

Global Existence for a viscoelastic plate equation with logarithmic nonlinearity

Bhargav Kumar Kakumani and Suman Prabha Yadav

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Corresponding Author: Bhargav Kumar Kakumani

Abstract The viscoelastic plate equation with the source of the form $|u|^{p-2} u \ln |u|$ with nonlinear frictional damping and memory terms is considered. We show local existence of solution of the model. Under additional hypothesis on the initial data, global existence of the problem is established.

1 Introduction

In this paper, we consider the viscoelastic plate equation with memory, damping, and a logarithmic nonlinearity source term. This kind of partial differential equation (PDE) simulates the dynamic behavior of a thin, elastic plate under the influence of nonlinear forcing, energy dissipation (damping), and time-dependent viscoelastic phenomena. These equations appear in a variety of physical and engineering applications where time-dependent material behavior and nonlinear feedback are important, such as in structural mechanics and material science. We denote $b(t)$ be the memory kernel that defines viscoelasticity, and $|u_t|^{m-2} u_t$ as the damping term that dissipates energy over time and is essential for stability and long-term behavior. This work deals with the local existence results for the following PDE:

$$\begin{cases} |u_t|^p u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^t b(t-s) \Delta^2 u(s) ds + |u_t|^{m-2} u_t = k |u|^{p-2} u \ln |u|, \\ x \in \Omega, t \in \mathbb{R}^+, \\ u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0, \quad x \in \partial\Omega, t \in \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \end{cases} \quad (1.1)$$

where Ω is the bounded domain of \mathbb{R}^n with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$, k is a positive constant, $m \geq 2$ and p satisfies $2 < p < +\infty$ if $n \leq 4$ and $2 < p < \frac{2n-4}{n-4}$ if $n \geq 5$.

In the case of linear damping (with $m = 2$), Levine [3, 4] was the first to investigate the combined effect of source terms and damping. He has proved that if the initial energy is negative, the solution will blow up in finite time. The results of Levine was extended to the situation of nonlinear damping ($m > 2$) by Georgiev and Todorova [9]. They determine criteria on m and p that ensure either global existence or blow-up. Specifically, they show that the solution exists if $m \geq p$, but blows-up in finite time if $p > m$ when the initial energy is sufficiently negative. In [8], the author in examines the following problem:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau) d\tau + a u_t |u_t|^{m-2} = b u |u|^{p-2},$$

associated with initial and Dirichlet boundary conditions. It is shown that, when $p > m$, any weak solution with negative initial energy blows-up in finite time under specific assumptions on the function g . Furthermore, with stronger damping, as long as $m \geq p$, solutions exist for all

time for any initial data in a suitable function space. The following equation is examined by the author in [1], where $m \geq p$ and $p > 2$:

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds - \epsilon_1 \Delta u_t + \epsilon_2 |u_t|^{m-2} u_t = \epsilon_3 |u|^{p-2} u, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\epsilon_1, \epsilon_2 \geq 0$ and $\epsilon_3 > 0$. By considering a strong damping $\epsilon_1 \Delta u_t$, they have proved the solution blows up in finite time. In [7], the authors considered the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u, & x \in \Omega, t \in \mathbb{R}^+, \\ u(x, t) = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.3}$$

and proved that solution exist globally when $m \geq p$, and blow-up occurs in finite time if $m < p$ and the initial energy is negative.

We denote c, C to be the generic constants throughout the paper. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar product and norms. To prove the results, we consider the following hypotheses on b and ρ :

(H1) Let $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a C^1 -nonincreasing function satisfying

$$b(0) > 0, \quad 1 - \int_0^\infty b(\tau)d\tau = l > 0.$$

(H2) Let ρ satisfies $0 \leq \rho \leq \frac{8}{n-4}$ if $n \geq 5$ and $\rho \geq 0$ if $n \leq 4$.

This article is organized as follows: In Section 2, we prove a local existence of solution of the model when $u_0, u_1 \in H_0^2(\Omega)$. We prove the global existence of problem (1.1) under some extra assumption the data u_0 in Section 3.

