

# Newton-like methods for solving non-linear equations: Another view

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**Abstract** The Newton-like method is a compelling numerical technique used in optimization problems and solving equations. It is a deeply rooted in the principles of iterative calculations, and the results converge promptly towards a solution. This paper considers a variant of Newton-Like method which is based on second-order approximation to the function  $f(x)$  by Taylor series expansion. Furthermore, it explores the elementary concepts of the Newton-like method, its mathematical formulation and applications in diverse fields. We delve into its advantages, limitations, cubic convergence and showcasing its efficiency by solving different examples. Furthermore, a comparative study between Newton and Newton-like method has been conducted.

## 1 Introduction

The challenge of finding the roots of a given equation

$$f(x) = 0, \tag{1.1}$$

where the  $f$  is adequately defined in the vicinity of a simple root  $\alpha \in (a, b)$  with  $f(a)f(b) < 0$ , arises commonly in many fields of science. In many problems it is challenging to find an analytical result of (1.1). Consequently, the utilization of numerical techniques for finding the solution to such problems becomes a major topic of substantial interest. The objective of [1] is to overcome and address limitations allied with the traditional Newton-Raphson method, offering an effective and robust numerical solution for nonlinear equations. Furthermore [1] discusses the methodology of combining Newton-Raphson and Adomian decomposition methods using different examples of proposed approach. [2, 3, 4, 5, 6] covers fundamental topics in numerical analysis and focuses on developing algorithms and computational methods for solving mathematical problems includes numerical methods for solving equations, interpolation, numerical integration, and the analysis of numerical errors. Mathematical foundations of Newton's method, algorithmic details, and considerations for its authentic application are conferred by [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In recent years, there have been some developments in the study on Newton-like methods. We are going to reconsider Newton-like method by theoretical analysis, it will be shown that the order of convergence is, atleast three even if the equation (1.1) contains multiple roots. Practical instances and case studies are included to illustrate the use of Newton's method in solving a variety of nonlinear equations.

### 2 Derivation of the Method

The Taylor expansion for the function  $f(x)$  in (1.1) at the point  $x_n$  gives,

$$f(x_n) + \frac{(x - x_n) f'(x_n)}{1!} + \frac{(x - x_n)^2 f''(x_n)}{2!} + \dots = 0. \tag{2.1}$$

By second-order approximation to this series, we may write,

$$f(x_n) + \frac{(x - x_n) f'(x_n)}{1!} + \frac{(x - x_n)^2 f''(x_n)}{2!} \cong 0. \tag{2.2}$$

That is,

$$(x - x_n)^2 f''(x_n) + 2(x - x_n) f'(x_n) + 2f(x_n) \cong 0. \tag{2.3}$$

If we solve this quadratic equation for  $(x - x_n)$ , we will obtain

$$(x - x_n) \cong - \frac{f'(x_n) \pm \sqrt{f'(x_n)^2 - 2f(x_n) f''(x_n)}}{f''(x_n)}. \tag{2.4}$$

Hence  $\alpha$  can be approximated by following iteration formulas,

$$x_{n+1} = x_n - \frac{f'(x_n) \pm \sqrt{f'(x_n)^2 - 2f(x_n) f''(x_n)}}{f''(x_n)}. \tag{2.5}$$

**Remark 2.1:** Since we must have least absolute value of numerator then we choose minus sign if  $f'(x_0) < 0$ , and plus sign if  $f'(x_0) > 0$ .

**Remark 2.2:** If  $\Delta(a) \Delta(b) < 0$ , we have a multiple zero, where  $\Delta = f'(x_n)^2 - 2f(x_n) f''(x_n)$ . In particular, if  $\lim_{n \rightarrow \infty} \Delta(x_n) \cong 0$  then we will get a zero with multiplicity two, because (2.5) becomes a kind of modified Newton’s formula that applied to find a multiple zero with multiplicity two [21].

