A STUDY OF NIL HECKE ALGEBRAS VIA HOPF ALGEBROIDS

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Abstract. We show how the comultiplication defined by Kostant and Kumar turns the affine nil Hecke algebra associated to a Coxeter system into a Hopf algebroid without an antipode. The proof relies on mixed dihedral braid relations between Demazure operators and simple reflections.

1　Introduction: Bialgebroids as bialgebras for certain fiber functors

Let k be a field, let H be a k-algebra and H-**Mod** its category of left H-modules. Pairs of a monoidal structure on H-**Mod** together with a fiber functor from H-**Mod** to some base monoidal category correspond to an algebraic structure on H generalizing that of a bialgebra, cf. [1]. The following tabular gives an overview of this correspondence:

base of fiber functor	setup	structure on H
$(\mathbf{Vect}_k, \otimes_k, k)$		k-bialgebra,
$(R\text{-}\mathbf{Mod}, \otimes_R, R)$	R commutative k-algebra,	(Sweedler) R-bialgebroid,
$(R\text{-}\mathbf{Bim}\text{-}R, \otimes_R, R)$	R any k-algebra	(Takeuchi) R-bialgebroid.

A bialgebroid H can be thought of as an extension of the notion of a bialgebra to two different less commutative settings. For Sweedler bialgebroids [2], the base algebra $R \hookrightarrow H$ is still commutative, but not necessarily central, while the base algebra R of a Takeuchi bialgebroid [3] is not-necessarily commutative and encoded in an algebra morphism $R \otimes_k R^{\mathrm{op}} \to H$. The framework of bialgebroids enables access to powerful Hopf-algebraic tools, such as the construction of dual Hopf algebroids and Drinfeld doubles [4], Tannakian reconstruction [5], Hopf modules [1], coideal subalgebras [6], and the Gerstenhaber algebra structure on cohomology [7]. Moreover the presence of a Hopf algebroid structure offers insights into algebro-combinatorial identities, analogous to results in the study of combinatorial Hopf algebras [8, 9]. This perspective is the one we take in this paper. We focus on Sweedler bialgebroids, which we simply call bialgebroids. Bialgebroids over a commutative k-algebra R are triples (H, Δ, ε), where H is a k-algebra H containing R with a comultiplication $\Delta \colon H \to H \otimes_R H$ and a counit $\varepsilon \colon H \to R$ satisying certain conditions (see Definition 2.6). These generalize the notation of an R-bialgebra, without requiring H to be an R-algebra. This is a generalization in three ways. First, the counit ε is not required to be an algebra morphism. This allows for more interesting actions of H on $\mathbb{1} = R$. Second, the comultiplication $\Delta \colon H \to H \otimes_R H$ is required to land in the Takeuchi product $H \times_R H \subseteq H \otimes_R H$. This algebraic condition is often non-trivial to check and corresponds in natural examples to some weak commutativity between elements of R and H. Finally, the existence of an antipode, which turns bialgebras into Hopf algebras, is not the condition one requires for bialgebroids to become Hopf algebroids. Instead, one requires the existence of an inverse for the Galois map. This inverse may be induced by one (or multiple!) antipodes $S \colon H \to H^{\mathrm{op}}$ or no antipode.

Overview and results

We establish that the nil Hecke algebra $\mathrm{n}\mathcal{H}(W, S)$ associated to a Coxeter system (W, S) is a Hopf algebroid without an antipode. Section 2 recalls the definitions and some elementary facts on Hopf algebroids. Experts on these topics should come there for the notation, specifically for Notation 2.2, where we explain what red and blue tensor symbols mean. Section 3 contains some of the most important examples of bialgebroids coming from ring theory, representation theory, and algebraic geometry. We discuss how the definition of bialgebra relates to that of a bialgebroid (Example 3.1), matrix algebras and semisimple algebras (Examples 3.2, 3.3), path algebras of quivers (Example 3.4), and finally Weyl algebras and differential operators (Examples 3.6, 3.7). Lastly, we mention two general constructions of bialgebroids – endomorphism algebras under freeness assumptions in Example 3.8, and skew group algebras in Example 3.9. Examples 3.6, 3.8, 3.9 all provide helpful intuition for Section 4. In Section 4 we show that the nil Hecke algebra introduced in [10] associated to a Coxeter system (W, S) and its geometric representation $\mathfrak{h}^*$ becomes a cocommutative Hopf algebroid over $R = \mathrm{Sym}(\mathfrak{h}^*)$ without antipode (Theorem 5.8, Corollary 5.11). This structure relies on mixed dihedral braid relations between Demazure operators and simple reflections; see Theorem 5.5. We prove these relations indirectly via the embedding $\mathrm{n}\mathcal{H} \hookrightarrow Q \star W$ of the nil Hecke algebra $\mathrm{n}\mathcal{H}$ into the twisted group algebra $Q \star W$

of W acting on the fraction field $Q = \mathbf{Frac}(R)$, and using the comultiplication on $Q \star W$ from [10]. In the case where W is finite, we conjecture that this Hopf algebroid structure corresponds to the monoidal structure $\otimes_R$ on $n\mathcal{H}$-**Mod**, which makes the Morita equivalence $n\mathcal{H}$-**Mod** $\to \mathrm{Sym}(\mathfrak{h}^*)^W$-**Mod** strong monoidal, see Conjecture 5.10.

2 Preliminaries on bialgebroids and Hopf algebroids

This section is a recollection of definitions and facts from [2]. We start with fixing some notation.

Notation 2.1. In this paper all rings/algebras are associative and unital, but not necessarily commutative. Ring and algebra morphisms are unital. Throughout this paper k denotes a field, R a commutative k-algebra, H a k-algebra with an embedding of algebras $R \subseteq H$. We regard H as R-R-bimodule via $r \cdot h \cdot r' = rhr'$ where $r, r' \in R$, $h \in H$. We set $H^{\mathrm{op}} \coloneqq H$ both as a k-vector space and R-R-bimodule, and view it as k-algebra via the opposite product $\cdot_{\mathrm{op}}$. In general, the bimodule structure and opposite product are not compatible; for instance, for $r \in R$, $h, h' \in H^{\mathrm{op}}$ we have $r \cdot (h \cdot_{\mathrm{op}} h') = rh'h$, while $(r \cdot h) \cdot_{\mathrm{op}} h' = h'rh$.

Notation 2.2 (Blue and red tensor products). Given $M = {}_R M, N = {}_R N$ two left R-modules we will write $M \otimes N \coloneqq {}_R M \otimes_{R \ R} N$ for the tensor product of left R-modules. If $M = M_R$ is a right R-module, we write $M \boxtimes N \coloneqq M_R \otimes_{R \ R} N$ for the balanced tensor product. The same color scheme applies to tensoring morphisms and elements. As a mnemonic compare blue circle $\simeq$ left, red square $\simeq$ right, where the left/right refers to the module M. For everyone reading the article in black and white: $\otimes$ will from now on be always as in this definition. Every other tensor product (for instance over k) will be highlighted (as e.g. $\otimes_k$).

Definition 2.3. We define subsets $H \times_R H \subseteq H \otimes H, H \times_{R^{\mathrm{op}}} H^{\mathrm{op}} \subseteq H \boxtimes H^{\mathrm{op}}$ by setting

$$H \times_R H \coloneqq \left\{ \sum_i h_i \otimes h'_i \; \middle| \; \forall r \in R \colon \sum_i (h_i r) \otimes h'_i = \sum_i h_i \otimes (h'_i r) \right\},$$

$$H \times_{R^{\mathrm{op}}} H^{\mathrm{op}} \coloneqq \left\{ \sum_i h_i \boxtimes h'_i \; \middle| \; \forall r \in R \colon \sum_i (r h_i) \boxtimes h'_i = \sum_i h_i \boxtimes (h'_i r) \right\}.$$

Both are called **Takeuchi products**.

The following remark motivates Definition 2.3 and Lemma 2.5.

