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Bianchi Type-V Cosmological Model in $f(R, T)$ Gravity: Exact Solutions and Dynamical Analysis

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Abstract. Here we have investigated modified $f(R,T)$ gravity for a Bianchi type-V model in Lyra's Geometry. To acquire the deterministic solution of the field equations within $f(R,T)$, we consider that $f(R,T) = R^{m_1} + T^{m_2}$, R is the Ricci scalar, T is the energy momentum tensor & m_1, m_2 are constants. The equation of state is $p = \omega\rho$, where p is the pressure, ρ is the energy density and ω is the equation of state parameter that depends on the type of matter. Some physical and geometrical features of the model are also be discussed.

1 Introduction

Cosmology is the mathematical study of the universe (or cosmos) as a whole. Cosmology is the study of structure, origin and development of the universe. The word "cosmology" is derived from the combination of two Greek words, Kosmos (meaning harmony or order) and logia (means words or disclosures) and cosmology deals with very large distances, objects that are very big and timescales that are very long. In addition, cosmology also deals with very small things. A cosmological model describes the origin, structure and evolution of the universe. By introducing or modifying the theory of gravity from general relativity, we can explain certain phenomena better than the standard cosmological model. In cosmology, $f(R,T)$ gravity is a modified theory of gravity which is an extension of general relativity, whose action depends on the Ricci scalar R and the trace T of the stress energy tensor. Modified gravity explains certain issues, like the accelerated expansion of the universe, dark energy and dark matter. $f(R,T)$ gravity explains the late time accelerated expansion of the universe, the effect of dark energy and is also used to study inflationary scenarios and other early-universe phenomena.

Lyra's geometry (which is a modified version of Riemannian geometry) used to study cosmology. It is also an important theory in modified theories of gravity, in which displacement field can be deemed a building block of the total energy that can play the role of dark energy. Lyra geometry was introduced by Lyra [1]. by introducing a gauge function ϕ_i into the geometry. Various researchers constructed cosmological models in the context of Lyra's geometry [2, 3, 4, 5, 6, 7, 8, 9]. The behavior of cosmological models with variable deceleration parameter, the scenario of two-fluid dark energy models in the Bianchi type-III universe and FLRW cosmological models with dynamic cosmological term in modified gravity are all discussed by Tiwari, Beesham, and Shukla [10, 11, 12, 13]. The hyperbolic scenario of the accelerating universe was covered by Khan et al. [14]. We have also studied the various deceleration parameter forms proposed by various researchers [15, 16, 17, 18]. By changing the geometrical components of Einstein's equation of motion, Starobinsky [19] and Kerner [20] proposed an alternative modified gravity theory, $f(R)$ gravity [21, 22]. Other modified theories are $f(T)$ gravity, where T is the torsion scalar [23, 24, 25], $f(G)$ gravity, where G is the Gauss Bonnet invariant [26, 27], scalar-tensor theories [28, 29] and higher-dimensional theories [30], Anisotropic model [31, 32]. In this paper, we have discussed the Bianchi type-V cosmological model in modified $f(R,T)$ gravity in Lyra's geometry with certain forms of a varying deceleration parameter.

2 Bianchi Type V Metric, Field Equations & Physical and Geometrical Properties

The Bianchi type-V metric is

$$ds^2 = -dt^2 + M^2 dx^2 + N^2 e^{2x}(dy^2 + dz^2) \tag{2.1}$$

where M and N are functions of cosmic time t only. The action S of $f(R,T)$ gravity is

$$S = \int \sqrt{-g}\left\{\frac{1}{16\pi} f(R,T) + L_m\right\} d^4x \tag{2.2}$$

where L_m is the Lagrangian density of matter. The value of the stress-energy tensor of the matter is defined as

$$T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{ij}} \tag{2.3}$$

The effective energy-momentum tensor T_{ij}^{eff} in $f(R,T)$ gravity is given by

$$T_{ij}^{\text{eff}} = \left(\rho + p + \frac{\partial f}{\partial T}\right) u_i u_j + \left(p + \frac{\partial f}{\partial T}\right) g_{ij} \tag{2.4}$$

