0)\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right)dx=-\int_{\Omega_\epsilon}\mu\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right).\nabla p_\epsilon\ dx.$$

The final equation for $R(\epsilon)$ becomes:

$$\begin{aligned}
R(\epsilon) &= \int_{\Omega_\epsilon}\left[\beta\left|\nabla\left(\frac{u_{\Omega_\epsilon}-u_{\Omega_0}}{\sqrt{s}}\right)\right|^2+\alpha\left|\frac{u_{\Omega_\epsilon}-u_{\Omega_0}}{\sqrt{s}}\right|^2\right]dx+\frac{1}{s}\left[\int_{\partial\Omega_\epsilon^m\cap\partial\overline{\omega}_\epsilon}\nabla u_{\Omega_0}.\nabla d\overline{\omega}\,p_0\,dH^{N-1}\right]dx \\
&- \frac{1}{s}\int_{\Omega_\epsilon}\mu\nabla\left(u_{\Omega_\epsilon}-u_{\Omega_0}\right).\nabla p_\epsilon\ dx.
\end{aligned}$$

□

Dirichlet condition

In this section, we consider the case of a Dirichlet condition, and the perturbed problem thus becomes:

$$\begin{cases} -\mu\Delta u_{\Omega_\epsilon} - (\lambda+\mu)\nabla(div\overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha\nabla\theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \Omega_\epsilon \\ u_{\Omega_\epsilon} = 0 \text{ in } \partial\Omega \\ u_{\Omega_\epsilon} = 0 \text{ on } \partial\overline{\omega_\epsilon}. \end{cases} \tag{5.19}$$

First of all, the calculations remain the same. The changes are simply in the considered spaces . The techniques for calculating the derivative of the Lagrangian with respect to the variables remain the same. And for this, we don't need to go into all the details, as we did in the case of a Neumann condition. And We can be extended to Ω by introducing the solution $\overline{\omega}_\epsilon^0 \longrightarrow \mathbb{R}$ of the problem

$$-\mu\Delta u_{\Omega_\epsilon} - (\lambda+\mu)\nabla(div\overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha\nabla\theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \overline{\omega}_\epsilon^0 \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\,\overline{\omega}_\epsilon. \tag{5.20}$$

We suppose that Ω_ϵ has two components Ω_ϵ^m and Ω_ϵ^0. Ω_ϵ^m is the component of Ω_ϵ for which $\partial\Omega$ is part of its boundary. Ω_ϵ^0 is the blind component of Ω_ϵ whose boundary has an empty intersection with $\partial\Omega$. The function u_{Ω_ϵ} is distributed between the two components Ω_ϵ^0 and Ω_ϵ^m as

$$-\mu\Delta u_{\Omega_\epsilon} - (\lambda+\mu)\nabla(div\overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha\nabla\theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \Omega_\epsilon^m, \tag{5.21}$$

$$\begin{cases} u_{\Omega_\epsilon} &= 0 \text{ on } \partial\Omega, \\ u_{\Omega_\epsilon} &= 0 \text{ on } \partial\Omega_\epsilon^m \cap \partial\overline{\omega}_\epsilon, \end{cases}$$

Since $\partial\overline{\omega}_\epsilon$ is made up of two disjoint boundary $\partial\Omega_\epsilon^0$ and $\partial\Omega_\epsilon^m \cap \partial\overline{\omega}_\epsilon$, we can construct an extension to Ω by defining the solution $\overline{\omega}_\epsilon^0 \longrightarrow \mathbb{R}$

$$-\mu\Delta u_{\Omega_\epsilon} - (\lambda+\mu)\nabla(div\overrightarrow{u_{\Omega_\epsilon}}) + 3k\alpha\nabla\theta_{\Omega_\epsilon} = \overrightarrow{f} \text{ in } \overline{\omega}_\epsilon^0 \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^m \cap \partial\overline{\omega}_\epsilon \;\; u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^0. \tag{5.22}$$