2 Local existence

In this section, we state and prove a local result existence for problem (1.1) in the cases where $\rho = 0$ and $\rho > 0$ independently. Using the fixed point arguments, we demonstrate the uniqueness of the solution when $\rho = 0$; however, if $\rho > 0$, uniqueness is not assured. The proof of the local existence part is a small adaptation of the work done in [2, 5]. Before we go to the main theorem, we define the weak solution for problem (1.1) as follows:

Definition 2.1. A function

$$u \in C([0, T], H_0^2(\Omega)) \cap C^1([0, T], L^{\rho+2} \cap H_0^2(\Omega)) \cap C^2([0, T], H^{-2}(\Omega))$$

is called a weak solution of (1.1) on $[0, T]$ if, for any $t \in [0, T]$ and for all $\eta \in H_0^2(\Omega)$, u satisfies

$$\begin{cases} \int_\Omega |u_t|^\rho u_{tt}(x, t)\eta(x)dx + \int_\Omega \Delta u(x, t)\Delta \eta(x)dx + \int_\Omega \Delta u_{tt}(x, t)\eta(x)dx \\ - \int_\Omega \Delta \eta(x) \int_0^t b(t-s)\Delta u(s)ds + \int_\Omega |u_t|^{m-2} u_t(x, t)\eta(x)dx \\ = k \int_\Omega |u(x, t)|^{p-2} u(x, t)\eta(x) \log(|u(x, t)|)dx, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{2.1}$$

Theorem 2.2. Let hypothesis (H1)–(H2) hold, $\rho = 0$ and $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$, then there is $T > 0$ such that problem (1.1) admits a unique solution on $[0, T]$.

Proof. We consider the space, for $T > 0$,

$$\mathcal{Y} := C([0, T], H_0^2(\Omega)) \cap C^1([0, T], L^2(\Omega)),$$

with the norm $\|u(t)\|_{\mathcal{Y}} = \left(\max_{t \in [0, T]} (\|\Delta u(t)\|_2^2 + \|u_t(t)\|_2^2) \right)^{\frac{1}{2}}$. The proof is divided into three steps: The existence of a weak solution to the modified problem is discussed in Step 1 by taking (1.1) with known source term (see (2.2)) into consideration. Step 2 discusses uniqueness of solution to problem (2.2). Lastly, in step 3, we demonstrate that equation (1.1) has a unique weak solution u .

Step 1: We now consider the following problem (for a given $u \in \mathcal{Y}$),

$$\begin{cases} v_{tt} + \Delta^2 v + \Delta^2 v_{tt} - \int_0^t b(t-s)\Delta^2 v(s)ds + |v_t|^{m-2} v_t = k |u|^{p-2} u \ln |u|, & \text{in } \Omega \times (0, \infty), \\ v(x, t) = \frac{\partial v}{\partial \nu}(x, t) = 0, & \text{in } \partial\Omega \times (0, \infty), \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & \text{in } \Omega. \end{cases} \tag{2.2}$$

We will prove that problem (2.2) admits a unique solution $v \in \mathcal{Y} \cap C^2([0, T], H^{-2}(\Omega))$ with $v_t \in L^m([0, T], L^m(\Omega))$. We set $V_q := \text{span}\{w_1, w_2, \dots, w_q\}$, where $\{w_j\}_{j=1}^\infty$ is the orthogonal complete system of eigenfunctions of Δ^2 in $H_0^2(\Omega)$ with $\|\Delta w_j\|_2 = 1$ for all j . Then, $\{w_j\}$ is orthogonal and complete in L^2 and H_0^2 . Denote,

$$u_{0q} = \sum_{j=1}^q (\int_\Omega \Delta u_0 \Delta w_j) w_j \quad \text{and} \quad u_{1q} = \sum_{j=1}^q (\int_\Omega \Delta u_1 \Delta w_j) w_j,$$

where $u_{0q}, u_{1q} \in V_q$, and as $q \rightarrow \infty$, $u_{0q} \rightarrow u_0, u_{1q} \rightarrow u_1$ in $H_0^2(\Omega)$. For each $q \in \mathbb{N}$ we search q functions g_{1q}, \dots, g_{qq} , such that $v_q(t) = \sum_{j=1}^q g_{jq} w_j$. Then any solution to equation (2.2) solve the following problem

$$\begin{cases} v_q'' + \Delta^2 v_q + \Delta^2 v_q'' - \int_0^t b(t-s)\Delta^2 v_q ds + |v_q'|^{m-2} v_q' = k |u|^{p-2} u \ln |u|, \\ v_q(0) = v_{0q}, \quad v_q'(0) = v_{1q}, & \text{in } \Omega. \end{cases} \tag{2.3}$$