where ρ is the energy density, p is the pressure of the matter and $u^i = (1,0,0,0)$ is the four velocity vector in comoving coordinates satisfying the condition

$$u_i u^i = -1 \tag{2.5}$$

The field equations obtained by varying the action S in equation (2.2) in $f(R,T)$ gravity with respect to the metric tensor g_{ij} are

$$f_R(R,T)R_{ij} - \frac{1}{2} f(R,T) g_{ij} + \left(g_{ij}\nabla^i\nabla_j - \nabla_i\nabla_j\right) f_R(R,T) = 8\pi T_{ij} - f_T(R,T)T_{ij} - f_T(R,T)\Theta_{ij} \tag{2.6}$$

where

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_m - 2g^{lm}\frac{\partial^2 L_m}{\partial g^{ij}\partial g^{lm}} \tag{2.7}$$

$$f_R(R,T) = \frac{\partial f(R,T)}{\partial R}, \quad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} \tag{2.8}$$

and ∇_i is the covariant derivative. Here, we are using $L_m = -p$, since there is no unique definition of the matter Lagrangian. Using this definition of L_m in equation (2.7), we get the value of Θ_{ij}, on which the physical nature of the matter field depends

$$\Theta_{ij} = -2T_{ij} - p g_{ij} \tag{2.9}$$

Harko et al. [33] have considered three possible forms for $f(R,T)$, which are

$$f(R,T) = \begin{cases} R + 2f_1(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases} \tag{2.10}$$

In this discussion, we are going to consider the second condition of equation (2.9), i.e.,

$$f(R,T) = f_1(R) + f_2(T) = R^{m_1} + T^{m_2} \tag{2.11}$$

where m_1 and m_2 are constants.

Now from equation(2.6), we get

$$R_{ij} - \frac{1}{2}R g_{ij} = (8\pi + 3)T_{ij} + \Lambda(t) g_{ij} \tag{2.12}$$

where $\Lambda(t) = p + \frac{T}{2}$ is the cosmological constant and depends upon the trace of the energy-momentum tensor T. This was proposed by Poplawski [34], where the cosmological constant in the gravitational Lagrangian is a function of the trace T of the energy-momentum tensor. Ahmed and Pradhan [35] studied the Bianchi type-V model in $f(R, T)$ gravity with $\Lambda(t)$. If the pressure of matter is neglected, $\Lambda(t)$ gravity, which is more general than Palatini $f(R)$, might be reduced to it [36, 37]. For the sake of simplicity, here we will take $m_1 = m_2 = 1$.

In this paper, at the beginning, standard $f(R, T)$ gravity is formulated in Riemannian geometry. We now wish to look at $f(R, T)$ gravity within the framework of Lyra's geometry. To modify the connection and curvature tensors, Lyra's displacement vector ϕ_i is introduced, i.e., in this work, the standard Levi-Civita connection is replaced by the Lyra connection containing a displacement vector as follows. The action of $f(R, T)$ in Lyra's geometry is given by (Singh and Singh [38])

$$S = \int \sqrt{-g}\left\{\frac{1}{16\pi}{}^*R + L_m\right\} d^4x \tag{2.13}$$

where ${}^*R = R + 3\phi^i{}_i + \frac{3}{2}\phi_i\phi^i$ so to incorporate $f(R, T)$ gravity into Lyra geometry we just have to add

$$\int \sqrt{-g}\frac{1}{16\pi}\left\{3\phi^i_{,i} + \frac{3}{2}\phi_i\phi^i\right\} d^4x \tag{2.14}$$

to the right side of the action (2.2).

Thus the subsequent field equations belong to the Lyra-geometric generalization of $f(R, T)$ gravity [39]. With this change in the gravitational field equations, curvature terms are changed and an extra contribution from ϕ_i is added. The resulting model is therefore a Lyra geometric extension of $f(R, T)$ gravity where ϕ_i includes novel geometric effects that could affect matter coupling and gravitational dynamics, which also enables accelerated expansion without the need for dark energy as in the ΛCDM model.