Considering a shape function J defined by (5.1), where $u_\Omega \in H_{div(\Omega,\mathbb{R}^3)}$ is solution to the variational problem

$$\mu\int_\Omega \nabla u_\Omega.\nabla v dx - (\lambda+\mu)\int_\Omega \nabla(\nabla\cdot u_\Omega)v dx - 3k\alpha\int_\Omega \nabla\theta_\Omega v dx = \int_\Omega f v dx \tag{5.23}$$

and

$$\int_\Omega \nabla\theta_\Omega\nabla\varphi dx = \int_\Omega g\varphi dx \tag{5.24}$$

for all $v \in H_{div(\Omega,\mathbb{R}^3)}$ and $\varphi \in H_{div(\Omega,\mathbb{R}^3)}$ for some given functions $f \in H^1(\mathbb{R}^3),\ g \in H^{-1}(\Omega)$. In the case where $div u_\Omega = 0$ in Ω, (5.23) is reduced to the following variational problem

$$\mu\int_\Omega \nabla u_\Omega.\nabla v dx = \int_\Omega f v dx + 3k\alpha\int_\Omega \nabla\theta_\Omega v dx \tag{5.25}$$

and θ_Ω solution to (5.24). The shape functional associated the perforated domain is given by

$$J(\Omega_\epsilon) = \alpha\int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 dx + \beta\int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 dx \tag{5.26}$$

where u_{Ω_ϵ} is solution of the variational problem

$$\mu\int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon}.\nabla v dx - (\lambda+\mu)\int_{\Omega_\epsilon} \nabla(\nabla\cdot u_{\Omega_\epsilon})v dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \tag{5.27}$$

We aim to compute the topological derivative of the functional $J(\Omega_\epsilon)$

$$DJ(\Omega) = \lim_{\epsilon\longrightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\alpha_{N-d} r^{N-d}}.$$

And for this purpose, we define the following set:

$$V_\epsilon = \left\{u_\Omega \in (L^2(\Omega_\epsilon))^d, div(u_\Omega) \in L^2(\Omega_\epsilon), u_\Omega = 0 \text{ on } \partial\Omega, u_{\Omega_\epsilon} = 0 \text{ on } \partial\overline{\omega_\epsilon}\right\}. \tag{5.28}$$

Our variational formulation (5.19) consists of finding $u_{\Omega_\epsilon} \in V_\epsilon$ such that

$$\mu \int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon}.\nabla v dx - (\lambda+\mu)\int_{\Omega_\epsilon} \nabla(\nabla\cdot u_{\Omega_\epsilon}) v dx - 3k\alpha \int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \tag{5.29}$$

In the case where $div(u_{\Omega_\epsilon}) = 0$ in Ω_ϵ we have

$$\mu \int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon}.\nabla v dx - 3k\alpha \int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx,\ \forall\ v \in V_\epsilon. \tag{5.30}$$

Thus, the Lagrangian dependent on ϵ will be written in the form :

$$L(\epsilon,\phi,\Phi) = \beta\int_{\Omega_\epsilon} |\nabla\phi|^2 dx + \alpha\int_{\Omega_\epsilon} |\phi-u_0|^2 dx + \mu\int_{\Omega_\epsilon} \nabla\phi.\nabla\Phi dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon}\Phi dx - \int_{\Omega_\epsilon} f\Phi dx \tag{5.31}$$

From this, we can now evaluate the derivative of the Lagrangian, dependent on ϵ, with respect to ϕ.

$$d_\phi L(\epsilon,\phi,\Phi,\phi') = \int_{\Omega_\epsilon} 2\beta\nabla\phi.\nabla\phi' dx + \int_{\Omega_\epsilon} 2\alpha(\phi-u_0)\phi' dx + \int_{\Omega_\epsilon} \mu\nabla\phi'.\nabla\Phi dx.$$

