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SCREEN TRANSVERSAL CAUCHY-RIEMANN BI-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of screen transversal Cauchy-Riemann bi-slant lightlike submanifolds of indefinite Kaehler manifolds. We not only present a comprehensive characterization theorem but also offer illuminating examples of these submanifolds, showcasing their non-trivial nature. Furthermore, we rigorously establish integrability conditions for the distributions D_1, D_2 and $Rad(TM)$ within such submanifolds. Delving deeper, we unveil the necessary and sufficient conditions for the foliations determined by these distributions to emerge as totally geodesic - a crucial insight into the geometric properties of these intriguing structures.

1 Introduction

In 1996, Duggal and Bejancu first introduced the theory concerning lightlike submanifolds within a semi-Riemannian manifold ([5]). Described as submanifolds where the induced metric g becomes degenerate, a lightlike submanifold $\mathcal{M}$ of a semi-Riemannian manifold $\bar{\mathcal{M}}$ has been pivotal in various applications, notably in general relativity, particularly within black hole theory. This significance prompted geometers to delve into the exploration of lightlike submanifolds within semi-Riemannian manifolds endowed with specific structures.

B. Y. Chen, in 1990, introduced the concept of slant immersions within complex geometry, representing a natural extension encompassing both totally real immersions and holomorphic immersions ([3], [4]). In ([1], [2]) A. Carriazo defined bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and introduced the notion of pseudo-slant submanifolds. Additionally, in 1994, N. Papaghuic contributed by introducing the notion of semi-slant submanifolds within Kaehler manifolds([9]). The concept of slant immersion of a Riemannian manifold into an almost contact metric manifold was coined by A. Lotta ([8]). Further developments unfolded as Sahin, in ([10], [11]), delved into the geometry of slant and screen-slant lightlike submanifolds within indefinite Hermitian manifolds. Explorations into the theory of slant, CR lightlike submanifolds, and SCR lightlike submanifolds within indefinite Kaehler manifolds and indefinite Sasakian manifolds have been documented in ([6], [15]).

This paper introduces a novel concept termed screen transversal Cauchy-Riemann (STCR) bi-slant lightlike submanifolds within indefinite Kaehler manifolds. The structure of the paper unfolds as follows: Section 2 presents foundational results, while Section 3 delves into the exploration of quasi-bi-slant lightlike submanifolds within an indefinite Kaehler manifold, accompanied by an illustrative example. Section 4 is dedicated to investigating foliations determined by distributions on screen transversal Cauchy-Riemann bi-slant lightlike submanifolds of indefinite Kaehler manifolds.

2 Preliminaries

A lightlike submanifold $(\mathcal{M}^m, g)$ immersed in a semi-Riemannian manifold $(\bar{\mathcal{M}}^{m+n}, \bar{g})$ is a submanifold in which induced metric g from $\bar{g}$ is degenerate. There exists locally a lightlike vector field $\xi \in \Gamma(T\mathcal{M}),\ \xi \neq 0$, such that $g(\xi, X) = 0$, for any $\mathcal{X} \in \Gamma(T\mathcal{M})$. Then, for each tangent space $T_p\mathcal{M}$ we have

$$T_p\mathcal{M}^\perp = \{u \in T_p\bar{\mathcal{M}} : \bar{g}(u, v) = 0, \forall\, v \in T_p\mathcal{M}\},$$

which is a degenerate n-dimension subspace of $T_p\bar{\mathcal{M}}$. The radical (null) subspace of $T_p\mathcal{M}$, denoted by $\mathcal{R}ad(T_p\mathcal{M})$, is defined by

$$\mathcal{R}ad(T_p\mathcal{M}) = \{\xi_p \in T_p\bar{\mathcal{M}},\ g(\xi_p, X) = 0,\ \mathcal{X} \in T_p\bar{\mathcal{M}}\}.$$

The dimension of $\mathcal{R}ad(T_p\bar{\mathcal{M}}) = T_p\bar{\mathcal{M}} \cap T_p\bar{\mathcal{M}}^\perp$ depends on $p \in \mathcal{M}$. For an r-lightlike submanifold, the mapping

$$\mathcal{R}ad(T\mathcal{M}) : p \in \mathcal{M} \to \mathcal{R}ad(T_p\bar{\mathcal{M}})$$

defines a smooth distribution on $\mathcal{M}$ of rank r, where $1 \leq r \leq m$. Now suppose that $S(T\mathcal{M})$ be a semi-Riemannian complementary distribution of $\mathcal{R}ad(T\mathcal{M})$ in $T\mathcal{M}$, called screen distribution, i.e.

$$T\mathcal{M} = \mathcal{R}ad(T\mathcal{M}) \oplus_{orth} S(T\mathcal{M}). \tag{2.1}$$

Let $S(T\mathcal{M}^\perp)$ be a semi-Riemannian complementary vector bundle of $\mathcal{R}ad(T\mathcal{M})$ in $T\mathcal{M}^\perp$, called screen transversal vector bundle. Since for any local basis $\{\xi_i\}$ of $\mathcal{R}ad(T\mathcal{M})$ there exists a local null frame $\{\mathcal{N}_i\}$ of sections with values in the orthogonal complement of $S(T\mathcal{M}^\perp)$ in $[S(T\mathcal{M})]^\perp$ such that $\bar{g}(\xi_i, \mathcal{N}_j) = \delta_{ij}$ and $\bar{g}(\mathcal{N}_i, \mathcal{N}_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(T\mathcal{M})$ locally spanned by $\{\mathcal{N}_i\}$. Let

$$tr(T\mathcal{M}) = ltr(T\mathcal{M}) \oplus_{orth} S(T\mathcal{M}^\perp). \tag{2.2}$$

Then, $tr(T\mathcal{M})$ is a complementary which is not orthogonal vector bundle to $T\mathcal{M}$ in $T\bar{\mathcal{M}}|_{\mathcal{M}}$, i.e.

$$T\bar{\mathcal{M}}|_{\mathcal{M}} = T\mathcal{M} \oplus tr(T\mathcal{M}) \tag{2.3}$$

and therefore

$$T\bar{\mathcal{M}}|_{\mathcal{M}} = S(T\mathcal{M}) \oplus_{orth} [\mathcal{R}ad(T\mathcal{M}) \oplus ltr(T\mathcal{M})] \oplus_{orth} S(T\mathcal{M}^\perp). \tag{2.4}$$