The field equations in Lyra's manifold with $8\pi G = 1$ and $c = 1$ are given by Sen [40] as

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}\phi_i\phi^i g_{ij} = (8\pi + 3)T_{ij} + \Lambda(t)g_{ij} \tag{2.15}$$

where ϕ_i is the displacement vector field in Lyra's manifold defined as

$$\phi_i = (0, 0, 0, \beta(t)) \tag{2.16}$$

For the metric (2.1), Einstein field equations obtained by taking $8\pi + 3 = b$ are

$$2\frac{\dot{M}\dot{N}}{MN} + \frac{\dot{N}^2}{N^2} - \frac{3}{M^2} - \frac{3}{4}\beta^2 - \Lambda(t) = b(p + 1) \tag{2.17}$$

$$2\frac{\ddot{N}}{N} + \frac{\dot{N}^2}{N^2} - \frac{1}{M^2} + \frac{3}{4}\beta^2 - \Lambda(t) = b(\rho + 1) \tag{2.18}$$

$$\frac{\ddot{M}}{M} + \frac{\ddot{N}}{N} + \frac{\dot{M}\dot{N}}{MN} - \frac{1}{M^2} + \frac{3}{4}\beta^2 - \Lambda(t) = b(p + 1) \tag{2.19}$$

Here an overhead dot indicates differentiation with respect to cosmic time t. Equations$(2.17) - -(2.19)$ are three equations in the five unknowns M, N, β, p, ρ. Therefore, to find a solution, two more conditions are required. The two conditions that we choose are

(i) Since the behavior of a power law solution is common in expanding or collapsing systems in cosmological models, we assume a solution of the system of equations of the form

$$\frac{\dot{M}}{M} = \frac{\dot{N}}{N} = \frac{a}{t^n} \tag{2.20}$$

where a, n are constants. We note that this assumption is equivalent to assuming $M = N$ in the metric(1.1). The form (2.20) suggests that the growth or decay rates of M and N follow a power law dependence on time. Integrating equation (2.20), we get

$$M = N = M_1 \exp\left\{\frac{a}{1-n}t^{1-n}\right\} \tag{2.21}$$

(ii) Since the power assumption (2.20) does not fully close the system, one more equation is needed to fully close the system. Therefore, we assume the equation of state

$$p = \omega\rho \tag{2.22}$$

For the equation of state of dark energy,

$$\omega = -1, \quad \text{i.e.,} \quad p = -\rho \tag{2.23}$$

The volume V is given by

$$V = M_1 \exp\left\{\frac{a}{1-n}t^{1-n}\right\} \tag{2.24}$$

The deceleration parameter q is given by

$$q = -1 \tag{2.25}$$

Since the deceleration parameter q is negative, this denotes an accelerating expanding universe. The shear scalar σ is given by

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = 0 \tag{2.26}$$

i.e., there is no shear. The expansion scalar is

$$\theta = \frac{3a}{t^n} \tag{2.27}$$

Also,

$$\lim_{t\to\infty} \frac{\sigma}{\theta} = 0 \tag{2.28}$$

Therefore, our model describes a non-shearing, non-rotating, continuously expanding universe.

For the sake of simplicity, taking $n = 1$, we get

$$\begin{aligned} p &= \frac{a}{t^2}(1-3a) \\ \rho &= \frac{a}{t^3}(3a-1) \end{aligned} \tag{2.29}$$

The cosmological parameter, which is a variable in this model, is given by

$$\Lambda = 3\left\{\frac{1}{2} - \frac{a}{t^2}(3a-1)\right\}$$

The displacement vector β has the value

$$\beta = \frac{2}{\sqrt{3}}\sqrt{\frac{1}{t^2}\left\{a(1-3a)(b+3) + (2-a^2)\right\} + b + \frac{3}{2}}$$

We now plot the volume V, pressure p, energy density ρ, expansion scalar θ and cosmological parameter Λ in the figures below:

In the graphical representation, the current epoch is represented by $t = 1$ and we have used dimensionless time t normalized by the Hubble time H_o^{-1} to remove the dependence on specific units, to make the results more general and to maintain the physical meaning of cosmological expressions. In all the above figures, the transition from deceleration ($t < 2$) to acceleration ($t > 2$) is depicted. The $t = 10$ endpoint of our figures represents a future de-Sitter state of the cosmological model, after which there is never-ending acceleration. The unit of time that we are using is that $t = 1$ corresponds to 13.7 Gyr, the current time.