Remark 2.4. Let M, N be two left H-modules. We want to turn $M \otimes N$ respectively $\mathrm{Hom}_R(M, N)$ into left H-modules. However, first, we want to turn them into modules over some tensor product $H \otimes_R H$ or $H \otimes_R H^{\mathrm{op}}$ of algebras and then, in the second step, pull back this action to H via some algebra morphism. At first sight, the candidates to act on these spaces are $H \otimes H$ respectively $H \boxtimes H^{\mathrm{op}}$. However, the naive component-wise multiplication is not well-defined for both. The Takeuchi products make this idea work.

The following lemma is straightforward to check.

Lemma 2.5. *The following hold:*
  i) Both $H \times_R H$ and $H \times_{R^{\mathrm{op}}} H^{\mathrm{op}}$ become k-algebras via the component-wise multiplication, moreover they contain R as subalgebra $R \cdot 1 \otimes 1 \subseteq H \otimes H$ respectively $R \cdot 1 \boxtimes 1 \subseteq H \boxtimes H^{\mathrm{op}}$.
  ii) Let M, N be two left H-modules. Then $M \otimes N$ is canonically a left $H \times_R H$-module and $\mathrm{Hom}_R(M, N)$ a left $H \times_{R^{\mathrm{op}}} H^{\mathrm{op}}$-module.

Proof. Part i) is true by direct inspection. The actions in part ii) are given on simple tensors by the following formulas
$$(h \otimes h')(m \otimes n) = (hm) \otimes (h'n), \qquad ((h \boxtimes h') \cdot f)(m) = hf(h'm).$$
Here $h \otimes h' \in H \times_R H$, $h \boxtimes h' \in H \boxtimes H$, $m \in M$, $n \in N$ and $f \in \mathrm{Hom}_R(M, N)$. $\qquad \square$

Definition 2.6 ([2, Definition 5.6]). We call H a **(left) bialgebroid** if it is equipped with morphisms of left R-modules $\Delta \colon H \to H \otimes H$, $\varepsilon \colon H \to R$, such that:
  i) The triple (H, Δ, ε) is a coassociative, counital coalgebra in left R-modules.
  ii) The comultiplication maps into the Takeuchi product $\Delta \colon H \to H \times_R H$ and is a k-algebra morphism when regarded in this way.
  iii) The map
  $$\rho_\varepsilon \colon H \to \mathrm{End}_k(R), \quad h \mapsto (r \mapsto h(r) \coloneqq \varepsilon(hr)) \tag{2.1}$$
  turns R into a left H-representation such that the restriction $\rho_\varepsilon\big|_R$ is the regular representation $\rho_{\mathrm{reg}} \colon R \to \mathrm{End}_k(R)$ of R on R.

A bialgebroid is called **Hopf** or simply **Hopf algebroid** if the Galois map given by the composition

$$\mathrm{Gal}_H \colon H \boxtimes H \xrightarrow{\Delta \boxtimes \mathrm{id}_H} H \otimes H \boxtimes H \xrightarrow{\mathrm{id}_H \otimes \mu} H \otimes H$$

is bijective, where $\mu \colon H \boxtimes H \to H$ is the multiplication map. We define the **translation map** $\mathrm{tak}_H \colon H \to H \boxtimes H$ by $\mathrm{tak}_H(h) \coloneqq \mathrm{Gal}_H^{-1}(h \otimes 1)$.

See Example 3.1 and picture 3.1 for the diagrammatic depiction of the Galois map, its inverse and the translation map in the Hopf algebra case.

Remark 2.7. Condition ii) is a natural assumption on Δ by the discussion in Remark 2.4. Condition iii) is a weakening of $\varepsilon\colon H \to R$ being an R-algebra morphism. Condition iii) can be rephrased in ring theoretic terms as two conditions. First, $\ker(\varepsilon)$ should be a left ideal of H, and second, the counit should satisfy $\varepsilon\big|_R = \mathrm{id}_R$.

Remark 2.8. The terminology "Galois map" comes from the theory of Hopf–Galois extensions. The translation map tak_H associated with a Hopf algebroid H is denoted τ in the literature. One can check that tak_H maps to $H \times_{R^{\mathrm{op}}} H$ and is an algebra morphism, when regarded as a map $\mathrm{tak}_H\colon H \to H \times_{R^{\mathrm{op}}} H^{\mathrm{op}}$. Since Gal_H is right H-linear, the inverse Gal_H^{-1} if it exists is also right H-linear. Hence Gal_H^{-1} is uniquely determined by tak_H, which can be defined on k-algebra generators of H.

Lemma 2.9. *Let (H, Δ, ε) be an R-bialgebroid. The category H-**Mod** of left H-modules becomes monoidal via the tensor product $\otimes$ with unit $\mathbb{1} = (R, \rho_\varepsilon)$. The forgetful functor $\mathrm{res}_R^H\colon H$-**Mod** $\to R$-**Mod** is strict monoidal. If H is Hopf, the R-module $\mathrm{Hom}_R(M, N)$ becomes an H-module and the internal Hom in $(H$-**Mod**, $\otimes, \mathbb{1})$.*

Proof. See [4, Theorem 3.5] for the original result and [11, Remark 4.26] for a short discussion of this fact using notation more similar to ours. □

3 Examples of bialgebroids

In this section we discuss several examples of Hopf algebroids. Unless stated otherwise they are special cases of general examples from [2].

Example 3.1. An R-bialgebra is exactly an R-bialgebroid which satisfies $R \subseteq \mathrm{Z}(H)$. In this case we have $H \times_R H = H \otimes H = H \boxtimes H = H \times_{R^{\mathrm{op}}} H$ as sets. In this sense, a bialgebroid is a less commutative object than a bialgebra. Additionally a bialgebra is a Hopf algebra, i.e. admits an antipode $S\colon H \to H$, if and only if it is a Hopf algebroid. Indeed, given a Hopf algebra one sets $\mathrm{tak}_H(h) = h_{(1)} \boxtimes S(h_{(2)})$, where $\Delta(h) = h_{(1)} \otimes h_{(2)}$ in Sweedler notation. On the other hand given tak_H one can recover the antipode as $S := (\varepsilon \boxtimes \mathrm{id}_H) \circ \mathrm{tak}_H$. Note that this is only well-defined since ε becomes right R-linear in this case. Once can summarize all involved maps diagrammatically as

$$\mathrm{Gal}_H = \;\;\includegraphics , \quad \mathrm{Gal}_H^{-1} = \;\;\boxed{S}\;\; , \quad \mathrm{tak}_H = \;\;\boxed{S}\;\; . \tag{3.1}$$

Note that these diagrams only make sense, since all the tensor products agree, i.e. we work in a consistent ambient monoidal category.

Example 3.2. Consider the embedding $R = k^n \subseteq H = \mathrm{M}_{n \times n}(k)$ of diagonal matrices in all $n \times n$-matrices. Then H is an R-bialgebroid via the comultiplication and counit

$$\Delta\colon \mathrm{M}_{n \times n}(k) \to \mathrm{M}_{n \times n}(k) \times_{k^n} \mathrm{M}_{n \times n}(k), \quad E_{ij} \mapsto E_{ij} \otimes E_{ij}$$

$$\varepsilon\colon \mathrm{M}_{n \times n}(k) \to k^n, \quad E_{ij} \mapsto e_i := E_{ii}$$

where $(E_{ij})_{1 \le i,j \le n}$ is the standard basis of H. The map ρ_ε becomes the algebra isomorphism $\mathrm{M}_{n \times n}(k) \cong \mathrm{End}_k(k^n)$ given by the choice of the standard basis of k^n. The counit ε "evaluates" a matrix at the identity matrix, i.e. right multiplies with the column vector $(1, \ldots, 1)^\intercal$. The map $\mathrm{tak}_H\colon E_{ij} \mapsto E_{ij} \boxtimes E_{ji}$ shows that $\mathrm{M}_{n \times n}(k)$ is even a Hopf algebroid. The monoidal structure on $\mathrm{M}_{n \times n}(k)$-modules is built in such a way that the Morita equivalence of vector spaces and left modules over $\mathrm{M}_{n \times n}(k)$

$$k^n \otimes_k - \colon \mathbf{Vect}_k \to \mathrm{M}_{n \times n}(k)\text{-}\mathbf{Mod}, \qquad V \mapsto k^n \otimes_k V$$

becomes a closed monoidal equivalence. One can alter this example by choosing a different basis of k^n.