Following this, we have the four possible cases for a lightlike submanifold:
Case 1: r-lightlike if $r \leq min(m, n)$,
Case 2: co-isotropic if $r = n \leq m,\ S(T\mathcal{M}^\perp) = \{0\}$,
Case 3: isotropic if $r = m \leq n,\ S(T\mathcal{M}) = \{0\}$,
Case 4: totally lightlike if $r = m = n,\ S(T\mathcal{M}) = S(T\mathcal{M}^\perp) = \{0\}$.
The Gauss and Weingarten formulae are given as

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + h(\mathcal{X}, \mathcal{Y}), \tag{2.5}$$

$$\bar{\nabla}_{\mathcal{X}}\mathcal{V} = -A_{\mathcal{V}}\mathcal{X} + \nabla^t_{\mathcal{X}}\mathcal{V}. \tag{2.6}$$

for all $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M})$ and $\mathcal{V} \in \Gamma(tr(T\mathcal{M}))$, where $\{\nabla_{\mathcal{X}}\mathcal{Y}, A_{\mathcal{V}}\mathcal{X}\}$ belong to $\Gamma(T\mathcal{M})$ and $\{h(\mathcal{X}, \mathcal{Y}), \nabla^t_{\mathcal{X}}\mathcal{V}\}$ belong to $\Gamma(tr(T\mathcal{M}))$. Here, ∇ and ∇^t are linear connections on $\mathcal{M}$ and on the vector bundle $tr(T\mathcal{M})$ respectively. The second fundamental form h is a symmetric $F(\mathcal{M})$-bilinear form on $\Gamma(T\mathcal{M})$ with values in $\Gamma(tr(T\mathcal{M}))$ and the shape operator $A_{\mathcal{V}}$ is a linear endomorphism of $\Gamma(T\mathcal{M})$. From (2.5) and (2.6) we have

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + h^l(\mathcal{X}, \mathcal{Y}) + h^s(\mathcal{X}, \mathcal{Y}),\ \ \forall\, \mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M}), \tag{2.7}$$

$$\bar{\nabla}_{\mathcal{X}}\mathcal{N} = -A_{\mathcal{N}}\mathcal{X} + \nabla^l_{\mathcal{X}}\mathcal{N} + D^s(\mathcal{X}, \mathcal{N}),\ \ \forall\, \mathcal{N} \in \Gamma(ltr(T\mathcal{M})), \tag{2.8}$$

$$\bar{\nabla}_{\mathcal{X}}\mathcal{W} = -A_{\mathcal{W}}\mathcal{X} + D^l(\mathcal{X}, \mathcal{W}) + \nabla^s_{\mathcal{X}}\mathcal{W},\ \ \forall\, \mathcal{W} \in \Gamma(S(T\mathcal{M}^\perp)). \tag{2.9}$$

where $h^l(\mathcal{X}, \mathcal{Y}) = L(h(\mathcal{X}, \mathcal{Y}))$, $h^s(\mathcal{X}, \mathcal{Y}) = S(h(\mathcal{X}, \mathcal{Y}))$, $D^l(\mathcal{X}, \mathcal{W}) = L(\nabla^t_{\mathcal{X}}\mathcal{W})$, $D^s(\mathcal{X}, \mathcal{N}) = S(\nabla^t_{\mathcal{X}}\mathcal{N})$, L and S are the projection morphisms of $tr(T\mathcal{M})$ on $ltr(T\mathcal{M})$ and $S(T\mathcal{M}^\perp)$ respectively. Thus h^l and h^s are $\Gamma(ltr(T\mathcal{M}))$-valued and $\Gamma(S(T\mathcal{M}^\perp))$-valued lightlike second fundamental form and screen second fundamental form of $\mathcal{M}$ respectively. On the other hand, ∇^l and ∇^s are linear connections on $ltr(T\mathcal{M})$ and $S(T\mathcal{M}^\perp)$ called the lightlike connection and screen transversal connection on $\mathcal{M}$ respectively. Now by using (2.5), (2.7)-(2.9) and metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^s(\mathcal{X},\mathcal{Y}),\mathcal{W}) + \bar{g}(\mathcal{Y}, D^l(\mathcal{X},\mathcal{W})) = g(A_{\mathcal{W}}\mathcal{X}), \tag{2.10}$$

$$\bar{g}(D^s(\mathcal{X},\mathcal{N}),\mathcal{W}) = g(\mathcal{N}, A_{\mathcal{W}}\mathcal{X}). \tag{2.11}$$

Suppose $\bar{P}$ is the projection of $T\mathcal{M}$ on $S(T\mathcal{M})$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_{\mathcal{X}}\bar{P}\mathcal{Y} = \nabla^*_{\mathcal{X}}\bar{P}\mathcal{Y} + h^*(\mathcal{X},\bar{P}\mathcal{Y}), \ \ \forall\, \mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M}), \tag{2.12}$$

$$\nabla_{\mathcal{X}}\xi = -A^*_{\xi}\mathcal{X} + \nabla^{*t}_{\mathcal{X}}\xi, \ \ \xi \in \Gamma(\mathcal{R}ad(T\mathcal{M})) \tag{2.13}$$

where $\{\nabla^*_{\mathcal{X}}\bar{P}\mathcal{Y}, -A^*_{\xi}\mathcal{X}\}$ and $\{h^*(\mathcal{X},\bar{P}\mathcal{Y}), \nabla^{*t}_{\mathcal{X}}\xi\}$ belong to $\Gamma(S(T\mathcal{M}))$ and $\Gamma(\mathcal{R}ad(T\mathcal{M}))$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on $S(T\mathcal{M})$ and $\mathcal{R}ad(T\mathcal{M})$ respectively. On the other hand, h^* and A^* are called the second fundamental forms of distributions $S(T\mathcal{M})$ and $\mathcal{R}ad(T\mathcal{M})$ respectively, which are $\Gamma(\mathcal{R}ad(T\mathcal{M}))$-valued and $\Gamma(S(T\mathcal{M}))$-valued $F(\mathcal{M})$-bilinear forms on $\Gamma(T\mathcal{M}) \times \Gamma(S(T\mathcal{M}))$ and $\Gamma(\mathcal{R}ad(T\mathcal{M})) \times \Gamma(T\mathcal{M})$. Now by using the above equations, we obtain

$$\bar{g}(h^l(\mathcal{X},\bar{P}\mathcal{Y}),\xi) = g(A^*_{\xi}\mathcal{X},\bar{P}\mathcal{Y}), \tag{2.14}$$

$$\bar{g}(h^*(\mathcal{X},\bar{P}\mathcal{Y}),\mathcal{N}) = g(A_{\mathcal{N}}\mathcal{X},\bar{P}\mathcal{Y}), \tag{2.15}$$

$$\bar{g}(h^l(\mathcal{X},\xi),\xi) = 0, \ A^*_{\xi}\xi = 0. \tag{2.16}$$

Here, it is important to note that the induced connection ∇ on $\mathcal{M}$ is not a metric connection in general. Since $\bar{\nabla}$ is a metric connection, by using (2.7) we get

$$(\nabla_{\mathcal{X}}g)(\mathcal{Y},\mathcal{Z}) = \bar{g}(h^l(\mathcal{X},\mathcal{Y}),\mathcal{Z}) + \bar{g}(h^l(\mathcal{X},\mathcal{Z}),\mathcal{Y}). \tag{2.17}$$