**Lemma2.1:** If  $\Delta(a) \Delta(b) > 0$  we will have complex zeros.

Let us write Eq. (2.5) as

$$x_{n+1} = x_n - \frac{1}{f''(x_n)} \left( f'(x_n) \pm \sqrt{f'(x_n)^2 - 2f(x_n) f''(x_n)} \right), \tag{2.6}$$

Now, if we rewrite it as

$$x_{n+1} = x_n - \frac{1}{f''(x_n)} \left( f'(x_n) \pm f'(x_n) \left[ 1 - \frac{2f(x_n) f''(x_n)}{f'(x_n)^2} \right]^{1/2} \right), \tag{2.7}$$

and impose to use

$$(1 - X)^{1/2} = 1 - \frac{1}{2}X - \frac{1}{8}X^2 - \frac{1}{16}X^3 - \dots \tag{2.8}$$

we obtain

$$x_{n+1} = x_n - \frac{1}{f''(x_n)}. \tag{2.9}$$

That is;

$$x_{n+1} = x_n - \frac{1}{f''(x_n)}. \tag{2.10}$$

Without loss of generality, let’s choose negative sign. We get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} - \dots \tag{2.11}$$

If we use first two terms in RHS, we obtain Newton formula. But if take first three terms we can claim that we have a new Newton-like method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}. \tag{2.12}$$

The derivation of the Newton-like method is the same as the derivation of the Newton method. Consequently the convergence criteria of the Newton method can be applied to the Newton-like method too. We have

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2} \frac{f(x)^2 f''(x)}{f'(x)^3}, \tag{2.13}$$

then

$$g'(x) = \frac{-f(x)^2 f''(x) [3f''(x) - f'(x)]}{2f'(x)^4} \tag{2.14}$$

And for convergence we require that

$$\left| \frac{f(x)^2 f''(x) [3f''(x) - f'(x)]}{2f'(x)^4} \right| < 1 \tag{2.15}$$

$$x_{n+1} = x_n - \frac{2f'(x_n)}{f''(x_n)} + \frac{f(x_n)}{f'(x_n)} + \frac{1}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} + \dots \tag{2.16}$$

$$x_{n+1} = x_n - \frac{2f'(x_n)}{f''(x_n)} + \frac{f(x_n)}{f'(x_n)} + \frac{1}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}. \tag{2.17}$$

$$g(x) = x - \frac{2f'(x)}{f''(x)} + \frac{f(x)}{f'(x)} + \frac{1}{2} \frac{f(x)^2 f''(x)}{f'(x)^3} \tag{2.18}$$

$$g(x) = 1 - \frac{2f'(x_n)}{f''(x_n)} + \frac{f(x_n)}{f'(x_n)} + \frac{1}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \tag{2.19}$$

For all  $x$  in some interval  $I$  containing the root. Since  $f(\alpha) = 0$ , the above condition is certainly satisfied at the root  $x = \alpha$  provided that  $f'(\alpha) \neq 0$ . Substituting  $x = \alpha$  in (2.14) show that  $g'(\alpha) = 0$ . Hence when  $f'(\alpha) \neq 0$ , the Newton-like method is at least a second-order process.

### 3 Numerical Examples and Convergence

This sections deals with some numerical test on some problems that been considered in several Numerical Analysis text books. We resolved them by Newton-like method and compare the results. In calculation of roots, the condition  $|f(x_n)| \ll 10^{-8}$  been implied to all examples, unless emphasis others.