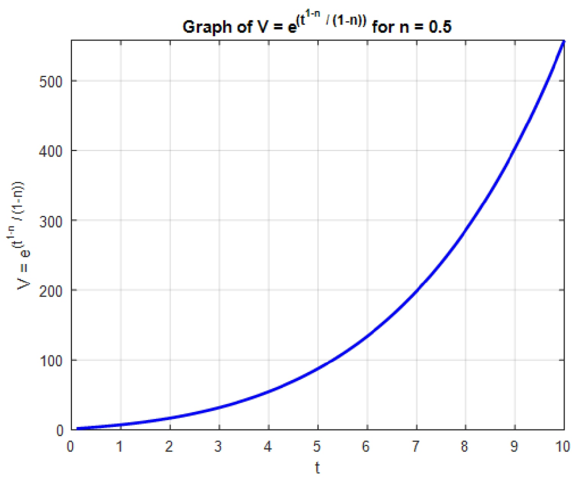

Figure 1. Variation of Spatial volume V against cosmic time t taking $M_1 = 1, a = 1$

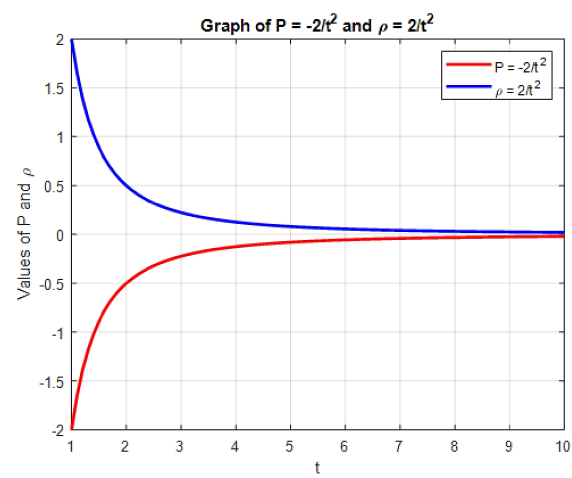

Figure 2. Variation of Pressure of matter and energy density against cosmic time t taking $a = 1$

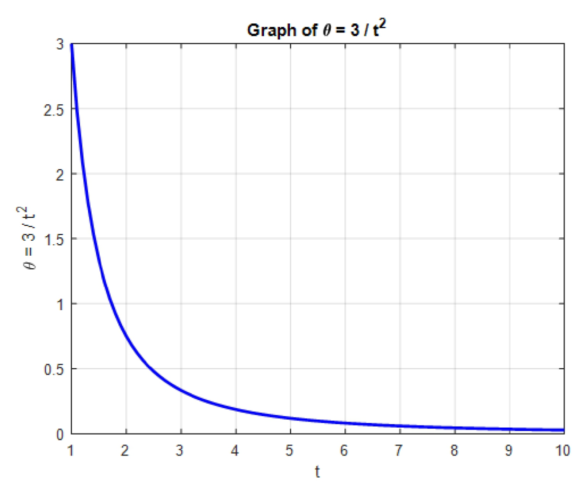

Figure 3. Variation of expansion scalar θ against cosmic time t taking $n = 1$

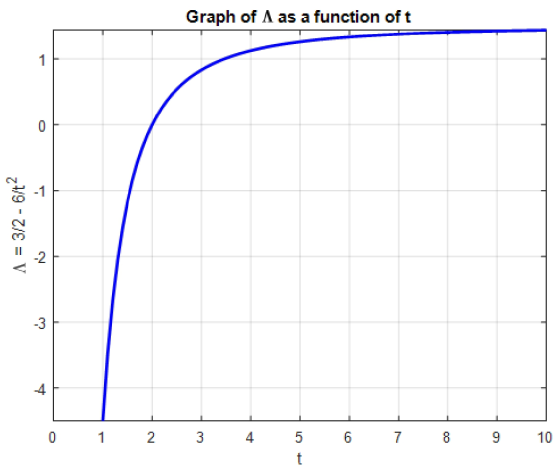

Figure 4. Variation of Cosmological constant Λ against cosmic time t taking $a = 1$

3 Conclusions

Here we discussed the Bianchi type-V cosmological model in $f(R,T)$ theory of gravity. It is observed that the shear scalar σ is zero, i.e., there is no shear scalar in the model.