A $2m$-dimensional semi-Riemannian manifold $(\bar{\mathcal{M}},\bar{g},\bar{\mathcal{J}})$ with constant index q, where $0 < q < 2m$, and a $(1,1)$ tensor field $\bar{\mathcal{J}}$ on $\bar{\mathcal{M}}$ is called an indefinite almost Hermitian manifold if the following conditions are satisfied:

$$\bar{\mathcal{J}}^2\mathcal{X} = -\mathcal{X}, \tag{2.18}$$

$$\bar{g}(\bar{\mathcal{J}}\mathcal{X},\bar{\mathcal{J}}\mathcal{Y}) = \bar{g}(\mathcal{X},\mathcal{Y}), \ \ \forall\, \mathcal{X},\mathcal{Y} \in \Gamma(T\bar{\mathcal{M}}). \tag{2.19}$$

An indefinite almost Hermitian manifold $(\bar{\mathcal{M}},\bar{g};\bar{\mathcal{J}})$ is called an indefinite Kaehler manifold if $\bar{\mathcal{J}}$ is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}})\mathcal{Y} = 0, \tag{2.20}$$

for all $\mathcal{X},\mathcal{Y} \in \Gamma(T\bar{\mathcal{M}})$, where $\bar{\nabla}$ is the Levi-Civita connection with respect to $\bar{g}$. The fundamental 2-form Ω of $\bar{M}$ is defined by

$$\Omega(\mathcal{X},\mathcal{Y}) = \bar{g}(\mathcal{X},\bar{J}\mathcal{Y}), \ \forall\, \mathcal{X},\mathcal{Y} \in \Gamma(T\bar{M}). \tag{2.21}$$

It is known that $d\Omega = 0$ for a Kaehler manifold $\bar{M}$, where d is the operator of exterior derivative. The Riemann curvature tensor field R satisfies

$$\bar{R}(\mathcal{X},\mathcal{Y})\bar{J} = \bar{J}\bar{R}(\mathcal{X},\mathcal{Y}), \ \ \bar{R}(\bar{J}\mathcal{X},\bar{J}\mathcal{Y}) = \bar{R}(\mathcal{X},\mathcal{Y}). \tag{2.22}$$

Suppose $\bar{M}(c)$ is an indefinite complex space form, with a constant holomorphic sectional curvature c. The curvature tensor of $\bar{M}(c)$ is given by

$$\bar{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} = \frac{c}{4}\{\bar{g}(\mathcal{Y},\mathcal{Z})\mathcal{X} - \bar{g}(\mathcal{X},\mathcal{Z})\mathcal{Y} + \bar{g}(\bar{J}\mathcal{Y},\mathcal{Z})\bar{J}\mathcal{X} - \bar{g}(\bar{J}\mathcal{X},\mathcal{Z})\bar{J}\mathcal{Y} + 2\bar{g}(\mathcal{X},\bar{J}\mathcal{Y})\bar{J}\mathcal{Z}\}, \ \forall\, \mathcal{X},\mathcal{Y} \in \Gamma(T\bar{\mathcal{M}}). \tag{2.23}$$

However, it is important to note that ∇ is a metric connection on $S(T\mathcal{M})$. The curvature tensor of a lightlike submanifold is denoted by R and the Gauss equation for lightlike submanifolds is given by

$$\begin{aligned}\bar{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} &= R(\mathcal{X},\mathcal{Y})\mathcal{Z} + A_{h^l(\mathcal{X},\mathcal{Z})}\mathcal{Y} - A_{h^l(\mathcal{Y},\mathcal{Z})}\mathcal{X} + A_{h^s(\mathcal{X},\mathcal{Z})}\mathcal{Y} - A_{h^s(\mathcal{Y},\mathcal{Z})}\mathcal{X}\\ &+ (\nabla_{\mathcal{X}} h^l)(\mathcal{Y},\mathcal{Z}) - (\nabla_{\mathcal{Y}} h^s)(\mathcal{X},\mathcal{Z}) + \mathcal{D}^l(\mathcal{X}, h^s(\mathcal{Y},\mathcal{Z})) - \mathcal{D}^l(\mathcal{Y}, h^s(\mathcal{X},\mathcal{Z})) +\\ &(\nabla_{\mathcal{X}} h^s)(\mathcal{Y},\mathcal{Z})(\nabla_{\mathcal{Y}} h^s)(\mathcal{X},\mathcal{Z}) + \mathcal{D}^s(\mathcal{X}, h^l(\mathcal{Y},\mathcal{Z}))\mathcal{D}^s(\mathcal{Y}, h^l(\mathcal{X},\mathcal{Z})), \quad (2.24)\end{aligned}$$

for any $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma(T\mathcal{M})$.

3 Screen Transversal Cauchy-Riemann Bi-slant Lightlike Submanifolds

In this section, we introduce the notion of screen transversal Cauchy-Riemann bi-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following lemma which was proved by Sahin. We shall use this lemma in defining the notion of screen transversal Cauchy-Riemann bi-slant lightlike submanifolds of indefinite Kaehler manifolds.