Example 3.3 (Generalization I of Example 3.2). Assume $k = \bar{k}$. Similarly to Example 3.2, every semisimple k-algebra H is an R-Hopf algebroid, where $R \subseteq H$ corresponds to the product of diagonal matrix subalgebras under an Artin–Wedderburn isomorphism

$$\prod_{i=1}^r k^{n_i} \cong R \subseteq H \cong \prod_{i=1}^r \mathrm{M}_{n_i \times n_i}(k),$$

where Δ and ε can be calculated explicitly, provided one knows the isomorphism. In this case, Δ defines an isomorphism $H \to H \times_R H$ of algebras. Tensoring any representation with an irreducible one over R returns the corresponding isotypical component of the given representation. As a special case, the Iwahori–Hecke algebra $\mathcal{H}_q(S_n)$ for generic q (see [12, Thm 7.4.7]) can be turned into a Hopf algebroid over its the subalgebra generated by the Jucys–Murphy elements (c.f. [13, §22.3]).

We are not aware of a reference for the following example and it has not been told to the author by anyone.

Example 3.4. For any quiver $Q = (Q_0, Q_1, s, t)$ the path algebra $H = kQ$ becomes a bialgebroid over $R = kQ_0$. The comultiplication Δ maps each path γ to $\gamma \otimes \gamma$, and the counit maps each γ to $e_{t(\gamma)}$ the idempotent constant path corresponding to the target vertex of γ. This bialgebroid structure corresponds on the side of representation theory to the monoidal structure which tensors two quiver representations vertex-wise; that is, for two quiver representations $V = ((V_i)_{i \in Q_0}, (\varphi_{\alpha, V})_{\alpha \in Q_1})$ one has $(V \otimes W)_i = V_i \otimes_k W_i$ for $i \in Q_0$ and $\varphi_{\alpha, V \otimes W} = \varphi_{\alpha, V} \otimes_k \varphi_{\alpha, W}$. The monoidal unit $\mathbb{1}$ is as vector space kQ_0, but not equipped with the trivial action, and instead equipped with the augmentation action ρ_ε, i.e. k is assigned to each vertex and the identity is assigned to each arrow. Almost no path algebras of quivers are Hopf algebroids. If there is any arrow $\alpha \in Q_1$, then $\alpha \otimes e_{t(\alpha)}$ has no preimage under the Galois map. Hence, a path algebra kQ is a Hopf algebroid if and only if $Q_1 = \emptyset$, i.e. Q is a disjoint union of points. In this case $R = kQ_0 = kQ = H$. The above point-wise tensor product of quiver representations has also been studied in [14], the bialgebroid approach can be understood as applying the concept of matrix algebras from [15] to [14, §4], see also the remark below.

Remark 3.5. We make three remarks on Example 3.4. As discussed, we have $\mathbb{1} = kQ_0$ as a vector space, but $\mathbb{1}$ is not the representation $kQ_0 = \bigoplus_{i \in Q_0} S_i$, which appears when studying Koszul duality. This fact also becomes obvious by the Eckmann–Hilton argument, which shows that the Ext-algebra of the monoidal unit $\mathbb{1}$ becomes graded commutative, which in general does not apply to the Koszul dual $(kQ)^!$. Second, note that the bialgebroid structure on kQ does not descend to a bialgebroid structure on quotients kQ/I. When I consists of monomial relations, Δ as in Example 3.4 is still well-defined, however ε is not. Hence, $(kQ/I\text{-Mod}, \otimes)$ becomes a monoidal category without a unit, also called a semigroup category. Finally Example 3.4 is just a special case of a "category algebra", i.e. the matrix algebra $[\mathcal{C}]$ (c.f. [15]) associated to any small category $\mathcal{C}$. Category algebras $[\mathcal{C}]$ with the diagonal comultiplication are always bialgebroids over $k\mathrm{Ob}(\mathcal{C})$ (the subalgebra spanned by all identity morphisms). Moreover they are Hopf if and only if $\mathcal{C}$ is a groupoid.

Example 3.6. Consider the Weyl algebra $\mathcal{W}_n$, that is the k-algebra with $2n$ generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ and relations
$$x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i x_j = x_j \partial_i + \delta_{i,j}.$$
Then $H = \mathcal{W}_n$ becomes a $k[x_1, \ldots, x_n]$-Hopf algebroid by setting
$$\Delta \colon \mathcal{W}_n \to \mathcal{W}_n \times_{k[x_1, \ldots, x_n]} \mathcal{W}_n, \quad \partial_i \mapsto \partial_i \otimes 1 + 1 \otimes \partial_i$$
$$\varepsilon \colon \mathcal{W}_n \to k[x_1, \ldots, x_n], x_1^{a_1} \cdots x_n^{a_n} \partial_1^{b_1} \cdots \partial_n^{b_n} \mapsto \delta_{0, b_1 + b_2 + \cdots + b_n} x_1^{a_1} \cdots x_n^{a_n}.$$
It is a standard but good exercise to check that Δ indeed maps into the Takeuchi-product. The corresponding representation ρ_ε is the polynomial representation of $\mathcal{W}_n$, where the ∂_i's act by partial derivatives on $k[x_1, \ldots, x_n]$. We have $\mathrm{tak}_{\mathcal{W}_n}(\partial_i) = \partial_i \boxtimes 1 - 1 \boxtimes \partial_i$.

Example 3.7. One may replace $k[x_1, \ldots, x_n]$, seen as the coordinate ring of the n-dimensional affine space $\mathbb{A}^n$, with the coordinate ring R of any smooth affine variety X. Then the algebra $H = \mathrm{Diff}(R)$ of differential operators on X becomes an R-Hopf algebroid. Concretely $\mathrm{Diff}(R)$ is defined as
$$\mathrm{Diff}(R) := \bigcup_{i \geq 0} \mathrm{Diff}_i(R), \quad \mathrm{Diff}_i(R) := \{f \in \mathrm{End}_k(R) \mid \ker(\mu)^{i+1} \cdot f = 0\},$$
where $\mu \colon R \otimes_k R \to R$ is the multiplication map (hence $\ker(\mu)^{i+1} \subseteq R \otimes_k R$), and the annihilator $\mathrm{Diff}_i(R)$ is taken with respect to the canonical action of $R \otimes_k R = R \otimes_k R^{\mathrm{op}}$ on $\mathrm{End}_k(R)$. Under the assumption that $\mathrm{Spec}(R)$ is smooth, $\mathrm{Diff}(R)$ is generated by the multiplication operators $\mathrm{Diff}_0(R) = \mathrm{End}_R(R) \cong R$ (note that $\ker(\mu)$ is generated by $r \otimes 1 - 1 \otimes r$ as R-R-bimodule) and the derivations $\mathrm{Der}_k(R) \subseteq \mathrm{Diff}_1(R)$. Moreover the only additional relations are the Weyl relation
$$r\partial = \partial r + \partial(r), \quad \text{where } r \in R, \partial \in \mathrm{Der}_k(R)$$
and the relation $(r\partial) = r\partial$ identifying the derivation $(r\partial)$ with the product of $r \in R$ with $\partial \in \mathrm{Der}(R)$ in $\mathrm{Diff}(R)$. The comultiplication is obtained by making every derivation of R primitive. In different language $\mathrm{Diff}(R)$ is canonically isomorphic to $\mathbf{U}(R, \mathrm{Der}_k(R))$, the universal enveloping algebra of the Lie–Rinehart algebra $(R, \mathrm{Der}_k(R))$, see [11, §2, §3] for an overview. Other helpful references on this topic are [2, §18], [16], and [17]. The non-smooth case is much more complicated, the results [11] on singular curves suggest that $\mathrm{Diff}(R)$ still becomes a Hopf algebroid, however with a highly non-trivial comultiplication.

We end this section by discussing two other families of examples of bialgebroids, closely related to the nil Hecke case later on. The first is the running example in [2] and generalizes Example 3.2.