Multiplying the equation (2.3) with w_{jq} and using integration by parts, we obtain the nonlinear system of ODE. The existence of unique $\{g_{jq}\}_{j=1}^q$ is ensured with the standard arguments available in the literature of ordinary differential equations. Therefore, there exists a unique v_q which satisfies (2.3). Multiply equation (2.3) with the test function $v_q'(t)$ and integrate over space and time to get

$$\begin{aligned} & \|v_q'\|_2^2 + \|\Delta v_q\|_2^2 + \|\Delta v_q'\|_2^2 - \int_0^t b(s)ds \|\Delta v_q\|_2^2 + (b \circ \Delta v_q) - \int_0^t (b' \circ \Delta v_q) + \int_0^t b(t) \|\Delta v_q\|_2^2 \\ & = 2 \int_0^t \int_\Omega k |u|^{p-2} u \ln |u| v_q' dx - 2 \int_0^t \|v_q'\|_m^m d\tau + \|v_{1q}\|_2^2 \\ & \quad + \|\Delta v_{0q}\|_2^2 + \|\Delta v_{1q}\|_2^2, \end{aligned} \tag{2.4}$$

where $(b \circ \Delta v_q) = \int_0^t b(t-s) \|\Delta v_q(s) - \Delta v_q(t)\|^2 ds$. We use the following inequality from Theorem 3.2 in [5], i.e.,

$$2 \int_0^t \int_\Omega k |u|^{p-2} u \ln |u| v_q' dx \leq CT + \int_0^t \|v_q'\|_m^m d\tau. \tag{2.5}$$

After rearranging the terms in the above two inequalities and using (H1), we get

$$\begin{aligned} & \|v_q'\|_2^2 + \|\Delta v_q'\|_2^2 + \left[1 - \int_0^t b(s)ds \right] \|\Delta v_q\|_2^2 + (b \circ \Delta v_q) \\ & - \int_0^t (b' \circ \Delta v_q) + \int_0^t b(t) \|\Delta v_q\|_2^2 + \int_0^t \|v_q'(\tau)\|_m^m d\tau \leq C, \end{aligned} \tag{2.6}$$

where $C > 0$ is independent of q . It follows from (2.6) that

$$\begin{cases} v_q(t) \text{ is bounded in } C(0, T; H_0^2(\Omega)), \\ v'_q(t) \text{ is bounded in } C(0, T; H_0^2(\Omega)), \\ v''_q(t) \text{ is bounded in } L^2(0, T; H^{-2}(\Omega)). \end{cases} \tag{2.7}$$

Therefore, we could pass the limit in (2.3) up to a subsequence and get a weak solution v of (2.2) (for details, see ([6, 2])). Thanks to the Aubin-Lions lemma, we have $v \in C(0, T; H_0^2(\Omega)) \cap C^1(0, T; L^2(\Omega))$ with $v_t \in L^m(0, T; L^m(\Omega))$. From (2.2), we get $v'' \in C^0(0, T; H^{-2}(\Omega))$. Consequently, the weak solution of problem (2.2) is obtained.

Step 2: We prove uniqueness by contradiction: Let w_1, w_2 be two solutions of (2.2) with the same initial data i.e., w_1, w_2 satisfy problem (2.2). On subtracting these two equations, and multiplying with $(w_{1t} - w_{2t})$ and integrating with respect to both space and time, we obtain

$$\begin{aligned} & \int_0^t \int_{\Omega} (w_{1tt} - w_{2tt}) (w_{1t} - w_{2t}) dxdt + \|\Delta w_1 - \Delta w_2\|_2^2 + \|\Delta w_{1tt} - \Delta w_{2tt}\|_2^2 \\ & + (1 - \int_0^t b(s) ds) \|\Delta w_1 - \Delta w_2\|_2^2 + b \circ \Delta (w_1 - w_2) \\ & + \int_0^t \int_{\Omega} (|w_{1t}|^{m-2} w_{1t} - |w_{2t}|^{m-2} w_{2t}) dxdt = 0. \end{aligned} \tag{2.8}$$