Lemma 3.1. *([11]) Let $\mathcal{M}$ be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$ of index $2q$. Suppose that there exists a screen distribution $S(T\mathcal{M})$ such that $\bar{\mathcal{J}}\mathcal{R}ad(T\mathcal{M}) \subset S(T\mathcal{M})$ and $\bar{\mathcal{J}}ltr(T\mathcal{M}) \subset S(T\mathcal{M})$. Then $\bar{\mathcal{J}}\mathcal{R}ad(T\mathcal{M}) \cap \bar{\mathcal{J}}ltr(T\mathcal{M}) = \{0\}$ and any complementry distribution to $\bar{\mathcal{J}}\mathcal{R}ad(T\mathcal{M}) \oplus \bar{\mathcal{J}}ltr(T\mathcal{M})$ in $S(T\mathcal{M})$ is Riemannian.*

Definition 3.2. Let $\mathcal{M}$ be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$ of index $2q$ such that $2q < dim(\mathcal{M})$. Then we say that $\mathcal{M}$ is a screen transversal Cauchy-Riemann bi-slant lightlike submanifold of $\bar{\mathcal{M}}$ if the following conditions are satisfied:
(i) $\mathcal{R}ad(T\mathcal{M})$ is a distribution on $\mathcal{M}$ such that

$$\mathcal{R}ad(T\mathcal{M}) = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

where $\bar{\mathcal{J}}\mathcal{D}_1 \subset S(T\mathcal{M})$ and $\bar{\mathcal{J}}\mathcal{D}_2 \subset S(T\mathcal{M}^\perp)$. Furthermore, we have $ltr(T\mathcal{M}) = L_1 \oplus L_2$ where $\bar{\mathcal{J}}L_1 \subset S(T\mathcal{M})$ and $\bar{\mathcal{J}}L_2 \subset S(T\mathcal{M}^\perp)$,
(ii) there exist non-degenerate orthogonal distributions $\mathcal{D}_1'$ and $\mathcal{D}_2'$ on $\mathcal{M}$ such that

$$S(T\mathcal{M}) = (\bar{\mathcal{J}}\mathcal{D}_1 \oplus \bar{\mathcal{J}}L_1) \oplus_{orth} \mathcal{D}_1' \oplus_{orth} \mathcal{D}_2',$$

where L_1 is a distribution of $ltr(T\mathcal{M})$,
(iii) the distribution $\mathcal{D}_1'$ exhibits slant behaviour with an angle θ_1, i.e. for each $x \in \mathcal{M}$ and each non-zero vector $\mathcal{X} \in (\mathcal{D}_1')_x$, the angle θ_1 between $\bar{\mathcal{J}}\mathcal{X}$ and the vector subspace $(\mathcal{D}_1')_x$ is a non-zero constant, which is independent of the choice of $x \in \mathcal{M}$ and $\mathcal{X} \in (\mathcal{D}_1')_x$,
(iv) the distribution $\mathcal{D}_2'$ is slant with angle θ_2, i.e. for each $x \in \mathcal{M}$ and each non-zero vector $\mathcal{X} \in (\mathcal{D}_2')_x$, the angle θ_2 between $\bar{\mathcal{J}}\mathcal{X}$ and the vector subspace $(\mathcal{D}_2')_x$ is a non-zero constant, which is independent of the choice of $x \in \mathcal{M}$ and $\mathcal{X} \in (\mathcal{D}_2')_x$.

These constant angles θ_1 and θ_2 are called the slant angles of distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. A generalized quasi-bi-slant lightlike submanifold is said to be proper if $\mathcal{D}_1 \neq \{0\}$; $\mathcal{D}_2 \neq \{0\}$ and $\theta_1 \neq \pi/2$; $\theta_2 \neq \pi/2$.

From the above definition, we have the following decomposition:

$$T\mathcal{M} = \mathcal{R}ad(T\mathcal{M}) \oplus_{orth} (\bar{\mathcal{J}}\mathcal{D}_1 \oplus \bar{\mathcal{J}}L_1) \oplus_{orth} \mathcal{D}_1' \oplus_{orth} \mathcal{D}_2'.$$

Let $(\mathbb{R}^{2m}_{2q}, \bar{g}, \bar{\mathcal{J}})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual $\mathcal{K}$aehler structure given by

$$\bar{g} = \frac{1}{4}(-\textstyle\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\bar{\mathcal{J}}(\textstyle\sum_{i=1}^{m}(\mathcal{X}_i \partial x_i + \mathcal{Y}_i \partial y_i)) = \sum_{i=1}^{m}(\mathcal{Y}_i \partial x_i - \mathcal{X}_i \partial y_i),$$

where the Cartesian coordinates on $\mathbb{R}^{2m}_{2q}$ are (x^i, y^i). Now we give an example of screen transversal Cauchy-Riemann bi-slant lightlike submanifolds of an indefinite Kaehler manifold.

Example 3.3. Suppose $(\mathbb{R}_4^{18},\ \bar{g},\ \bar{\mathcal{J}})$ is an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-,\ -,\ +,\ +,\ +,\ +,\ +,\ +,\ +,\ +,\ -,\ +,\ -,\ +,\ +,\ +,\ +,\ +,\ +,\ +)$ with respect to the canonical basis $\{\partial x_1,\ \partial x_2,\ \partial x_3,\ \partial x_4,\ \partial x_5,\ \partial x_6,\ \partial x_7,\ \partial x_8,\ \partial x_9,\ \partial y_1, \partial y_2,\ \partial y_3,\ \partial y_4,\ \partial y_5,\ \partial y_6,\ \partial y_7,\ \partial y_8,\ \partial y_9\}$. Suppose $\mathcal{M}$ is a submanifold of $\mathbb{R}_4^{18}$ given by $x^1 = \sin u_2$, $y^1 = -\cos u_2$, $x^2 = u_1$, $y^2 = u_3 - \frac{u_4}{2}$, $x_3 = u_2$, $y_3 = 0$, $x^4 = u_1$, $y^4 = u_3 + \frac{u_4}{2}$, $x^5 = u_5 \cos u_6$, $y^5 = u_5 \sin u_6$, $x^6 = k_1 \cos u_5$, $y^6 = k_1 \sin u_5$, $x^7 = u_7$, $y^7 = u_8$, $x^8 = k_2 \sin u_8$, $y^8 = k_2 \cos u_8$, where k_1 and k_2 are constants. Here the local frame of $T\mathcal{M}$ is given by $\{\mathcal{Z}_1,\ \mathcal{Z}_2,\ \mathcal{Z}_3,\ \mathcal{Z}_4,\ \mathcal{Z}_5,\ \mathcal{Z}_6,\ \mathcal{Z}_7,\ \mathcal{Z}_8\}$, where

$$\mathcal{Z}_1 = 2(\partial x_2 + \partial x_4), \quad \mathcal{Z}_2 = 2(\cos u_2 \partial x_1 + \sin u_2 \partial y_1 + \partial x_3),$$
$$\mathcal{Z}_3 = 2(\partial y_2 + \partial y_4), \quad \mathcal{Z}_4 = (-\partial y_2 + \partial y_4),$$
$$\mathcal{Z}_5 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 - k_1 \sin u_5 \partial x_6 + k_1 \cos u_5 \partial y_6),$$
$$\mathcal{Z}_6 = 2(-u_5 \sin u_6 \partial x_5 + u_5 \cos u_6 \partial y_5),$$
$$\mathcal{Z}_7 = 2(\partial x_7), \quad \mathcal{Z}_8 = 2(\partial y_7 + k_2 \cos u_8 \partial x_8 - k_2 \sin u_8 \partial y_8).$$