Example 3.8. Let $R' \subseteq R$ be commutative rings such that R is finite free as an R'-module. Famously, one has a Morita equivalence between categories of left modules
$$R \otimes_{R'} - \colon R'\text{-Mod} \xrightarrow{\cong} \mathrm{End}_{R'}(R)\text{-Mod}, \qquad V \mapsto R \otimes_{R'} V.$$
Pulling the (symmetric) monoidal structure $\otimes_{R'}$ on R'-**Mod** back through the equivalence defines a (symmetric) monoidal structure on $\mathrm{End}_{R'}(R)$-**Mod**, which agrees with $\otimes$ coming from the following cocommutative R-bialgebroid structure on $\mathrm{End}_{R'}(R)$. First, we identify
$$\mathrm{End}_{R'}(R) \cong R \otimes_{R'} R^*, \text{ where } R^* = \mathrm{Hom}_{R'}(R, R').$$
Since R is a finite free algebra over R', R^* becomes a finite free coalgebra over R' with comultiplication
$$\mu^* \colon R^* \to (R \otimes_{R'} R)^* \cong R^* \otimes_{R'} R^*.$$

Extension of scalars from R' to R gives a map

$$\Delta = \mathrm{id} \otimes_{R'} \mu^* \colon R \otimes_{R'} R^* \to R \otimes_{R'} R^* \otimes_{R'} R^* \cong (R \otimes_{R'} R^*) \otimes (R \otimes_{R'} R^*).$$

One can check that this left R-linear map indeed ends up in the Takeuchi product $(R \otimes_{R'} R^*) \times_R (R \otimes_{R'} R^*)$. Translating everything back through $\mathrm{End}_{R'}(R) \cong R \otimes_{R'} R^*$ one obtains the bialgebroid structure on $\mathrm{End}_{R'}(R)$. The corresponding representation ρ_ε is the inclusion $\mathrm{End}_{R'}(R) \hookrightarrow \mathrm{End}_k(R)$. As discussed in [2] $\mathrm{End}_{R'}(R)$ is a Hopf algebroid over R.

The following example is a special case of [2, §7, Example $A\#H$] and will be important in the final section.

Example 3.9. Let G be a group and R a commutative k-algebra on which G acts by k-algebra automorphisms. Consider the skew group algebra $R \star G \coloneqq R \otimes_k kG$. Using the short notation $rg \coloneqq r \otimes_k g$, the multiplication becomes $(rg)(r'g') = rg(r')gg'$, where $r, r' \in R$, $g, g' \in G$. The algebra $R \star G$ becomes a Hopf algebroid over R when equipped with the comultiplication given by $\Delta(g) = g \otimes g$ and counit given by $\varepsilon(rg) = r$. The translation map is then given by $\mathrm{tak}_{R \star G}(rg) = rg \boxtimes g^{-1} = g \boxtimes g^{-1}r$. In [18, §5], the construction of the Hopf algebroid $R \star G$ was generalized to a Hopf algebroid $V\#A$, where kG is replaced by an arbitrary Hopf algebra A, and R by a commutative algebra V in the Drinfeld center of A. .

Remark 3.10. Note that in Examples 3.1, 3.2, 3.6, 3.9 an antipode induces the translation map. More precisely there is an anti-isomorphism $S_H \colon H \to H^{\mathrm{op}}$ which is the identity on R, such that $\mathrm{tak}_H = (\mathrm{id}_H \otimes S_H) \circ \Delta$, or in Sweedler notation $\mathrm{tak}_H = h_{(1)} \boxtimes S_H(h_2)$, where $\Delta(h) = h_{(1)} \otimes h_{(2)}$ cf. Example 3.1. These are for Example 3.2 matrix transposition, for Example 3.6 the antiautomorphism $\partial_i \mapsto -\partial_i$ of the Weyl algebra and for Example 3.9 $g \mapsto g^{-1}$. An antipode cannot exist for Example 3.8, unless R is Frobenius over R', see [2, Theorem 12.4]. In Corollary 5.11, we will see an explicit example of a Hopf algebroid without antipode.

4 Preliminaries on nil Hecke algebras

This section is all about nil Hecke algebras for Coxeter systems. We recall the definitions surrounding Coxeter groups; for the general theory of Coxeter groups see for instance [19].

Notation 4.1. From now on we work with the ground field $k = \mathbb{R}$. Additionally we fix (W, S) a Coxeter system, i.e. a group W together with a choice of generators $S \subset W$ such that W has a presentation of the form $\langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \text{ for } s \neq t \in S \rangle$ for $m_{st} = m_{ts} = \mathrm{ord}(st) \geq 2$ (we write $m_{st} = \infty$, when no relation on st is imposed). An expression is a finite sequence of letters from S and will be denoted $\underline{w} = s_1 \cdots s_r$ with $s_i \in S$ for $1 \leq i \leq r$. Given an expression $\underline{w}$ leaving out the underline means the corresponding element $w = s_1 \cdots s_r \in W$. We denote by $|\underline{w}| \coloneqq r$ the length of the expression. We call an expression $\underline{w}$ of w reduced if it has minimal length among the expressions yielding w.

Next, we recall the definition of the "root system" associated to the geometric representation $\mathfrak{h}^*$ of (W, S), see for instance [20, Theorem 1.3.11].

Notation 4.2. Let $\mathfrak{h}^* \coloneqq \bigoplus_{s \in S} \mathbb{R}\,\alpha_s$ and call the elements α_s simple roots. Consider the symmetric bilinear form on $\mathfrak{h}^*$ defined by $(\alpha_s, \alpha_t) = -\cos(\frac{\pi}{m_{st}})$ (reading $m_{ss} = 1$). We define for $\lambda \in \mathfrak{h}^*, s \in S$ the scalar $\langle \lambda, \alpha_s^\vee \rangle \coloneqq \frac{2(\lambda, \alpha_s)}{(\alpha_s, \alpha_s)}$. For $s \in S$ and $\lambda \in \mathfrak{h}^*$ the formulas $s(\lambda) = \lambda - \langle \lambda, \alpha_s^\vee \rangle \alpha_s$ define a faithful representation of W on $\mathfrak{h}^*$, which we call the geometric representation of (W, S). The elements of $\Phi \coloneqq \{w(\alpha_s) \mid s \in S, w \in W\} \subset \mathfrak{h}^* \setminus \{0\}$ are called roots. Let $\Phi^+ \subseteq \Phi$ be the set of positive roots, i.e. those roots which are positive linear combinations of simple roots. Let $R = \mathrm{Sym}(\mathfrak{h}^*)$. We view R as a graded algebra with $\mathfrak{h}^*$ concentrated in degree 2. We denote by $R^W \subseteq R$ the algebra of W-invariants with respect to the action of W on R induced by the action of W on $\mathfrak{h}^*$.

Example 4.3. Let $(W, S) = (S_n, \{s_i = (i, i + 1) \mid 1 \leq i \leq n - 1\})$ and consider the action of S_n on $V \coloneqq \mathfrak{h}^*(\mathfrak{gl}_n) \coloneqq \bigoplus_{i=1}^n \mathbb{R}\, x_i$ permuting the n basis vectors $x_1, \ldots, x_n$. One can identify the geometric representation $\mathfrak{h}^*$ of S_n with the quotient $\mathfrak{h}^*(\mathfrak{sl}_n) \coloneqq \mathfrak{h}^*(\mathfrak{gl}_n)/U$ by the 1-dimensional subrepresentation U spanned by $\sum_{i=1}^n x_i$. Under this identification the simple root $\alpha_i \coloneqq \alpha_{s_i}$ associated to $s_i = (i, i + 1)$ becomes $(x_i - x_{i+1}) + U$. There is no harm in replacing the geometric representation $\mathfrak{h}^*$ by $\mathfrak{h}^*(\mathfrak{gl}_n)$ setting $\alpha_i = x_i - x_{i+1}$. In this case one defines the symmetric bilinear form $(-, -)$ in such a way that $\{x_i \mid 1 \leq i \leq n\}$ becomes an orthonormal basis. In particular one has $(\alpha_i, \alpha_i) = 2$. Using this choice of bilinear form, the coefficients $\langle \alpha_i, \alpha_j^\vee \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ remain the same as for the geometric representation. Moreover this redefinition of $\langle -, \alpha_i^\vee \rangle$ induces the permutation action of S_n on V. We will never mention $(-, -)$ again and just use $\langle -, \alpha_i^\vee \rangle$. For our purposes $\mathfrak{h}^* = \mathfrak{h}^*(\mathfrak{sl}_n)$ and $\mathfrak{h}^*(\mathfrak{gl}_n)$ behave like they are the same. Hence, we will work with $\mathfrak{h}^*(\mathfrak{gl}_n)$ when we discuss the S_n case for the nil Hecke algebra in Example 4.6.

