ON BZS COMPLETELY 0-SIMPLE SEMIGROUPS

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Abstract Following the work of [6], we investigate a class of semigroups whose elements are either idempotent or nilpotent of index 2. These semigroups are known as *Boolean Zero-Square* semigroups, or BZS semigroups. In particular, we present a method to count the number of idempotents and nilpotents in any completely 0-simple semigroup, or $C0S$ semigroup. We developed a program in C++ to generate BZS $C0S$ semigroups.

1 Introduction

A *semigroup* S is a set with a single binary associative operation. An *ideal* I in a semigroup S is a subset of S which is is closed under multiplication by elements of S; that is, if $s \in S$ and $i \in I$, then $si \in S$ and $is \in S$. We denote these conditions by writing $sI \subseteq I$ and $Is \subseteq I$. A semigroup S which contains a zero element 0 is *0-simple* if S and 0 are the only ideals in S.

An element $e \in S$ is *idempotent* if $e^2 = e$. Let E denote the set of idempotents in E. We can define a partial order on E as follows: if $e, f \in S$, then $e \le f$ iff $e = ef = fe$. This partial order is *trivial* if $e \le f$ implies $e = f$.

A semigroup is *completely 0-simple* if S is 0-simple and the partial order on E is trivial. Such semigroups have been characterized by the following theorem, known as the *Rees Representation Theorem*.

Theorem 1.1. *[1, 5] Let I, Λ be two sets. Let S be the set of all $I \times \Lambda$ matrices with at most one nonzero element from a group G, together with the zero matrix. A nonzero matrix is denoted (i, g, λ) while the matrix consisting entirely of zeros is denoted 0.*

Let P be a $\Lambda \times I$ matrix with at least one entry from G in each row and column and zeros elsewhere. The matrix P is referred to as the sandwich matrix. *Define multiplication * on S as follows: if $(i, g, \lambda), (j, h, \mu) \in S$ then*

$$(i, g, \lambda) * (j, h, \mu) = (i, g, \lambda)(\lambda, k, j)(j, h, \mu)$$

This product is (i, gkh, μ) if $k \ne 0$ and 0 otherwise. Then S is a $C0S$ semigroup. Conversely, any $C0S$ semigroup can be represented by such a set of matrices.

A semigroup S is *Boolean Zero Square* or BZS if every element $s \in S$ satisfies either $s^2 = s$ or $s^2 = 0$. Semigroups and rings with these properties are studied in [2, 3, 4, 6]. In this paper we characterize those $C0S$ semigroups which are BZS.

In this paper we focus on $C0S$ semigroups whose entries are either 0 or 1. In Section 2 we present the main results of this paper. In Section 3 we give examples of our results. Finally, in Section 4 we present the pseudocode for the computer program that we wrote that generates BZS $C0S$ semigroups and serves to verify our results. We also provide a link to the actual program.

2 Main Results

Notation We investigate $C0S$ semigroups whose matrices have entries from $\{0,1\}$. If $A = (i, 1, \lambda)$, we denote A as $E_{i,\lambda}$. The set of idempotents in S is denoted by E, and the set of nilpotents in S is denoted by N. The cardinality of these sets is denoted by $|E|$ and $|N|$.

Lemma 2.1. *The matrix $E_{i,\lambda}$ is nilpotent if and only if the (λ, i) entry in P is 0.*
The matrix $E_{i,\lambda}$ is idempotent if and only if the (λ, i) entry in P is 1.