Hence $\mathcal{R}ad(T\mathcal{M}) = span\{\mathcal{Z}_1, \mathcal{Z}_2\}$. Also it is easy to see that $\mathcal{D}_1 = span\{\mathcal{Z}_1\}$ and $\mathcal{D}_2 = span\{\mathcal{Z}_2\}$, where $\bar{\mathcal{J}}\mathcal{Z}_1 = -\mathcal{Z}_3 \in \Gamma(S(T\mathcal{M}))$ and $\bar{\mathcal{J}}\mathcal{Z}_2 = \mathcal{W}_2 \in \Gamma(S(T\mathcal{M}^\perp))$. Moreover $S(T\mathcal{M}) = span\{\mathcal{Z}_3,\ \mathcal{Z}_4,\ \mathcal{Z}_5,\ \mathcal{Z}_6,\ \mathcal{Z}_7,\ \mathcal{Z}_8\}$. Also $\mathcal{D}_1' = span\{\mathcal{Z}_5, \mathcal{Z}_6\}$ and $\mathcal{D}_2' = span\{\mathcal{Z}_7, \mathcal{Z}_8\}$ are slant distributions with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1+k_1^2})$ and $\theta_2 = \cos^{-1}(1/\sqrt{1+k_2^2})$ respectively. On the other hand the lightlike transversal bundle $ltr(T\mathcal{M})$ is spanned by

$$\mathcal{N}_1 = (-\partial x_2 + \partial x_4), \ \ \mathcal{N}_2 = (-\cos u_2 \partial x_1 - \sin u_2 \partial y_1 + \partial x_3).$$

From this we have $ltr(T\mathcal{M}) = span\{\mathcal{N}_1, \mathcal{N}_2\}$, where $L_1 = span\{\mathcal{N}_1\}$ and $L_2 = span\{\mathcal{N}_2\}$. Here $\bar{\mathcal{J}}\mathcal{N}_1 = -\mathcal{Z}_4 \in \Gamma(S(T\mathcal{M}))$ and $\bar{\mathcal{J}}\mathcal{N}_2 = \mathcal{W}_1 \in \Gamma(S(T\mathcal{M}^\perp))$. Also $S(T\mathcal{M}^\perp)$ is spanned by

$$\mathcal{W}_1 = 2(-\sin u_2 \partial x_1 + \cos u_2 \partial y_1 - \partial y_3), \quad \mathcal{W}_2 = 2(\sin u_2 \partial x_1 - \cos u_2 \partial y_1 - \partial y_3),$$
$$\mathcal{W}_3 = 2(k_1^2 \cos u_6 \partial x_5 + k_1^2 \sin u_6 \partial y_5 + k_1 \sin u_5 \partial x_6 - k_1 \cos u_5 \partial y_6),$$
$$\mathcal{W}_4 = 2(u_5 \cos u_5 \partial x_6 + u_5 \sin u_5 \partial y_6),$$
$$\mathcal{W}_5 = 2(k_2 \sin u_8 \partial x_8 + k_2 \cos u_8 \partial y_8), \quad \mathcal{W}_6 = 2(k_2^2 \partial y_7 - k_2 \cos u_8 \partial x_8 + k_2 \sin u_8 \partial y_8).$$

Hence $\mathcal{M}$ is a proper screen transversal Cauchy-Riemann bi-slant 2-lightlike submanifold of $\mathbb{R}_4^{18}$.

Example 3.4. Suppose $(\mathbb{R}_4^{18},\ \bar{g},\ \bar{\mathcal{J}})$ is an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-,\ -,\ +,\ +,\ +,\ +,\ +,\ +,\ +,\ +,\ -,\ +,\ -,\ +,\ +,\ +,\ +,\ +,\ +,\ +)$ with respect to the canonical basis $\{\partial x_1,\ \partial x_2,\ \partial x_3,\ \partial x_4,\ \partial x_5,\ \partial x_6,\ \partial x_7,\ \partial x_8,\ \partial x_9,\ \partial y_1, \partial y_2,\ \partial y_3,\ \partial y_4,\ \partial y_5,\ \partial y_6,\ \partial y_7,\ \partial y_8,\ \partial y_9\}$. Suppose $\mathcal{M}$ is a submanifold of $\mathbb{R}_4^{18}$ given by $x^1 = -\cos u_2$, $y^1 = \sin u_2$, $x^2 = u_1$, $y^2 = u_3 - \frac{u_4}{2}$, $x_3 = u_2$, $y_3 = 0$, $x^4 = u_1$, $y^4 = u_3 + \frac{u_4}{2}$, $x^5 = u_5 \sin u_6$, $y^5 = u_5 \cos u_6$, $x^6 = k_1 \sin u_5$, $y^6 = k_1 \cos u_5$, $x^7 = u_7$, $y^7 = u_8$, $x^8 = k_2 \cos u_8$, $y^8 = k_2 \sin u_8$, where k_1 and k_2 are constants. Here the local frame of $T\mathcal{M}$ is given by $\{\mathcal{Z}_1,\ \mathcal{Z}_2,\ \mathcal{Z}_3,\ \mathcal{Z}_4,\ \mathcal{Z}_5,\ \mathcal{Z}_6,\ \mathcal{Z}_7,\ \mathcal{Z}_8\}$, where

$$\mathcal{Z}_1 = 2(\partial x_2 + \partial x_4), \quad \mathcal{Z}_2 = 2(\sin u_2 \partial x_1 + \cos u_2 \partial y_1 + \partial x_3),$$
$$\mathcal{Z}_3 = 2(\partial y_2 + \partial y_4), \quad \mathcal{Z}_4 = (-\partial y_2 + \partial y_4),$$
$$\mathcal{Z}_5 = 2(\sin u_6 \partial x_5 + \cos u_6 \partial y_5 + k_1 \cos u_5 \partial x_6 - k_1 \sin u_5 \partial y_6),$$
$$\mathcal{Z}_6 = 2(u_5 \cos u_6 \partial x_5 - u_5 \sin u_6 \partial y_5),$$
$$\mathcal{Z}_7 = 2(\partial x_7), \quad \mathcal{Z}_8 = 2(\partial y_7 - k_2 \sin u_8 \partial x_8 + k_2 \cos u_8 \partial y_8).$$