Remark 4.4. If (W, S) happens to be the Weyl group of a Kac–Moody Lie algebra over $\mathbb{C}$ associated with a generalized Cartan matrix of size $|S| \times |S|$, there is no harm in replacing $k = \mathbb{R}$ by $k = \mathbb{C}$ and the geometric representation $\mathfrak{h}^*$ by the dual of the Cartan, which is a $(|S| + \mathrm{corank}(A))$-dimensional complex vector space.

Definition 4.5. The **nil Hecke algebra** $n\mathcal{H}$ is the $\mathbb{R}$-algebra generated by $\mathfrak{h}^*$ (weights) and $\{\partial_s \mid s \in S\}$ (nil Coxeter generators) subject to the relations

i) $\lambda\mu = \mu\lambda$ for $\lambda, \mu \in \mathfrak{h}^*$.

ii) The twisted Weyl relations

$$\partial_s \lambda = s(\lambda)\partial_s + \langle \lambda, \alpha_s^\vee \rangle \text{ for } s \in S, \lambda \in \mathfrak{h}^*.$$
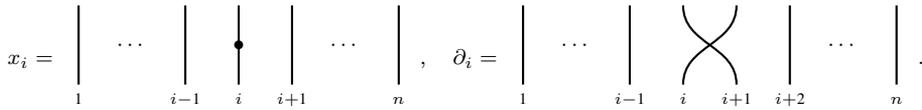
iii) The nil Coxeter relations

$$\partial_s^2 = 0, \quad \underbrace{\partial_s\partial_t\partial_s\cdots}_{m_{st}-\text{factors}} = \underbrace{\partial_t\partial_s\partial_t\cdots}_{m_{st}-\text{factors}} \text{ for } s,t \in S$$

The subalgebra of n$\mathcal{H}$ generated by $\{\partial_s \mid s \in S\}$ is called the **nil Coxeter algebra** and denoted by n$\mathcal{C}$ = n$\mathcal{C}(W,S)$.

Example 4.6. Consider again the case $W = S_n$ and replace the geometric representation $\mathfrak{h}^*$ by $\mathfrak{h}^*(\mathfrak{gl}_n)$ as discussed in Example 4.3. In particular we set $R = \text{Sym}(\mathfrak{h}^*(\mathfrak{gl}_n))$, which is the polynomial ring $k[x_1, \ldots, x_n]$. Writing $\partial_i = \partial_{s_i}$ this definition yields the usual type A nil Hecke algebra from the literature with generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_{n-1}$ and relations

$$\partial_i^2 = 0, \quad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}, \quad \partial_i\partial_j = \partial_j\partial_i \quad (\text{for } |i-j| \geq 2),$$

$$x_ix_j = x_jx_i \quad (\text{for all } i,j), \quad \partial_ix_i = x_{i+1}\partial_i + 1, \quad x_i\partial_i = \partial_ix_{i+1} + 1,$$

$$\partial_ix_j = x_j\partial_i \quad (\text{for } |i-j| \geq 2),$$

for all indices i,j, such that the above expressions make sense. In the literature, the nil Hecke algebra is viewed as a diagram algebra, with

$$x_i = \left|\begin{array}{c}1\end{array}\right. \cdots \left|\begin{array}{c}i-1\end{array}\right. \left|\begin{array}{c}\bullet\\i\end{array}\right. \left|\begin{array}{c}i+1\end{array}\right. \cdots \left|\begin{array}{c}n\end{array}\right. , \quad \partial_i = \left|\begin{array}{c}1\end{array}\right. \cdots \left|\begin{array}{c}i-1\end{array}\right. \left.\begin{array}{c}\times\\i \quad i+1\end{array}\right| \left|\begin{array}{c}i+2\end{array}\right. \cdots \left|\begin{array}{c}n\end{array}\right. .$$

We gather some facts on the nil Hecke algebra.

Notation 4.7. Let $w \in W$ and let $\underline{w} = s_1 \cdots s_r$ be a expression of w. We write $\partial_{\underline{w}} := \partial_{s_1} \cdots \partial_{s_r} \in$ n$\mathcal{H}$ for the nil Coxeter monomial dependent on $\underline{w}$, where by convention $\partial_{\underline{\emptyset}} = 1$.

Lemma 4.8. *Let $w \in W$ and let $\underline{w} = s_1 \cdots s_r$ a expression. The following are equivalent:*

i) $\partial_{\underline{w}} \neq 0$,

ii) $\underline{w}$ is a reduced expression.

Moreover, given two reduced expressions $\underline{w}, \underline{w}'$ we have $\partial_{\underline{w}} = \partial_{\underline{w}'}$ if and only if $w = w'$.

Proof. See [20, Theorem 11.1.2 b), d)]. $\square$

Notation 4.9. By Lemma 4.8, we can define for $w \in W$ the nil Coxeter monomial $\partial_w := \partial_{\underline{w}}$, where $\underline{w}$ is any reduced expression of w.

Proposition 4.10. *The following statements about n$\mathcal{H}$ hold:*

i) Denote by $Q = \text{Quot}(R)$ the field of fractions of $R = \text{Sym}(\mathfrak{h}^*)$. The action of W on R extends to an action on Q. We have R-linear embeddings of k-algebras

$$R \star W \hookrightarrow n\mathcal{H} \hookrightarrow Q \star W,$$

$$s \mapsto 1 - \alpha_s\partial_s, \quad \partial_s \mapsto \frac{1}{\alpha_s}(1-s).$$

The set $\{\partial_w \mid w \in W\}$ forms an R-basis of n$\mathcal{H}$ and a Q-basis of $Q \star W$.

ii) The assignments on generators n$\mathcal{H} \to \text{End}_k(R)$

$$\lambda \mapsto \rho_{n\mathcal{H}}(\lambda) = \lambda \cdot -, \quad \partial_s \mapsto \bar{\partial}_s := \rho_{n\mathcal{H}}(\partial_s) \colon f \mapsto \frac{f - s(f)}{\alpha_s}.$$

*define a faithful R^W-linear representation $\rho_{n\mathcal{H}} \colon$ n$\mathcal{H} \to \text{End}_{R^W}(R)$ of n$\mathcal{H}$ on R. The operators $\bar{\partial}_s$ are called **Demazure**, **BGG–Demazure**, or **divided difference operators**, and satisfy the twisted Leibniz rule*

$$\bar{\partial}_s(f_1)f_2 + s(f_1)\bar{\partial}_s(f_2) = \bar{\partial}_s(f_1f_2) = \bar{\partial}_s(f_1)s(f_2) + f_1\bar{\partial}_s(f_2) \tag{4.1}$$

for all $f_1, f_2 \in R$.

iii) If W is finite, one has an isomorphism of n$\mathcal{H}$-modules

$$R \cong n\mathcal{H}e_{\text{triv}}, \quad \text{where} \quad e_{\text{triv}} = \frac{1}{|W|}\sum_{w \in W} w = \frac{1}{|W|}\partial_{w_o}\prod_{\alpha \in \Phi^+}\alpha,$$

where $w_o \in W$ denotes the longest element in W.

iv) If W is finite, $\rho_{n\mathcal{H}} \colon$ n$\mathcal{H} \to \text{End}_{R^W}(R)$ is an isomorphism.