Proof. $E_{i,\lambda} * E_{i,\lambda} = E_{i,\lambda} P E_{i,\lambda} = E_{i,\lambda} E_{\lambda,i} E_{i,\lambda}$. This element is zero iff the (λ, i) entry in P is zero, and this element is 1 iff the (λ, i) entry in P is 1.   □

Theorem 2.2. *If S is a $C0S$ semigroup, then S is BZS iff the entries in S and the sandwich matrix P are from $\{0, 1\}$.*

Proof. ($\Rightarrow$) By contrapositive. Let $(i, g, \lambda) \in S$ where $g \neq 0$ and $g \neq 1$. Let the (λ, i) entry in P be h where $h \neq 0$ and $h \neq g^{-1}$. Then $(i, g, \lambda) * (i, g, \lambda) = (i, g, \lambda)(\lambda, h, i)(i, g, \lambda) = (i, ghg, \lambda)$ and $ghg \neq g$ and $ghg \neq 0$.
($\Leftarrow$) This follows from Lemma 2.1.   □

See also Theorem 6 in [6].

Theorem 2.3. *Let S be a BZS $C0S$ semigroup over $\{0, 1\}$. Then*

(1) every element of N is nilpotent of index 2;

(2) $|E|$ equals the number of $1's$ in P;

(3) $|N|$ equals the number of $0's$ in P.

Moreover, the matrix $A = E_{i,\lambda}$ is idempotent iff the the $(\lambda, 1)$ entry in P is 1; otherwise, $A^2 = 0$.

Proof. This follows from Lemma 2.1 and Theorem 2.2.   □

Corollary 2.4. $max\{|I|, |\Lambda|\} \leq |E| \leq |I||\Lambda|$

Proof. If every element of P is 1, then P contains $|I||\Lambda|$ 1's. Hence $|E| \leq |I||\Lambda|$ by Theorem 2.3(2).
    The minimal value of $|E|$ equals the minimal number of 1's in P by Theorem 2.3. Recall that each element of P has a least one 1 in each row and column. Therefore, to find the minimal value of $|E|$ we need to count the minimal number of 1's in any sandwich matrix P. Take as matrix P the matrix with elements in positions $(1, 1), (2, 2), \ldots, (|I|, |I|)$. There are three cases to consider.
    If $|I| = |\Lambda|$, then this matrix has $|I| = |\Lambda|$ 1's, and there is a single 1 in each row and column. Any other matrix with $|I| = |\Lambda|$ can have fewer 1's. Hence $|E| = |I| = |\Lambda|$.
    If $|I| > |\Lambda|$, then we can arbitrarily put a single 1 anywhere in each of the remaining rows. In this case $|E| = |I|$.
    If $|I| < |\Lambda|$ then we can arbitrarily put a single 1 in each of the remaining columns. In this case $|E| = |\Lambda|$.   □

    We generalize the previous results to arbitrary $C0S$ semigroups. The proof is virtually identical to that of Theorem 2.3.

Theorem 2.5. *Let S be a $C0S$ semigroup over $G \cup 0$ for an arbitrary group G. Then*

(1) $|E|$ equals the number of nonzero entries in P;

(2) if k is the number of $0's$ in P, then $|N| = |G|k$.

Moreover, the matrix (i, g, λ) is idempotent iff the (λ, i) entry of P is g^{-1}, and (i, g, λ) is nilpotent of index 2 iff the (λ, i) entry of P is 0.

    Section 4, below, describes a computational algorithm that generates all BZS $C0S$ semigroups for any I and Λ. The program yields the following results for $2 \leq I \leq \Lambda \leq 5$.

I	Λ	Number of BZS $C0S$ Semigroups
2	2	7
2	3	25
2	4	79
2	5	241
3	3	265
3	4	2161
3	5	16,081
4	4	41,503
4	5	693,601
5	5	24,997,921

3 Examples

In this section we provide examples of the semigroups in the previous section.

Example 3.1. Let S be the set of all 2×2 matrices over $\{0,1\}$ with at most one nonzero entry, and let the sandwich matrix P be the identity matrix, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $S = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\} \cup \{0\}$. Denote these matrices by A, B, C, D respectively. The semigroup S is given in the following table.