Hence $\mathcal{R}ad(T\mathcal{M}) = span\{\mathcal{Z}_1, \mathcal{Z}_2\}$. Also it is easy to see that $\mathcal{D}_1 = span\{\mathcal{Z}_1\}$ and $\mathcal{D}_2 = span\{\mathcal{Z}_2\}$, where $\bar{\mathcal{J}}\mathcal{Z}_1 = -\mathcal{Z}_3 \in \Gamma(S(T\mathcal{M}))$ and $\bar{\mathcal{J}}\mathcal{Z}_2 = \mathcal{W}_2 \in \Gamma(S(T\mathcal{M}^\perp))$. Moreover $S(T\mathcal{M}) = span\{\mathcal{Z}_3,\ \mathcal{Z}_4,\ \mathcal{Z}_5,\ \mathcal{Z}_6,\ \mathcal{Z}_7,\ \mathcal{Z}_8\}$. Also $\mathcal{D}_1' = span\{\mathcal{Z}_5, \mathcal{Z}_6\}$ and $\mathcal{D}_2' = span\{\mathcal{Z}_7, \mathcal{Z}_8\}$

are slant distributions with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1+k_1^2})$ and $\theta_2 = \cos^{-1}(1/\sqrt{1+k_2^2})$ respectively. On the other hand the lightlike transversal bundle $ltr(T\mathcal{M})$ is spanned by

$$\mathcal{N}_1 = (-\partial x_2 + \partial x_4),\ \ \mathcal{N}_2 = (-\sin u_2 \partial x_1 - \cos u_2 \partial y_1 + \partial x_3).$$

From this we have $ltr(T\mathcal{M}) = span\{\mathcal{N}_1, \mathcal{N}_2\}$, where $L_1 = span\{\mathcal{N}_1\}$ and $L_2 = span\{\mathcal{N}_2\}$. Here $\bar{\mathcal{J}}\mathcal{N}_1 = -\mathcal{Z}_4 \in \Gamma(S(T\mathcal{M}))$ and $\bar{\mathcal{J}}\mathcal{N}_2 = \mathcal{W}_1 \in \Gamma(S(T\mathcal{M}^\perp))$. Also $S(T\mathcal{M}^\perp)$ is spanned by

$$\mathcal{W}_1 = 2(-\cos u_2 \partial x_1 + \sin u_2 \partial y_1 - \partial y_3),\quad \mathcal{W}_2 = 2(\cos u_2 \partial x_1 - \sin u_2 \partial y_1 - \partial y_3),$$
$$\mathcal{W}_3 = 2(k_1^2 \sin u_6 \partial x_5 + k_1^2 \cos u_6 \partial y_5 - k_1 \cos u_5 \partial x_6 + k_1 \sin u_5 \partial y_8),$$
$$\mathcal{W}_4 = 2(u_5 \sin u_5 \partial x_6 + u_5 \cos u_5 \partial y_6),$$
$$\mathcal{W}_5 = 2(k_2 \cos u_8 \partial x_8 + k_2 \sin u_8 \partial y_8),\quad \mathcal{W}_6 = 2(k_2^2 \partial y_7 - k_2 \sin u_8 \partial x_8 + k_2 \cos u_8 \partial y_8).$$

Hence $\mathcal{M}$ is a proper screen transversal Cauchy-Riemann bi-slant 2-lightlike submanifold of $\mathbb{R}_4^{18}$.

Now, for any vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we put

$$\bar{\mathcal{J}}\mathcal{X} = P\mathcal{X} + F\mathcal{X}, \tag{3.1}$$

where $P\mathcal{X}$ and $F\mathcal{X}$ are the tangential and transversal parts of $\bar{\mathcal{J}}\mathcal{X}$ respectively. We denote the projections on $\mathcal{D}_1$, $\mathcal{D}_2$, $\bar{\mathcal{J}}\mathcal{D}_1$, $\bar{\mathcal{J}}L_1$, $\mathcal{D}'_1$ and $\mathcal{D}'_2$ in $T\mathcal{M}$ by P_1, P_2, P_3, P_4, P_5, and P_6 respectively. Similarly, we denote the projections of $tr(T\mathcal{M})$ on $ltr(T\mathcal{M})$ and $S(T\mathcal{M}^\perp)$ by Q and R respectively. Thus, for any $\mathcal{X} \in \Gamma(T\mathcal{M})$, we get

$$\mathcal{X} = P_1\mathcal{X} + P_2\mathcal{X} + P_3\mathcal{X} + P_4\mathcal{X} + P_5\mathcal{X} + P_6\mathcal{X}, \tag{3.2}$$

Now applying $\bar{\mathcal{J}}$ to (3.2), we have

$$\bar{\mathcal{J}}\mathcal{X} = \bar{\mathcal{J}}P_1\mathcal{X} + \bar{\mathcal{J}}P_2\mathcal{X} + \bar{\mathcal{J}}P_3\mathcal{X} + \bar{\mathcal{J}}P_4\mathcal{X} + \bar{\mathcal{J}}P_5\mathcal{X} + \bar{\mathcal{J}}P_6\mathcal{X}, \tag{3.3}$$

which gives

$$\bar{\mathcal{J}}\mathcal{X} = \bar{\mathcal{J}}P_1\mathcal{X} + \bar{\mathcal{J}}P_2\mathcal{X} + \bar{\mathcal{J}}P_3\mathcal{X} + \bar{\mathcal{J}}P_4\mathcal{X} + fP_5\mathcal{X} + FP_5\mathcal{X} + fP_6\mathcal{X} + FP_6\mathcal{X}, \tag{3.4}$$

where $fP_5\mathcal{X}$ and $FP_5\mathcal{X}$ (resp. $fP_6\mathcal{X}$ and $FP_6\mathcal{X}$) denotes the tangential and transversal component of $\bar{\mathcal{J}}P_5\mathcal{X}$ (resp. $\bar{\mathcal{J}}P_6\mathcal{X}$). Thus we get $\bar{\mathcal{J}}P_1\mathcal{X} \in \Gamma(S(T\mathcal{M}))$, $\bar{\mathcal{J}}P_2\mathcal{X} \in \Gamma(\bar{\mathcal{J}}\mathcal{D}_2) \subset \Gamma(S(T\mathcal{M}^\perp))$, $\bar{\mathcal{J}}P_3\mathcal{X} \in \Gamma(\mathcal{D}_1)$, $\bar{\mathcal{J}}P_4\mathcal{X} \in \Gamma(L_1) \subset \Gamma(ltr(T\mathcal{M}))$, $fP_5\mathcal{X} \in \Gamma(\mathcal{D}'_1)$, $FP_5\mathcal{X} \in \Gamma(CR_3\mathcal{W}) \subset \Gamma(S(T\mathcal{M}^\perp)$, $fP_6\mathcal{X} \in \Gamma(\mathcal{D}'_2)$ and $FP_6\mathcal{X} \in \Gamma(CR_4\mathcal{W}) \subset \Gamma(S(T\mathcal{M}^\perp)$.