Proof. See [20, §XI.1]. One has to carefully check that the methods of their proof, which are done for $\mathfrak{h}$ being the Cartan subalgebra of a Kac–Moody Lie algebra, and which rely on real roots, apply in the same way to the geometric representation and its roots. $\square$

Remark 4.11. One may use iv) in Proposition 4.10 to conclude for finite Coxeter groups W that n$\mathcal{H}$ is a Hopf algebroid via the construction from Example 3.8. Note that R is indeed free as a R^W-module by the theorem of Chevalley–Sheppard–Todd (see e.g. [19, §3.6]). Theorem 5.8 will tell us that this holds also for infinite ones.

5 Mixed dihedral braid relations and structure theorem

This is the section containing the new results. Recall that $R \star W$ is a Hopf algebroid over R, and $Q \star W$ is a Hopf algebroid over Q with the structure discussed in Example 3.9. From now on we view $s \in S$ as the element $s = 1 - \alpha_s \partial_s \in n\mathcal{H}$ in the nil Hecke algebra where using Proposition 4.10.

Notation 5.1. Fix $s, t \in S$ two simple reflections with $m = m_{st} < \infty$. Denote by $W_{s,t} \subseteq W$ the parabolic subgroup generated by s, t, which is a dihedral group of order $2m$. We fix the [21] shorthand notation for expressions, which alternate i times between s and t

$$_s\hat{\underline{i}} := \underbrace{\underline{sts\cdots}}_{i\text{ letters}}, \quad _t\hat{\underline{i}} := \underbrace{\underline{tst\cdots}}_{i\text{ letters}}.$$

Denote by $w_{o,s,t} \in W_{s,t}$ the longest element of the parabolic subgroup $W_{s,t}$ that is the element with the two reduced expressions $_s\hat{\underline{m}}_{st}$ and $_t\hat{\underline{m}}_{st}$. For two expressions $\underline{y}$ and $\underline{w}$ we write $\underline{y} \subseteq \underline{w}$, if $\underline{y}$ is a subexpression, i.e. $\underline{y}$ is obtained by leaving out some letters of $\underline{w}$. We write $\iota : \underline{y} \hookrightarrow \underline{w}$ for an embedding, which is a fixed choice of left-out letters. An embedding ι is equivalent to a function $\chi_\iota : \{1, \ldots, |\underline{w}|\} \to \{0, 1\}$, which picks out which letters are in the subexpression ($\chi_\iota(j) = 1$) and which are not ($\chi_\iota(j) = 0$). An expression $\underline{w}$ has exactly $2^{|\underline{w}|}$ embedded subexpressions.

Example 5.2. We have $_s\hat{\underline{3}} = \underline{sts}$, $_s\hat{\underline{4}} = \underline{stst}$ and $_t\hat{\underline{7}} = \underline{tststst}$. If $m_{st} = 5$ we have $ststs = tstst$, which implies $_t\hat{\underline{7}} = tststst = stststst = sts = {}_s\hat{\underline{3}}$. The expression $_s\hat{\underline{3}}$ has 7 subexpressions $\underline{\emptyset}, \underline{s}, \underline{t}, \underline{ts}, \underline{ss}, \underline{st}, \underline{sts}$ and 8 embedded subexpressions $\underline{\hat{s}\hat{t}\hat{s}}, \underline{\hat{s}\hat{t}s}, \underline{s\hat{t}\hat{s}}, \underline{s\hat{t}s}, \underline{\hat{s}ts}, \underline{s\hat{t}\hat{s}}, \underline{st\hat{s}}, \underline{sts}$, the gray letters with a hat on top are those we left out of the expression. Every subexpression except $\underline{ss}$ is reduced.

Definition 5.3. Let $\iota : \underline{w} \hookrightarrow \underline{w}' = s_1 \cdots s_r$ be an embedding of a subexpression. Let χ_ι as in Notation 5.1. We define the **mixed monomial** $\mathrm{mix}_\iota \in n\mathcal{H}$ as the product

$$\mathrm{mix}_\iota := \mathrm{mix}_{\iota,1} \mathrm{mix}_{\iota,2} \cdots \mathrm{mix}_{\iota,r} \in n\mathcal{H}, \quad \text{where} \quad \mathrm{mix}_{\iota,j} = \begin{cases} s_j, & \text{if } \chi_\iota(j) = 1, \\ \partial_{s_j}, & \text{if } \chi_\iota(j) = 0. \end{cases}$$

Example 5.4. Consider $_s\hat{\underline{7}} = \underline{stststs}$. For $\iota_1 : \underline{\hat{s}t\hat{s}\hat{t}sts} \hookrightarrow {}_s\hat{\underline{7}}$ we have $\mathrm{mix}_{\iota_1} = \partial_s t \partial_s \partial_t sts$. For $\iota_2 : \underline{\hat{s}t\hat{s}ts\hat{t}\hat{s}} \hookrightarrow {}_s\hat{\underline{7}}$ we have $\mathrm{mix}_{\iota_2} = \partial_s t \partial_s ts \partial_t \partial_s$.

Theorem 5.5 (Mixed dihedral braid relations). *Let $s, t \in S$ be two simple reflections with $m = m_{st} < \infty$. Let $w \in W_{s,t}$ be a fixed element. Then the relation*

$$\mathrm{Rel}_w : \sum_{\substack{\iota : \underline{w} \subseteq {}_s\hat{\underline{m}}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota = \sum_{\substack{\iota : \underline{w} \subseteq {}_t\hat{\underline{m}}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota$$

holds in $n\mathcal{H}$.

Before we prove the theorem we consider some examples.

Example 5.6. The relation Rel_1 is always

$$\mathrm{Rel}_1 : \partial_{s\hat{\underline{m}}_{st}} = \underbrace{\partial_s \partial_t \partial_s \cdots}_{m_{st} \text{ factors}} = \underbrace{\partial_t \partial_s \partial_t \cdots}_{m_{st} \text{ factors}} = \partial_{t\hat{\underline{m}}_{st}},$$

i.e. the full dihedral braid relation for the Demazure operators. The relation $\mathrm{Rel}_{w_{o,s,t}}$ is on the other hand

$$\mathrm{Rel}_{w_{o,s,t}} : {}_s\hat{m}_{st} = \underbrace{sts\cdots}_{m_{st} \text{ factors}} = \underbrace{tst\cdots}_{m_{st} \text{ factors}} = {}_t\hat{m}_{st}$$

$_s\hat{m}_{st} = {}_t\hat{m}_{st}$, i.e. the full dihedral braid relation for the simple reflections. The other relations are mixed versions of the braid relation in the dihedral group $W_{s,t}$. For each fixed number m_{st} the relations can be checked on products of positive roots. For instance:

i) For $m_{st} = 2$ we obtain the relations

$$\mathrm{Rel}_s : s\partial_t = \partial_t s, \qquad \mathrm{Rel}_t : \partial_s t = t\partial_s$$

ii) For $m_{st} = 3$ we obtain the type A relations

$$\mathrm{Rel}_s : s\partial_t \partial_s + \partial_s \partial_t s = \partial_t s \partial_t, \qquad \mathrm{Rel}_{st} : st\partial_s = \partial_t st.$$

as well as analogous relations Rel_t and Rel_{ts}.

iii) For $m_{st} = 4$ we obtains the type B/C relations

$$\mathrm{Rel}_s : s\partial_t \partial_s \partial_t + \partial_s \partial_t s \partial_t = \partial_t s \partial_t \partial_s + \partial_t \partial_s \partial_t s,$$

$$\mathrm{Rel}_{st} : st\partial_s \partial_t + s\partial_t \partial_s t + \partial_s \partial_t st = \partial_t st \partial_s, \qquad \mathrm{Rel}_{sts} : sts\partial_t = \partial_t sts,$$

and as before, the ones obtained by swapping s and t.