Using the following known inequality

$$\left(|\alpha_1|^{m-2} \alpha_1 - |\alpha_2|^{m-2} \alpha_2 \right) (\alpha_1 - \alpha_2) \geq C |\alpha_1 - \alpha_2|^m \text{ for } m \geq 2,$$

in (2.8), we conclude that $w = v$, i.e., problem (2.2) has a unique weak solution.

Step 3: In this step, we prove that the solution to (1.1) exist. For $u_0, u_1 \in H_0^2(\Omega)$ we denote

$$M_T = \{u \in \mathcal{Y} : u(0) = u_0, u_t(0) = u_1, \|u\|_{\mathcal{Y}} \leq R\},$$

for all $T > 0$, where

$$R^2 = \|\Delta u_0\|_2^2 + \|\Delta u_1\|_2^2.$$

Now, for any $u \in M_T$, we define a map $\phi : \mathcal{Y} \rightarrow \mathcal{Y}$ such that $v = \phi(u)$ is the solution to problem (2.2). For small $T > 0$, we have

- (i) ϕ maps a ball with radius R in M_T into itself;
- (ii) ϕ is a contraction in M_T , indicating that there is a unique $u \in M_T$ satisfying $u = \phi(u)$, a solution to (1.1).

A proof of the above two points follows from the similar lines given in Theorem 3.2 in [5], so we omit the details. Hence the theorem is proved. □

Theorem 2.3. Assume $\rho > 0$, hypothesis (H1)–(H2) hold and $u_0, u_1 \in H_0^2(\Omega)$, then problem (1.1) has a weak solution on $[0, T]$ for some $T > 0$.

Proof. For $T > 0$, we consider the space

$$\mathcal{X} := C([0, T], H_0^2(\Omega)) \cap C^1([0, T], L^{\rho+2} \cap H_0^2(\Omega)),$$

endowed with the norm

$$\|u(t)\|_{\mathcal{X}} = \left(\max_{t \in [0, T]} (\|\Delta u(t)\|_2^2 + \|\Delta u_t(t)\|_2^2 + \|u_t(t)\|_{\rho+2}^{\rho+2}) \right)^{\frac{1}{2}}$$

We use the FaedoGalerkin approach to prove that problem (1.1) has a local weak solution. The proof proceeds along the same lines as Step 1 in Theorem 2.2 and [2]. We therefore leave out the specifics here. □

3 Global existence

The global existence result of equation (1.1) under an extra assumption on the initial data u_0 is proved in this section. We provide a few definitions which are helpful to prove global existence. At first we introduce the energy functional $J(u)$ and the Nehari functional $I(u)$ on $C^1((0, T); H_0^2 \setminus \{0\})$ by

$$J(u)(t) = \frac{1}{2} \left(1 - \int_0^t b(s) ds \right) \|\Delta u\|_2^2 + \frac{1}{2} \|\Delta u_t\|_2^2 + \frac{1}{2} (b \circ \Delta u) - \frac{1}{p} \int_{\Omega} k u^p \ln |u| dx + \frac{k}{p^2} \|u\|_p^p, \tag{3.1}$$

and

$$I(u)(t) = \left(1 - \int_0^t b(s) ds \right) \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + (b \circ \Delta u) - \frac{2}{p} \int_{\Omega} k u^p \ln |u| dx. \tag{3.2}$$

We now define the mountain pass level d (i.e., the depth of the potential well for the functional J) as:

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{3.3}$$

where \mathcal{N} is Nehari manifold defined as $\mathcal{N} = \{u : u \in H_0^2(\Omega) \setminus \{0\}, I(u) = 0\}$.