Also, for any $\mathcal{W} \in \Gamma(tr(T\mathcal{M}))$, we have

$$\mathcal{W} = Q\mathcal{W} + R\mathcal{W}, \tag{3.5}$$

Applying $\bar{\mathcal{J}}$ to (3.5), we obtain

$$\bar{\mathcal{J}}\mathcal{W} = \bar{\mathcal{J}}Q\mathcal{W} + \bar{\mathcal{J}}R\mathcal{W}, \tag{3.6}$$

which gives

$$\bar{\mathcal{J}}\mathcal{W} = \bar{\mathcal{J}}Q_1\mathcal{W} + \bar{\mathcal{J}}Q_2\mathcal{W} + \bar{\mathcal{J}}R_1\mathcal{W} + \bar{\mathcal{J}}R_2\mathcal{W} + BR_3\mathcal{W} + CR_3\mathcal{W} + BR_4\mathcal{W} + CR_4\mathcal{W}, \tag{3.7}$$

where, $BR_3\mathcal{W}$ and $CR_3\mathcal{W}$ (resp. $BR_4\mathcal{W}$ and $CR_4\mathcal{W}$) denotes the tangential and transversal component of $\bar{\mathcal{J}}R_3\mathcal{W}$ (resp. $\bar{\mathcal{J}}R_4\mathcal{W}$). Thus we get $\bar{\mathcal{J}}Q_1\mathcal{W} \in \Gamma(\mathcal{D}_1)$, $\bar{\mathcal{J}}Q_2\mathcal{W} \in \Gamma(S(T\mathcal{M}^\perp))$, $\bar{\mathcal{J}}R_1\mathcal{W} \in \Gamma(\mathcal{D}_2)$, $\bar{\mathcal{J}}R_2\mathcal{W} \in \Gamma(L_2)$, $BR_3\mathcal{W} \in \Gamma(\mathcal{D}'_1)$, $CR_3\mathcal{W} \in \Gamma(S(T\mathcal{M}^\perp))$, $BR_4\mathcal{W} \in \Gamma(\mathcal{D}'_2)$ and $CR_4\mathcal{W} \in \Gamma(S(T\mathcal{M}^\perp))$.

Now, by using (2.20), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $\mathcal{D}_1$, $\mathcal{D}_2$, $\bar{\mathcal{J}}\mathcal{D}_1$, $\bar{\mathcal{J}}L_1$, $\mathcal{D}'_1$, $\mathcal{D}'_2$, L_1, L_2 and $S(T\mathcal{M}^\perp)$, we obtain

$$P_1(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_1\mathcal{Y}) + P_1(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Y}) + P_1(\nabla_{\mathcal{X}}fP_5\mathcal{Y}) + P_1(\nabla_{\mathcal{X}}fP_6\mathcal{Y}) = P_1(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X}) + P_1(A_{\bar{\mathcal{J}}P_4\mathcal{Y}}\mathcal{X}) + P_1(A_{FP_5\mathcal{Y}}\mathcal{X}) + P_1(A_{FP_6\mathcal{Y}}\mathcal{X}) + \bar{\mathcal{J}}P_3\nabla_{\mathcal{X}}\mathcal{Y}, \quad (3.8)$$

$$P_3(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_1\mathcal{Y}) + P_3(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Y}) + P_3(\nabla_{\mathcal{X}}fP_5\mathcal{Y}) + P_3(\nabla_{\mathcal{X}}fP_6\mathcal{Y}) = P_3(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X}) + P_3(A_{\bar{\mathcal{J}}P_4\mathcal{Y}}\mathcal{X}) + P_3(A_{FP_5\mathcal{Y}}\mathcal{X}) + P_3(A_{FP_6\mathcal{Y}}\mathcal{X}) + \bar{\mathcal{J}}P_1\nabla_{\mathcal{X}}\mathcal{Y}, \quad (3.9)$$

$$P_4(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_1\mathcal{Y}) + P_4(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Y}) + P_4(\nabla_{\mathcal{X}}fP_5\mathcal{Y}) + P_4(\nabla_{\mathcal{X}}fP_6\mathcal{Y}) = P_4(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X}) + P_4(A_{\bar{\mathcal{J}}P_4\mathcal{Y}}\mathcal{X}) + P_4(A_{FP_5\mathcal{Y}}\mathcal{X}) + P_4(A_{FP_6\mathcal{Y}}\mathcal{X}) + \bar{\mathcal{J}}Q_1h^l(\mathcal{X},\mathcal{Y}), \quad (3.10)$$

$$P_5(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_1\mathcal{Y}) + P_5(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Y}) + P_5(\nabla_{\mathcal{X}}fP_5\mathcal{Y}) + P(\nabla_{\mathcal{X}}fP_6\mathcal{Y}) = fP_5\nabla_{\mathcal{X}}\mathcal{Y} + P_5(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X}) + P_5(A_{\bar{\mathcal{J}}P_4\mathcal{Y}}\mathcal{X}) + P_5(A_{FP_5\mathcal{Y}}\mathcal{X}) + P_5(A_{FP_6\mathcal{Y}}\mathcal{X}) + BR_3h^s(\mathcal{X},\mathcal{Y}), \quad (3.11)$$

$$P_6(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_1\mathcal{Y}) + P_6(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Y}) + P_6(\nabla_{\mathcal{X}}fP_5\mathcal{Y}) + P_6(\nabla_{\mathcal{X}}fP_6\mathcal{Y}) = fP_6\nabla_{\mathcal{X}}\mathcal{Y} + P_6(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X}) + P_6(A_{\bar{\mathcal{J}}P_4\mathcal{Y}}\mathcal{X}) + P_6(A_{FP_5\mathcal{Y}}\mathcal{X}) + P_6(A_{FP_6\mathcal{Y}}\mathcal{X}) + BR_4h^s(\mathcal{X},\mathcal{Y}), \quad (3.12)$$