A common strategy to prove such relations is to use the faithfulness of the action $\rho\colon \mathrm{n}\mathcal{H} \to \mathrm{End}_{R^W}(R)$ and a Chevalley–Sheppard–Todd basis for R as R^W-module. For instance, consider the case $m_{st} = 3$, such that $\langle s, t\rangle = S_3$. A R^{S_3}-basis of R is given by the following 6 vectors
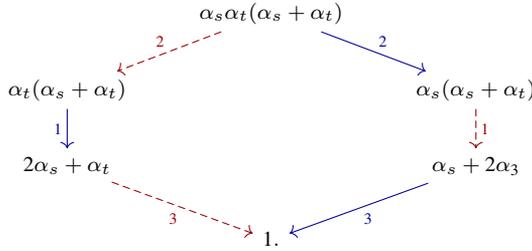
Here, we indicated the action of ∂_s in dashed red and the action of ∂_t in blue. The commutativity of the picture shows the full braid relation $\partial_s\partial_t\partial_s = \partial_t\partial_s\partial_t$. If one wants to check relation Rel$_s$, which is a relation between degree -2 elements in $\mathrm{n}\mathcal{H}$ it is enough to check it on the upper 3 of the 6 basis vectors for degree reasons. An alternative way is to write out both sides of each relation explicitly in $Q \star W$.

Checking these relations gets harder as m_{st} grows. The following example shows that there is a non-trivial connection between the mixed dihedral braid relations and identities between sums of products of positive roots.

Example 5.7. Already for $m_{st} = 4$ the calculation for relation Rel$_{st}$ is quite long and boils down to the following relation

$$t(\alpha_s)s(\alpha_s) = \alpha_s\alpha_t + s(\alpha_t)\alpha_s + t(\alpha_s)\alpha_t.$$

This equation holds both for the realization from Lie theory given by $\mathfrak{h}^*_{\mathfrak{so}_5}$, where $s(\alpha_t) = \alpha_t + 2\alpha_s$ and $t(\alpha_s) = \alpha_s + \alpha_t$ are asymmetrical, as well as for the geometric realization, where both $s(\alpha_t) = \alpha_t + \sqrt{2}\alpha_s$ and $t(\alpha_s) = \alpha_s + \sqrt{2}\alpha_t$ are symmetrical.

Proof of Theorem 5.5. Our proof relies on the embedding $\mathrm{n}\mathcal{H} \subseteq Q \star W$ and on the Q-linear comultiplication on $Q \star W$. We change the color tone slightly and write $\otimes_Q := \otimes_Q$ to distinguish it from $\otimes = \otimes_R$. First, we compute that the comultiplication $\Delta\colon Q \star W \to (Q \star W) \otimes_Q (Q \star W)$ from Example 3.9 maps ∂_s to $\partial_s \otimes_Q s + 1 \otimes_Q \partial_s$. Indeed one has

$$\partial_s \otimes_Q s + 1 \otimes_Q \partial_s = \frac{1}{\alpha_s}(1 - s) \otimes_Q s + 1 \otimes_Q \frac{1}{\alpha_s}(1 - s)$$

$$= \frac{1}{\alpha_s}(1 \otimes_Q s - s \otimes_Q s + 1 \otimes_Q 1 - 1 \otimes_Q s)$$

$$= \frac{1}{\alpha_s}(1 \otimes_Q 1 - s \otimes_Q s) = \Delta(\partial_s).$$

Let $s, t \in S$ with $m := m_{st} < \infty$. In $(Q \star W) \otimes_Q (Q \star W)$ we have the equality

$$\Delta(\underbrace{\partial_s\partial_t\cdots}_{m_{st} \text{ factors}}) = \Delta(\underbrace{\partial_t\partial_s\cdots}_{m_{st} \text{ factors}}).$$

Using that Δ is an algebra morphism into $(Q \star W) \otimes_Q (Q \star W)$ this implies that

$$\underbrace{\Delta(\partial_s)\Delta(\partial_t)\cdots}_{m_{st} \text{ factors}} = \underbrace{\Delta(\partial_t)\Delta(\partial_s)\cdots}_{m_{st} \text{ factors}}.$$

Using the formula for $\Delta(\partial_s)$ and $\Delta(\partial_t)$ and multiplying everything out on both sides, one obtains

$$\sum_{\iota:\,\underline{w} \subseteq_s \hat{m}_{st}} \partial_{\underline{w}} \otimes_Q \mathrm{mix}_\iota = \sum_{\iota:\,\underline{w} \subseteq_t \hat{m}_{st}} \partial_{\underline{w}} \otimes_Q \mathrm{mix}_\iota,$$

where the sums runs over all embedded subexpressions $\underline{w}$ of $_s\hat{m}_{st}$ respectively $_t\hat{m}_{st}$. By Lemma 4.8, we have $\partial_{\underline{w}} = 0$, unless $\underline{w}$ is a reduced expression in W. Hence the equality becomes

$$\sum_{w \in W_{s,t}} \partial_w \otimes_Q \sum_{\substack{\iota:\,\underline{w} \subseteq_s \hat{m}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota = \sum_{w \in W_{s,t}} \partial_w \otimes_Q \sum_{\substack{\iota:\,\underline{w} \subseteq_t \hat{m}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota$$

after reordering summands. Since $\{\partial_w \mid w \in W\}$ is a Q-basis of $Q \star W$ by Proposition 4.10 we can compare coefficients to deduce that for $w \neq w_{o,s,t}$

$$\sum_{\substack{\iota:\,\underline{w} \subseteq_s \hat{m}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota = \sum_{\substack{\iota:\,\underline{w} \subseteq_t \hat{m}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota$$

in $Q \star W$ and in particular in $\mathrm{n}\mathcal{H}$. $\qquad\square$

Theorem 5.8 (Structure theorem). *The assignments* $n\mathcal{H} \to n\mathcal{H} \otimes n\mathcal{H}$

$$\Delta_{n\mathcal{H}}(\lambda) := \lambda \cdot 1 \otimes 1, \quad \Delta_{n\mathcal{H}}(\partial_s) := \partial_s \otimes s + 1 \otimes \partial_s = \partial_s \otimes 1 + s \otimes \partial_s,$$

where $\lambda \in \mathfrak{h}^*$, $s \in S$, *extend to an algebra morphism* $\Delta_{n\mathcal{H}} \colon n\mathcal{H} \to n\mathcal{H} \times_R n\mathcal{H}$. *Together with the map* $\varepsilon \colon n\mathcal{H} \to R$, $\varepsilon(h) := \rho_{n\mathcal{H}}(h)(1)$ *this turns* $n\mathcal{H}$ *into a cocommmutative Hopf algebroid. The map* $\mathrm{tak}_{n\mathcal{H}} \colon n\mathcal{H} \times_{R^{\mathrm{op}}} n\mathcal{H}^{\mathrm{op}}$ *is given by*

$$\mathrm{tak}_{n\mathcal{H}}(\lambda) = \lambda 1 \boxtimes 1, \quad \mathrm{tak}_{n\mathcal{H}}(\partial_s) = \partial_s \boxtimes s + 1 \boxtimes \partial_s = \partial_s \boxtimes 1 - s \boxtimes \partial_s.$$

The subalgebra $R \star W$ *is a Hopf subalgebroid of* $n\mathcal{H}$.