	0	A	B	C	D
0	0	0	0	0	0
A	0	A	B	0	0
B	0	0	0	A	B
C	0	C	D	0	0
D	0	0	0	C	D

The sandwich matrix has two 0's and two 1's. As from Theorem 2.3 we have two idempotents where $E = \{A, D\}$, and two nilpotents where $N = \{B, C\}$.

Example 3.2. Let S be as in Example 3.1 and let $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The semigroup S is given in the following table.

	0	A	B	C	D
0	0	0	0	0	0
A	0	0	0	A	B
B	0	A	B	A	B
C	0	0	0	C	D
D	0	C	D	C	D

The sandwich matrix has one 0 and three 1's. As from Theorem 2.3 we have three idempotents where $E = \{B, C, D\}$, and one nilpotent where $N = \{A\}$.

Example 3.3. Let S be the set of all 2×3 matrices over $\{0,1\}$ with at most one nonzero entry, and let $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $S = \{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,1}, E_{2,2}, E_{2,3}\} \cup \{0\}$. Denote these matrices by A, B, C, D, E, F respectively. The multiplication table is:

	0	A	B	C	D	E	F
0	0	0	0	0	0	0	0
A	0	A	B	C	A	B	C
B	0	A	B	C	0	0	0
C	0	0	0	0	A	B	C
D	0	D	E	F	D	E	F
E	0	D	E	F	0	0	0
F	0	D	E	F	D	E	F

The sandwich matrix has two 0's and four 1's. As from Theorem 2.3 we have four idempotents where $E = \{A, B, D, F\}$, and two nilpotents where $N = \{C, E\}$.

4 PseudoCode

Here we highlight some pseudocode for the program that we used to discover and check our results. These algorithms were instrumental in locating the patterns described in the previous sections. The program can be found at https://github.com/LoyolaResearch/Simple SandwichMatrix and is publicly available. We implemented these algorithms in C++, using Microsoft's Visual Studio™ to afford high performance and fidelity.

Our program uses mostly brute-force approaches to generate all possible $BZS\ C0S$ semigroups. However, we developed some novel optimizations to simplify and speed up the processing. First, we dynamically stored and generated every potential matrix of a given size (X and Y) in a one-dimensional array of characters. Since we are only considering the group with only one element, we simply count from zero to the $2^{(x*y)}$ number of possible matrices, and for each value, we use the reducing modulus values to generate the array of ones and zeros.

```
for (i = 0; i < count; i++) {
   for (j = 0; j < X * Y; j++) {
      if (i % 2 == 0) matrix[j] = '0';
      else matrix[j] = '1';
      num /= 2;
   }
}
```

Next, we simply test each generated matrix and store, in the list of possible center sandwich matrices, those with at least one nonzero digit in each row and column. This is quickly computed by adding the values in each row or column and ignoring those with a sum of zero.

```
   isValid = true;
   for (j = 0; j < Y; j++) {
      int rowZeros = 0;
      for (k = 0; k < X; l++)
         rowZeros += (matrix[(j * X) + k] != '0');
      if (rowZeros == 0) isValid = false;
   }
   for (j = 0; j < X; j++) {
      int colZeros = 0;
      for (k = 0; k < Y; l++)
         colZeros += (matrix[(j * Y) + k] != '0');
      if (colZeros == 0) isValid = false;
   }
   if (isValid)
      for (j = 0; j < X * Y; j++)
         sandwichMatrices[sandwichCount++][j] = matrix[j];
}
```

Finally, we repeat the same basic method to generate and store the set of matrices in the semigroup.

```
for (i = 1; i < count; i *= 2) {
   for (j = 0; j < Y * X; j++) {
      if (i % 2 == 0) matrices[matricesCount++][j] = '0';
      else matrices[matricesCount++][j] = '1';
      num /= 2;
   }
}
```

Although somewhat complicated to generate all possible iterations of the matrices, this method is executed in $O(n^2)$ time. From here, we convert the single-dimensional matrix structures to standard two-dimensional arrays and use common matrix multiplication techniques.

References

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