Hence

$$\lim_{t\to\infty} \frac{\sigma}{\theta} = 0.$$

The model describes a non-shearing, non-rotating, continuously expanding universe. Also it is observed that, as time t increases, the expansion scalar θ decreases.

The energy density of the vacuum is linked to the cosmological parameter Λ, which affects the pace of expansion of the universe. When the cosmic time $t > 2$, the cosmological parameter $\Lambda > 0$, and the universe experiences accelerated expansion similar to the dark energy model. When $t < 2$, $\Lambda < 0$, and our model experiences deceleration. The negative value of the cosmological parameter reflects an attractive force. This contributes to the deceleration of the universe in the past. At the point $t = 2$, we get $\Lambda = 0$. This point is known as the transition point where the universe shifts from deceleration to acceleration.

References

[1] G. Lyra, über-eine Modifikation der Riemannschen Geometrie. Mathematische Zeitschrift 54 (1951), 52-64.

[2] F. Rahaman, J. Bera, Higher dimensional cosmological model in Lyra geometry. Int. J. Mod. Phys. 10 (2010), 729.

[3] F. Rahaman, S. Chakraborty, J. Bera, Inhomogeneous cosmological model in Lyra geometry. Int. J. Mod. Phys. (2002), 11, 1501.

[4] J. Baro, P.K. Singh, T. Singh, A. Higher Dimensional LRS Bianchi Type-I String Cosmological Model with Bulk Viscosity in General Relativity. Indian J. Sci. Tech. (2021) 14, 46.

[5] G. Mohanty, K. L. Mahanta, R. R. Sahoo, Non-Existence of Five Dimensional Perfect Fluid Cosmological Model in Lyra Manifold. Astrophys. Space Sci. (2006) 306, 269.

[6] G. Mohanty, K. L. Mahanta, B. K. Bishi, Five Dimensional Cosmological Models in Lyra Geometry with Time Dependent Displacement Field. Astrophys. Space Sci. (2007), 310, 273.

[7] G. Mohanty, K. L. Mahanta, Five-Dimensional Axially Symmetric String Cosmological Model in Lyra Manifold. Astrophys. Space Sci. (2007), 312, 301.

[8] P. K. Singh, M. R. Mollah, Higher dimensional LRS Bianchi type-I Cosmological model universe interacting with perfect fluid in Lyra Geometry. The African Rev. Phys. (2016), 11, 33.

[9] B. P. Brahma and M. Dewri, Role of $f(R,T)$ Gravity in Bianchi Type-V Dark Energy Model with Electromagnetic Field based on Lyra Geometry. J. Sci. Res. (2022), 14, 721.

[10] R. K. Tiwari, A. Beesham, B. K. Shukla, Scenario of two-fluid dark energy models in Bianchi type-III Universe. Int. J. Geom. Methods Mod. Phys. (2018), 15, 1850189.

[11] R. K. Tiwari, A. Beesham, B. Shukla, Cosmological models with viscous fluid and variable deceleration parameter. Eur. Phys. J. Plus (2017), 132, 20.

[12] R. K. Tiwari, A. Beesham, B. Shukla, Behaviour of the cosmological model with variable deceleration parameter. Eur. Phys. J. Plus (2016), 131, 1–9.

[13] R. K. Tiwari, A. Beesham, B. K. Shukla, FLRW Cosmological Models with Dynamic Cosmological Term in Modified Gravity. Universe (2021), 7, 319.

[14] R. A. A. Khan, R. K. Tiwari, J. Bharali, A. Bouali, Yildiz, G.D.A.; Güdekli, E. Hyperbolic Scenario of Accelerating Universe in Modified Gravity. Symmetry (2023), 15, 1238.

[15] M. S. Berman, F. de Mello Gomide, Cosmological models with constant deceleration parameter. Gen. Relativ. Gravit. (1988), 20, 191–198.