**Example 3.1:** Evaluate cubic root of 31, correct to four significant figures (4D).

**Solution.** Suppose  $\sqrt[3]{31} = x$  whence  $f(x) = x^3 - 31 = 0$ . If we start with  $x_0 = 3$ , then Newton iterative formula gives

$$x_1 = x_0 - \frac{x_0^3 - 31}{3x_0^2} = 3.148, \tag{3.1}$$

$$x_2 = x_1 - \frac{x_1^3 - 31}{3x_1^2} = 3.142, \tag{3.2}$$

$$x_3 = x_2 - \frac{x_2^3 - 31}{3x_2^2} = 3.141, \tag{3.3}$$

$$x_4 = x_3 - \frac{x_3^3 - 31}{3x_3^2} = 3.141. \tag{3.4}$$

$$\therefore \sqrt[3]{31} \cong 3.141. \tag{3.5}$$

But by Newton-like method we obtain

$$x_1 = x_0 - \frac{x_0^3 - 31}{3x_0^2} - \frac{1}{2} \frac{f'(x_0)^2 f''(x_0)}{f'(x_0)^3} = 3 + \frac{4}{27} - \frac{288}{39366} = 3.141. \tag{3.6}$$

**Example 3.2:** The equation  $x + \cos x = 0$  has been considered in several text books has a root in  $(-1, 0)$ . Approximate this root with  $x_0 = -0.5$ .

**Solution.** By Newton formula we will obtain:

$$x_1 = -0.75522, x_2 = -0.73914, x_3 = -0.73909, \tag{3.7}$$

and by Newton-like method we get  $x_1 = -0.73590, x_2 = -0.73909$ .

Note that since  $\Delta(a) \Delta(b) > 0$  we will have complex zeros. It has been shown that one of these roots with correct to (4d) is  $x = 2.4869 - 1.8094i$  if we start with  $x_0 = -0.5$ .

There is no doubt if this equation has any other complex roots, we can find them by changing starting point. For example, if we start with  $x_0 = 10$ , instead of  $x_0 = -0.5$ , by using (2.5) we come to  $x = 9.1100 - 2.9502i$ , after six iterations. But if we use formula (2.12) we get the same result as we get by  $x_0 = -0.5$ , but with twelve iterations [21].

We continue our numerical examples with multiple roots.

**Example 3.3:** Find the roots of the equation given below

$$x^3 - x^2 - x + 1 = 0.$$

**Solution.** This example has a multiple root  $x = 1$ , and a simple root  $x = -1$ . It was chosen from [16, 17]. By taking initial value  $x_0 = 0.8$ , we get the following results:

$$x_1 = 0.9685, x_2 = 0.9942, x_3 = 0.9989, x_4 = 0.9998, x_5 = 1.0000.$$

By the Newton method after twelve iterations we obtain  $x_{12} = 1.0000$ .

With  $a = 0$  and  $b = 2$  we have  $\Delta(a) \Delta(b) < 0$ , therefore there is a multiple zero in  $(0, 2)$  with multiplicity two. In this paper we try to introduce a modified formula whenever we have multiple roots. But before that, it can be proved that:

**Theorem 3.1.** *If  $\alpha$  is the exact value of the root  $f(x) = 0$  with multiplicity  $m$ , then  $\alpha$  will be a simple root of equation  $f^{(m-1)}(x) = 0$  and thus the sequence  $\{x_n\}_0^\infty$  given by*

$$x_{n+1} = x_n - \frac{f^{(m)}(x_n)}{f^{(m+1)}(x_n)} - \frac{[f^{(m-1)}(x_n)]^2 \cdot f^{(m+1)}(x_n)}{2 [f^{(m)}(x_n)]^3} \tag{3.8}$$

*converges, at least, with order of three.*

*Proof.* It is sufficient to assume  $F(x) = f^{(m-1)}(x)$  and then show the sequence obtain from the following relation converges, at least with order three.