Lemma 3.1. *Suppose $\alpha \in (0, \alpha^*)$, and $r(\alpha) := \left(\frac{pl\alpha}{2kc}\right)^{\frac{1}{p+\alpha-2}}$, where c is a constant that comes from the Sobolev embedding, $H_0^2(\Omega) \hookrightarrow L^q(\Omega)$ with q satisfies $1 \leq q \leq \frac{2n}{n-4}$ if $n \geq 5$ and $q \geq 1$ if $n \leq 4$. If $\|\Delta u\|_2 \leq r(\alpha)$, and for any $u \in H_0^2(\Omega) \setminus \{0\}$, we have $I(u) > 0$.*

Proof. Using the definition of $I(u)$ given in (3.2), we observe that

$$\begin{aligned} I(u) &= \left(1 - \int_0^t b(s) ds \right) \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + (b \circ \Delta u) - \frac{2}{p} \int_{\Omega} k u^p \ln |u| dx \\ &\geq l \|\Delta u\|_2^2 - \frac{2}{p} \int_{\Omega} k u^p \ln |u| dx \\ &> l \|\Delta u\|_2^2 - \frac{2k}{p\alpha} \|u\|_{p+\alpha}^{p+\alpha} \\ &\geq l \|\Delta u\|_2^2 - \frac{2kc}{p\alpha} \|\Delta u\|_2^{p+\alpha} \\ &= \frac{2kc}{p\alpha} \|\Delta u\|_2^2 \left(\frac{l p \alpha}{2kc} - \|\Delta u\|_2^{p+\alpha-2} \right) \\ &\geq 0. \end{aligned}$$

Hence, the lemma is proved. □

Let u be a solution of (1.1). We define the total energy $E(u)$ for problem (1.1):

$$E(u)(t) = \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} \left(1 - \int_0^t b(s) ds \right) \|\Delta u\|_2^2 + \frac{1}{2} \|\Delta u_t\|_2^2 + \frac{1}{2} (b \circ \Delta u) - \frac{1}{p} \int_{\Omega} k u^p \ln |u| dx + \frac{k}{p^2} \|u\|_p^p, \tag{3.4}$$

where

$$(b \circ \Delta u)(t) = \int_0^t b(t-s) \|\Delta u(s) - \Delta u(t)\|_2^2 ds.$$

Differentiating E with respect to t , we get

$$\frac{d}{dt} E(u) = \frac{1}{2} (b' \circ \Delta u) - \frac{1}{2} b(t) \|\Delta u\|_2^2 - \|u_t\|_m^m \leq 0. \tag{3.5}$$

We define a subset W of $H_0^2(\Omega)$ related to problem (1.1),

$$W = \{u \in H_0^1(\Omega) : J(u) < d, I(u) > 0\} \tag{3.6}$$

Lemma 3.2. Let $u_0, u_1 \in H_0^2(\Omega)$, $E(0) < d$ and u be the weak solution of problem (1.1) in $[0, T)$. If $I(u_0) > 0$, then $u \in W$.

The proof of the above lemma follows from the similar arguments given in [5]. So, we omit the details here.

Theorem 3.3. Assume the hypothesis of Lemma 3.2 and let $u_0 \neq 0$ with $\|\Delta u_0\|_2 \leq r(\alpha)$, then u is the global solution to problem (1.1).

Proof. Since $\|\Delta u_0\|_2 \leq r(\alpha)$, from Lemma 3.1 we get $I(u_0) > 0$. Thanks to Lemma 3.2, it follows that $u(t) \in W$ on $[0, T)$. From equations (3.4)–(3.2), we have the following estimate

$$\begin{aligned} d > E(0) > E(u) &= \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + J(u) \\ &= \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} I(u) + \frac{k}{p^2} \|u\|_p^p \\ &> \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + c \|\Delta u\|_2^2. \end{aligned}$$

Hence, we have, $\|u_t\|_{\rho+2}^{\rho+2} + \|\Delta u\|_2^2 < c < +\infty$. Therefore, the global existence result follows from the continuation principle. \square

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Author information

Bhargav Kumar Kakumani, Department of Mathematics, Birla Institute of Technology and Science-Pilani, Hyderabad Campus, Hyderabad, India.

E-mail: bhargav@hyderabad.bits-pilani.ac.in

Suman Prabha Yadav, Department of Mathematics, Birla Institute of Technology and Science-Pilani, Hyderabad Campus, Hyderabad, India

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