[16] O. Akarsu, T. Dereli, Cosmological models with linearly varying deceleration parameter. Int. J. Theor. Phys. (2012), 51, 612–621.

[17] Y. L. Bolotin, V. Cherkaskiy, O. Lemets, D. Yerokhin, L. Zazunov, Cosmology in terms of the deceleration parameter. Part I. arXiv (2015), arXiv:1502.00811.

[18] A. Bouali, H. Chaudhary, U. Debnath, A. Sardar, Mustafa, G. Data Analysis of three parameter models of deceleration parameter in FRW Universe. arXiv (2023), arXiv:2304.13137.

[19] A. A. Starobinsky, A new type of isotropic cosmological models without singularity. Phys. Lett. B (1980), 91, 99–102.

[20] R. Kerner, Cosmology without singularity and nonlinear gravitational Lagrangians. Gen. Relativ. Gravit. (1982), 14, 453–469.

[21] S. Nojiri, S. D. Odintsov, Unified cosmic history in modified gravity: From $F(R)$ theory to Lorentz non-invariant models. Phys. Rep. (2011), 505, 59–144.

[22] S. Nojiri, S. Odintsov, V. Oikonomou, Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution. Phys. Rep. (2017), 692, 1–104.

[23] G. R. Bengochea, R. Ferraro, Dark torsion as the cosmic speed-up. Phys. Rev. D (2009), 79, 124019.

[24] E. V. Linder, Einstein's Other Gravity and the Acceleration of the Universe. Phys. Rev. D (2010), 81, 127301.

[25] K. Bamba, C. Q. Geng, Thermodynamics of cosmological horizons in $f(T)$ gravity. J. Cosmol. Astropart. Phys. (2011), 2011, 008.

[26] K. Bamba, S. D. Odintsov, L. Sebastiani, S. Zerbini, Finite-time future singularities in modified Gauss-Bonnet and (R, G) gravity and singularity avoidance. Eur. Phys. J. C (2010), 67, 295–310.

[27] M. V. de S. Silva, M. E. Rodrigues, Regular black holes in $f(G)$ gravity. Eur. Phys. J. C (2018), 78, 1–18.

[28] C. Brans, R. H. Dicke, Mach's principle and a relativistic theory of gravitation. Phys. Rev. (1961), 124, 925.

[29] D. Saez, V. Ballester, A simple coupling with cosmological implications. Phys. Lett. A (1986), 113, 467–470.

[30] M. J. Duff, Kaluza-Klein theory in perspective. In Proceedings of the Symposium: The Oskar Klein Centenary; World Scientific: Singapore, (1994); pp. 22–35.

[31] S. N. K. Sharma, Anisotropic cosmological model involving null radiation flow and magnetic field. Pal. J. Math (2022), 11, 477–484.

[32] A. Pradhan, S. S. Kumar, K. Jotania, Anisotropic Bianchi Type-I Massive String Cosmological Models in General Relativity. Pal. J. Math (2012), 1, 117–132.

[33] T. Harko, F. S. Lobo, S. Nojiri, S. D. Odintsov, $f(R, T)$ gravity. Phys. Rev. D (2011), 84, 024020.

[34] N. J. Poplawski, Interacting dark energy in $f(R)$ gravity. Arxiv 20060608031.

[35] N. Ahmed, A. Pradhan, Bianchi Type-V cosmology in $f(R, T)$ gravity with $\Lambda(T)$. arxiv (2014), 1303.3000v2.

[36] G. Magnano, Are there metric theories of gravity other than General Relativity? arxiv (1995), 9511027.

[37] N. J. Porlawski, A Lagrangian description of interacting dark energy. Class Quantum Grav. (2006), 23, 2011.

[38] T. Singh and G. P. Singh, Lyra's Geometry and cosmology: A Review, (1993) 41, 737.

[39] B. P. Brahma, M. Dewri, Bianchi type-V Modified $f(R, T)$ gravity model in Lyra geometry with variable deceleration parameter (2021), 11, 1018.

[40] D. K. Sen, A static cosmological model. Z. Phys. (1957), 149, 311.

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