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} - \frac{[F(x_n)]^2 \cdot F''(x_n)}{2[F'(x_n)]^3} \tag{3.9}$$

The relation (3.9) can be helpful to approximate zero with multiplicity  $m$  if we know the value of  $m$ . The following theorems can be used to have an idea of finding  $m$ . □

**Theorem 3.2.** *If  $\alpha$  is a root of the equation  $f(x) = (x - \alpha)^m q(x)$  with multiplicity  $m$ , then*

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{x_n - \alpha} = \frac{2m^2 - 3m + 1}{2m^2}.$$

*Proof.* As  $f(x) = (x - \alpha)^m q(x)$ , for  $x \neq \alpha$ , where  $\lim_{x \rightarrow \alpha} q(x) \neq 0$ . Then,

$$f'(x) = m(x - \alpha)^{m-1} q(x) + (x - \alpha)^m q'(x),$$

$$f''(x) = m(m - 1)(x - \alpha)^{m-2} q(x) + 2m(x - \alpha)^{m-1} q'(x) + (x - \alpha)^m q''(x).$$

If we put  $x = x_n$ , and substitute  $f, f'$  and  $f''$  in (2.12), we can write;

$$x_{n+1} - \alpha = x_n - \alpha - \frac{2qA(mq + Aq')^2 + Aq^2 [m(m - 1)q + 2mq'A + q''A^2]}{2(mq + A^3q')}$$

where,  $A = x_n - \alpha$ . Dividing both sides by  $x_n - \alpha$ , we will get,

$$\frac{x_{n+1} - \alpha}{x_n - \alpha} = 1 - \frac{2q(mq + Aq')^2 + q^2 [m(m - 1)q + 2mq'A + q''A^2]}{2(mq + A^3q')}$$

Since  $\lim_{n \rightarrow \infty} (x_n - \alpha) = 0$ , then

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1} - \alpha}{x_n - \alpha} \right| = \left| 1 - \frac{2m^2q^3 + m(m - 1)q^3}{2m^3q^3} \right|.$$

By simplifying RHS, we conclude that

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1} - \alpha}{x_n - \alpha} \right| = \frac{2m^2 - 3m + 1}{2m^2}. \tag{3.10}$$

□

**Remark 3.1:** With conditions on theorem 3.2;

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \right| = \frac{2m^2 - 3m + 1}{2m^2}. \tag{3.11}$$

It is not difficult to prove this remark. Later we will use this relation for example 5.

Now we are going to introduce main modified formula by the remark given below.

**Remark 3.2:** If  $\alpha$  is a root of  $f(x) = 0$  with multiplicity  $m$ , then the sequence  $\{x_n\}_0^\infty$  given by

$$x_{n+1} = x_n - \frac{mf(x_n)}{f'(x_n)} - \frac{f(x_n)^2 f''(x_n)}{m(m + 1)f'(x_n)^3} \tag{3.12}$$

converges, at least with order three.

To prove this remark, reader may refer to [21]. The proof is the same.

Now back to example (3.3) We use Eq. (3.12) with  $m = 2$ . We obtain

$$x_1 = 1.0222, \quad x_2 = 0.9990, \quad x_3 = 1.0000. \tag{3.13}$$

**Example 3.4:** The equation  $x - \sin x = 0$ , has a root with multiplicity 3 at  $\alpha = 0$ . Use modified method to approximate this root.

**Solution.** This example was chosen from [2]. In this book by the Newton’s method we obtained following results with starting value,  $x_0 = 0.5$ :

$$x_1 = 0.33197, x_2 = 0.22091, x_3 = 0.14717, x_4 = 0.09817$$

$$x_5 = 0.06540, x_6 = 0.04364, x_7 = 0.02909.$$

Let use remark 3.2 with  $m=3$ , we get

$$x_1 = 0.0134000, x_2 = 0.0002486.$$

We have  $f(x_2) = -2.5622e - 12 \leq 10^{10}$ .

Remark 3.2 can be useful to find value of  $m$ . Let’s try to find this value. We end this section with an example with a multiple zero.

**Example 3.5:** The function

$$f(x) = x^4 - 2.4x^3 - 23.56x^2 + 60x - 36,$$

has a multiple zero in the interval  $(1, 2)$ . Use  $x_0 = 2$  to determine this zero correct to (4D).