Subsequently, we obtain the variational formulation of the adjoint state equation given by $d_\phi L(0, u_{\Omega_0}, p_0, \phi') = 0$, where $u_{\Omega_0} = u_{\Omega_\epsilon}$ for $\epsilon = 0$. Find $p_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega 2\beta\nabla u_{\Omega_0}.\nabla\phi' dx + \int_\Omega 2\alpha(u_{\Omega_0} - u_0)\phi' dx + \int_\Omega \mu\nabla\phi'.\nabla p_0\ dx = 0. \tag{5.32}$$

Next, we derive the Lagrangian with respect to Φ.

$$d_\Phi L(\epsilon,\phi,\Phi,\Phi') = \int_{\Omega_\epsilon} \mu\nabla\phi.\nabla\Phi'\, dx - \int_{\Omega_\epsilon} f\Phi' dx - 3k\alpha\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon}\Phi' dx.$$

The initial state $u_{\Omega_0} = u_\Omega$ is a solution of $d_\Phi L(0, u_{\Omega_0}, 0, \Phi') = 0\ \forall\ \Phi' \in H_0^1$ and in this case, we have:

$$\int_\Omega \mu\nabla u_{\Omega_0}.\nabla\Phi'\, dx - \int_\Omega f\Phi' dx - 3k\alpha\int_\Omega \nabla\theta_\Omega\Phi' dx = 0.$$

Then, we have:

$$\int_\Omega \left[\mu\nabla u_{\Omega_0}.\nabla\Phi' - f\Phi' - 3k\alpha\nabla\theta\Phi'\right] dx = 0.$$

The state u_{Ω_ϵ} for all $\epsilon \geq 0$ satisfies

$$\int_{\Omega_\epsilon} \left[\mu\nabla u_{\Omega_\epsilon}.\nabla\Phi' - f\Phi' - 3k\alpha\nabla\theta\Phi'\right] dx = 0,\ \ \forall\Phi' \in V_\epsilon.$$

In the following, we aim to find the derivative of the Lagrangian, with respect to ϵ. To achieve this, let us first compute the quotient

$$\frac{L(\epsilon,\phi,\Phi) - L(0,\phi,\Phi)}{s}.$$

$$L(\epsilon,\phi,\Phi) - L(0,\phi,\Phi) \quad = \quad -\left[\int_{\overline{\omega}_\epsilon} \beta|\nabla\phi|^2 + \alpha|\phi-u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dx.$$

For $d = 0$, $\overline{\omega} = \{x_0\}$, $\overline{\omega}_\epsilon = \{x \in \mathbb{R}^N : |x - x_0| \leq \epsilon\} = \overline{B}(x_0,\epsilon)$.

$$\begin{aligned} d_s L(0,\phi,\Phi) \quad &= \quad -\beta|\nabla\phi(x_0)|^2 - \alpha|\phi(x_0) - u_0(x_0)|^2 - \mu\nabla\phi(x_0).\nabla\Phi(x_0) \\ &+ \quad f\Phi(x_0) + 3k\alpha\nabla\theta_\Omega\Phi(x_0). \end{aligned}$$

By evaluating the last equation at the point u_{Ω_0}, p_0, we obtain:

$$\begin{aligned} d_s L(0, u_{\Omega_0}, p_0) &= -\beta|\nabla u_{\Omega_0}(x_0)|^2 - \alpha|u_{\Omega_0}(x_0) - u_0(x_0)|^2 - \mu\nabla u_{\Omega_0}(x_0).\nabla p_0(x_0) \\ &+ f p_0(x_0) + 3k\alpha\nabla\theta_\Omega p_0(x_0). \end{aligned}$$

Hence, if $0 < d \leq N-1$, we have:

$$\frac{L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi)}{s} \longrightarrow -\left[\int_{\overline{\omega}} \beta|\nabla\phi|^2 + \alpha|\phi - u_0|^2 + \mu\nabla\phi.\nabla\Phi - f\Phi - 3k\alpha\nabla\theta_\Omega\Phi\right] dH^d.$$