Proof. We first check $\partial_s \otimes s + 1 \otimes \partial_s = \partial_s \otimes 1 + s \otimes \partial_s$. By R-linearity of $\otimes$ we have

$$\partial_s \otimes 1 + s \otimes \partial_s = \partial_s \otimes 1 + (1 - \alpha_s \partial_s) \otimes \partial_s$$

$$= \partial_s \otimes 1 + 1 \otimes \partial_s - \partial_s \otimes \alpha_s \partial_s = \partial_s \otimes s + 1 \otimes \partial_s.$$

Since R is generated by $\mathfrak{h}^*$, it is enough to check the Takeuchi condition with elements of $\lambda \in \mathfrak{h}^*$. We calculate

$$\partial_s \lambda \otimes 1 + s\lambda \otimes \partial_s = (s(\lambda)\partial_s + \langle \lambda, \alpha_s^\vee \rangle) \otimes 1 + s(\lambda)s \otimes \partial_s$$

$$= \partial_s \otimes s(\lambda) + 1 \otimes \langle \lambda, \alpha^\vee \rangle + s \otimes s(\lambda)\partial_s.$$

Now we use the twisted Weyl relation on the rightmost summand and obtain

$$\partial_s \otimes s(\lambda) + 1 \otimes \langle \lambda, \alpha_s^\vee \rangle + s \otimes (\partial_s \lambda - \langle \lambda, \alpha_s^\vee \rangle)$$

$$= \partial_s \otimes (\lambda - \langle \lambda, \alpha_s^\vee \rangle \alpha_s) + 1 \otimes \langle \lambda, \alpha_s^\vee \rangle + s \otimes (\partial_s \lambda - \langle \lambda, \alpha_s^\vee \rangle)$$

$$= \partial_s \otimes \lambda + s \otimes \partial_s \lambda,$$

where we used in the last step that

$$-\partial_s \otimes \langle \lambda, \alpha_s^\vee \rangle \alpha_s = -\alpha_s \partial_s \otimes \langle \lambda, \alpha_s^\vee \rangle = (s - 1) \otimes \langle \lambda, \alpha_s^\vee \rangle.$$

When checking that the comultiplication $\Delta_{n\mathcal{H}} \colon n\mathcal{H} \to n\mathcal{H} \times_R n\mathcal{H}$ is well-defined, the only non-trivial part is

$$\underbrace{\Delta_{n\mathcal{H}}(\partial_s)\Delta_{n\mathcal{H}}(\partial_t)\cdots}_{m_{st} \text{ factors}} = \underbrace{\Delta_{n\mathcal{H}}(\partial_t)\Delta_{n\mathcal{H}}(\partial_s)\cdots}_{m_{st} \text{ factors}},$$

which is equivalent by the same computation as in the proof of Theorem 5.5 to the following equality in $n\mathcal{H} \otimes n\mathcal{H}$

$$\partial_{s\hat{m}_{st}} \otimes {}_s\hat{m} + \sum_{w \in W_{s,t}} \partial_w \otimes \sum_{\substack{\iota : \underline{w} \subseteq {}_s\hat{m}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota = \partial_{t\hat{m}_{st}} \otimes {}_t\hat{m} + \sum_{w \in W_{s,t}} \partial_w \otimes \sum_{\substack{\iota : \underline{w} \subseteq {}_t\hat{m}_{st} \\ \underline{w} \text{ is reduced}}} \mathrm{mix}_\iota.$$

By Theorem 5.5, this is true. The rest of the definition is trivial, note that $\varepsilon(\partial_s) = 0$ for all $s \in S$ to check that $n\mathcal{H}, \Delta, \varepsilon$ is a coalgebra over R. To check that $n\mathcal{H}$ is a Hopf algebroid, one checks that the assignments we wrote extend to a well-defined algebra morphism $\mathrm{tak}_{n\mathcal{H}} \colon n\mathcal{H} \times_{R^{\mathrm{op}}} n\mathcal{H}^{\mathrm{op}}$, which induces $\mathrm{Gal}_{n\mathcal{H}}^{-1}$. $\square$

Remark 5.9. The descent theorem [11, §5.5] gives a general condition on how to restrict a Hopf algebroid structure on some algebra to a given subalgebra. In our setting, the "original" Hopf algebroid is $Q \star W$, and the subalgebra is the nil Hecke algebra. From this point of view, the freeness argument behind the proof of the mixed relations Theorem 5.5 is there to show the conditions written in [11, Lemma 5.14]. Indeed, Theorem 5.8 can be seen as a special case of the descent theorem.

We pose a conjecture.

Conjecture 5.10. Assume W is finite. Then the symmetric monoidal structure on $n\mathcal{H}$-**Mod** coming from the cocommutative R-bialgebroid structure on $n\mathcal{H}$ corresponds under the equivalence

$$R \otimes_{R^W} - \colon R^W\text{-}\mathbf{Mod} \to n\mathcal{H}\text{-}\mathbf{Mod}$$

to the symmetric monoidal product $\otimes_{R^W}$ on R^W-**Mod**.

It is straight-forward, but little enlightening to check it in the S_2-case. We suspect it follows in general by a case by case distinction over all the finite groups or a delicate analysis of products of R^W-basis elements $\rho(\partial_w)(\prod_{\alpha \in \Phi^+} \alpha)$ of R.

Corollary 5.11. *Nil Hecke algebras of non-trivial Coxeter systems are Hopf algebroids without an antipode. More precisely* $\mathrm{tak}_{n\mathcal{H}} \neq (\mathrm{id}_{n\mathcal{H}} \otimes S) \circ \Delta$ *for any k-algebra morphism* $S \colon n\mathcal{H} \to n\mathcal{H}^{\mathrm{op}}$, *which is the identity on* R.

Proof. Since the Coxeter system is non-trivial, we can fix a simple reflection s. To simplify notation we write $\alpha := \alpha_s$ and $\partial := \partial_s$, in particular $s = 1 - \alpha\partial = \partial\alpha - 1$. The proof has three steps. First we calculate that $\mathrm{tak}_{n\mathcal{H}}(s) = s \boxtimes s$. Assuming that an antipode S exists, we show in the second step that $S(s) = s$. Finally, we let $S(\partial_s)\alpha$ act on R and use degree reasons to get a contradiction. In step 1, we calculate

$$\mathrm{tak}_{n\mathcal{H}}(1 - \alpha\partial) = 1 \boxtimes 1 - \alpha(\partial \boxtimes s + 1 \boxtimes \partial)$$

$$= 1 \boxtimes (1 + s - s) - \alpha\partial \boxtimes s - \alpha 1 \boxtimes \partial$$

$$= s \boxtimes s + 1 \boxtimes (1 - s) - \alpha 1 \boxtimes \partial = s \boxtimes s,$$

where we used $1 - s = \alpha\partial$. Now assume an antipode S inducing $\mathrm{tak}_{n\mathcal{H}}$ exists. We know that $\Delta(s) = s \otimes s$, hence $s \boxtimes s = \mathrm{tak}_{n\mathcal{H}}(s) = s \boxtimes S(s)$ by assumption on the description of $\mathrm{tak}_{n\mathcal{H}}$. Since $n\mathcal{H}$ is free as a right as well as a left R-module, $s \neq 0$, and R is an integral domain, we can compare coefficients to conclude that $s = S(s)$. To come to a contradiction, consider $s = S(s) = S(1 - \alpha\partial) = 1 - S(\partial)\alpha$, which implies $1 - s = S(\partial)\alpha$. Now consider the action of both sides on $\alpha \in R$. We have $2\alpha = (1 - s)(\alpha) = (S(\partial)\alpha)(\alpha) = S(\partial)(\alpha^2) = \alpha^2 S(\partial)(1)$. Now $S(\partial)(1) \in R$ is some polynomial in α which multiplied with α^2 yields 2α, which is a contradiction. $\square$

Remark 5.12. In the S_2-case we have $n\mathcal{H} \cong \operatorname{End}_{k[\alpha^2]}(k[\alpha])$ which under appropiate identifications becomes Example 3.8. In [2, Theorem 12.4], it is discussed that if an antipode on $\operatorname{End}_{R'}(R)$ exists, then R has to be Frobenius as an algebra over R'. This is the problem here: there is no isomorphism of $k[\alpha]$-modules $k[\alpha] \to k[\alpha]^* = \operatorname{Hom}_{k[\alpha^2]}(k[\alpha], k[\alpha^2])$, since $k[\alpha]^*$ contains torsion elements for α like $1^\vee : 1 \mapsto 1, \alpha \mapsto 0$. However, the moral point of our proof is that $Q \star W$ has an antipode (as Hopf algebroid over Q), which does not map $n\mathcal{H}$ into itself. Indeed the antipode is the algebra morphism $Q \star W \to (Q \star W)^{\operatorname{op}}$, which maps maps s to s and α to α, hence $\partial = \frac{1}{\alpha}(1-s)$ is mapped to $(1-s)\frac{1}{\alpha}$, which is not in $n\mathcal{H}$, since it does not preserve $R = k[\alpha] \subset Q = k(\alpha)$.

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