**Solution.** If we use Newton-like formula we get  $x_7 = 1.2000$ . The exact zero is  $x = 1.2$ .(repeated root with  $m=2$ ). If we use Eq. (3.12), we obtain

$$x_1 = 1.3984, x_2 = 1.2491, x_3 = 1.2122, x_4 = 1.2031, x_5 = 1.2008.$$

$$\left| \frac{x_5 - x_4}{x_4 - x_3} \right| = \frac{2m^2 - 3m + 1}{2m^2},$$

$$\left| \frac{1.2008 - 1.2031}{1.2031 - 1.2122} \right| = 0.2527 = \frac{2m^2 - 3m + 1}{2m^2}$$

whence  $m = 1.5851$ . But since  $m$  is an integer then  $m = 2$ . We use this value and use Eq. (3.12) we get  $x_1 = 1.1059, x_2 = 1.2035, x_3 = 1.2000$ . In Newton-Raphson’s extended formula given by (2.12), we can replace some other term (or terms) rather than  $f''(x)$ . We want to replace this term by  $F''(x)$ , where

$$F''(x_i) = \frac{-6}{h_i^2} [f(x_i) - f(x_{i-1})] + \frac{2}{h_i} [2f'(x_i) + f'(x_{i-1})], h_i = x_i - x_{i-1}.$$

Hence (2.12) will become

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{1}{2} \frac{f(x_n)^2}{f'(x_n)^3} \left\{ \frac{6}{h_n^2} [f(x_n) - f(x_{n-1})] - \frac{2}{h_n} [2f'(x_n) + f'(x_{n-1})] \right\}, \tag{3.14}$$

where the second derivative in (2.12) has been replaced by its approximation by differentiating a two-point Hermite interpolation formula for  $f(x)$ , with two points being  $x_i \wedge x_{i-1}$ . This is a modified formula with order of convergence  $1 + \sqrt{3} \cong 2.7321$  [9]. A reader may wish to extend Eq. (3.14) to find a relation for solution of the system of non-linear equations. Here, one guess can be

$$x^{(k+1)} = x^{(k)} - J^{-1}(x^{(k)}) f(x^{(k)}) - 0.5 * [J^{-1}(x^{(k)})]^3 [f(x^{(k)})]^2 \left[ \left[ -\frac{6}{h_i^2} [f(x^{(k)})] \right. \right. \\ \left. \left. - f(x^{(k-1)}) + \frac{2}{h_i} [2J(x^{(k)}) + J(x^{(k-1)})] \right]; k = 0, 1, 2, \dots \tag{3.15}$$

which can be driven as Newton formula for solving a system of non-linear equations given by;

$$x^{(k+1)} = x^{(k)} - J^{-1}(x^{(k)}) f(x^{(k)}); k = 0, 1, 2, \dots$$

We leave it as an open problem for readers.

## 4 Conclusion

### Efficacy of the Newton-like Method

- The method demonstrates its effectiveness in approximating zeros, also capable of approximating and identifying multiple zeros of a given function.

### Handling Multiple Zeros

- In Example 3.4, the method is applied to a function with a multiple zero.
- Even without knowing the exact value of the multiplicity  $m$ , the method provides a reliable approximation that is very close to the exact value.

### Adapted Formulation for $m \gg 3$

- Two modified formulas are introduced for cases where the multiplicity  $m$  is significantly larger than 3.
- Moreover, these modifications likely enhance the convergence or accuracy of the method for higher multiplicities.

### Practicality of Relation (3.12)

- Notably suggested that Relation (3.12) is a useful formula, chiefly in cases where more differentiations become impractical.
- This suggests that, in specific scenarios, an abridged formula (Relation (3.12)) is effective and sufficient for practical applications.

In conclusion, the Newton-like method showcased in the article demonstrates versatility in handling multiple zeros and provides effective approximations, even when specific parameters like multiplicity are not precisely known. The introduction of modified formulas further extends its applicability, and the simplification in Relation (3.12) offers practical utility, especially when dealing with higher multiplicities.

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