Therefore, taking the last result at the point u_{Ω_0}, p_0 becomes:

$$d_s L(0, u_{\Omega_0}, p_0) = -\left[\int_{\overline{\omega}} \beta|\nabla u_{\Omega_0}|^2 + \alpha|u_{\Omega_0} - u_0|^2 + \mu\nabla u_{\Omega_0}.\nabla p_0 - f p_0 - 3k\alpha\nabla\theta_\Omega p_0\right] dH^d.$$

We define $R(\epsilon)$ by

$$R(\epsilon) = \int_0^1 d_x\, L\left(\epsilon, u_{\Omega_0} + \Psi\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right), p_0, \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s}\right)\right) d\Psi.$$

The final expression of $R(\epsilon)$ is geven by:

$$\begin{aligned} R(\epsilon) &= \int_{\Omega_\epsilon} \left[\beta\left|\nabla\left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}}\right)\right|^2 + \alpha\left|\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}}\right|^2\right] dx + \frac{1}{s}\left[\int_{\partial\Omega_\epsilon^m \cap \partial\overline{w}_\epsilon} \nabla u_{\Omega_0}.\nabla d\overline{\omega}\, p_0\, dH^{N-1}\right] dx \\ &- \frac{1}{s}\int_{\Omega_\epsilon} \mu\nabla\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right).\nabla p_\epsilon\, dx. \end{aligned}$$

Theorem 5.9. *Let $0 \leq d < N$, $E = \overline{w}$ and $s = \alpha_{N-d} r^{N-d}$. The topological derivative exists if and only if the following limit :*

$$L = \lim_{\epsilon \longrightarrow 0} \left(L_0(\epsilon) + L_1(\epsilon)\right),$$

exists with

$$L_0(\epsilon) = \int_{\Omega_\epsilon} \beta\left|\nabla\left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}}\right)\right|^2 + \alpha\left|\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}}\right|^2$$

and

$$L_1(\epsilon) = \frac{1}{s}\left[\int_{\partial\Omega_\epsilon^m \cap \partial\overline{w}_\epsilon} \nabla u_{\Omega_0}.\nabla d\overline{\omega}\, p_0\, dH^{N-1}\right] dx - \frac{1}{s}\int_{\Omega_\epsilon} \mu\nabla\left(u_{\Omega_\epsilon} - u_{\Omega_0}\right).\nabla p_\epsilon\, dx.$$

Moreover, the topological derivative of the function is given by the expression :

$$\begin{aligned} DJ(\Omega) &= \lim_{\epsilon \longrightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\alpha_{N-d} r^{N-d}} \\ &= L - \left[\int_{\overline{\omega}} \beta|\nabla u_{\Omega_0}|^2 + \alpha|u_{\Omega_0} - u_0|^2 + \mu\nabla u_{\Omega_0}.\nabla p_0 - f p_0 - 3k\alpha\nabla\theta_\Omega p_0\right] dH^d. \end{aligned}$$

where p_0, u_{Ω_0} are solutions of systems

$$\int_\Omega 2\beta\nabla u_{\Omega_0}.\nabla\phi' dx + \int_\Omega 2\alpha(u_{\Omega_0} - u_0)\phi' dx + \int_\Omega \mu\nabla\phi'.\nabla p_0\, dx = 0.$$

6 Conclusion remarks

In this work, we studied a linear thermoelasticity problem using optimization methods. After giving a thermoelasticity model and recalling the existence of a weak solution to this problem, we established the existence of an optimal shape of the minimization problem. Finally, the shape and topological derivatives have also been established using the minmax method.

As future work, we intend to study the regularity problems and numerical methods of the proposed problem.

Conflict of interest

The authors declare that they have no conflict ot interest.

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