$$Q_1h^l(\mathcal{X},\bar{\mathcal{J}}P_1\mathcal{Y}) + Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_2\mathcal{Y}) + Q_1h^l(\mathcal{X},\bar{\mathcal{J}}P_3\mathcal{Y}) + Q_1h^l(\mathcal{X},fP_5\mathcal{Y}) + Q_1h^l(\mathcal{X},fP_6\mathcal{Y}) = \bar{\mathcal{J}}P_4\nabla_{\mathcal{X}}\mathcal{Y} + \bar{\mathcal{J}}P_4h^l(\mathcal{X},\mathcal{Y}) - Q_1\nabla^l_{\mathcal{X}}\bar{\mathcal{J}}P_4\mathcal{Y} - Q_1\mathcal{D}^l(\mathcal{X},FP_5\mathcal{Y}) - Q_1\mathcal{D}^l(\mathcal{X},FP_6\mathcal{Y}), \quad (3.13)$$

$$Q_2h^l(\mathcal{X},\bar{\mathcal{J}}P_1\mathcal{Y}) + Q_2\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_2\mathcal{Y}) + Q_2h^l(\mathcal{X},\bar{\mathcal{J}}P_3\mathcal{Y}) + Q_2h^l(\mathcal{X},fP_5\mathcal{Y}) + Q_2h^l(\mathcal{X},fP_6\mathcal{Y}) = Q_2\nabla^l_{\mathcal{X}}\bar{\mathcal{J}}P_4\mathcal{Y} - Q_2\mathcal{D}^l(\mathcal{X},FP_5\mathcal{Y}) - Q_2\mathcal{D}^l(\mathcal{X},FP_6\mathcal{Y}) + \bar{\mathcal{J}}R_2h^s(\mathcal{X},\mathcal{Y}), \quad (3.14)$$

$$R_1h^s(\mathcal{X},\bar{\mathcal{J}}P_1\mathcal{Y}) + R_1h^s(\mathcal{X},\bar{\mathcal{J}}P_3\mathcal{Y}) + R_1\mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}P_4\mathcal{Y}) + R_1h^s(\mathcal{X},fP_5\mathcal{Y}) + R_1h^s(\mathcal{X},fP_6\mathcal{Y}) = \bar{\mathcal{J}}P_2\nabla_{\mathcal{X}}\mathcal{Y} - R_1\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y} - R_1\nabla^s_{\mathcal{X}}FP_5\mathcal{Y} - R_1\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}, \quad (3.15)$$

$$R_2h^s(\mathcal{X},\bar{\mathcal{J}}P_1\mathcal{Y}) + R_2h^s(\mathcal{X},\bar{\mathcal{J}}P_3\mathcal{Y}) + R_2\mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}P_4\mathcal{Y}) + R_2h^s(\mathcal{X},fP_5\mathcal{Y}) + R_2h^s(\mathcal{X},fP_6\mathcal{Y}) = \bar{\mathcal{J}}Q_2h^l(\mathcal{X},\mathcal{Y}) - R_2\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y} - R_2\nabla^s_{\mathcal{X}}FP_5\mathcal{Y} - R_2\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}, \quad (3.16)$$

$$R_3h^s(\mathcal{X},\bar{\mathcal{J}}P_1\mathcal{Y}) + R_3h^s(\mathcal{X},\bar{\mathcal{J}}P_3\mathcal{Y}) + R_3\mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}P_4\mathcal{Y}) + R_3h^s(\mathcal{X},fP_5\mathcal{Y}) + R_3h^s(\mathcal{X},fP_6\mathcal{Y}) = FP_5\nabla_{\mathcal{X}}\mathcal{Y} - R_3\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y} - R_3\nabla^s_{\mathcal{X}}FP_5\mathcal{Y} - R_3\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}, \quad (3.17)$$

$$R_4h^s(\mathcal{X},\bar{\mathcal{J}}P_1\mathcal{Y}) + R_4h^s(\mathcal{X},\bar{\mathcal{J}}P_3\mathcal{Y}) + R_4\mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}P_4\mathcal{Y}) + R_4h^s(\mathcal{X},fP_5\mathcal{Y}) + R_4h^s(\mathcal{X},fP_6\mathcal{Y}) = FP_6\nabla_{\mathcal{X}}\mathcal{Y} - R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y} - R_4\nabla^s_{\mathcal{X}}FP_5\mathcal{Y} - R_4\nabla^s_{\mathcal{X}}FP_6\mathcal{Y} + CR_4h^s(\mathcal{X},\mathcal{Y}). \quad (3.18)$$

Theorem 3.5. *Let $\mathcal{M}$ be a q-lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$ of index $2q$. Then $\mathcal{M}$ is a STCR bi-slant lightlike submanifold if and only if*
(i) $\mathcal{R}ad(T\mathcal{M})$ is a distribution on $\mathcal{M}$ such that

$$\mathcal{R}ad(T\mathcal{M}) = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

where $\bar{\mathcal{J}}\mathcal{D}_1 \subset S(T\mathcal{M})$ *and* $\bar{\mathcal{J}}\mathcal{D}_2 \subset S(T\mathcal{M}^{\perp})$. *Furthermore, we have* $ltr(T\mathcal{M}) = L_1 \oplus L_2$ *where* $\bar{\mathcal{J}}L_1 \subset S(T\mathcal{M})$ *and* $\bar{\mathcal{J}}L_2 \subset S(T\mathcal{M}^{\perp})$.
(ii) the screen distribution $S(T\mathcal{M})$ *can be split as a direct sum*

$$S(T\mathcal{M}) = (\bar{\mathcal{J}}\mathcal{D}_1 \oplus \bar{\mathcal{J}}L_1) \oplus_{orth} \mathcal{D}'_1 \oplus_{orth} \mathcal{D}'_2,$$

(iii) there exists a constant $\lambda_1 \in [0,1)$ *such that* $P^2\mathcal{X} = -\lambda_1\mathcal{X}$, *for all* $\mathcal{X} \in \Gamma(\mathcal{D}'_1)$, *where* $\lambda_1 = cos^2\theta_1$ *and* θ_1 *is the slant angle of* $\mathcal{D}'_1$,
(iv) there exists a constant $\lambda_2 \in [0,1)$ *such that* $P^2\mathcal{X} = -\lambda_2\mathcal{X}$, *for all* $\mathcal{X} \in \Gamma(\mathcal{D}'_2)$, *where* $\lambda_2 = cos^2\theta_2$ *and* θ_2 *is the slant angle of* $\mathcal{D}'_2$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then the distribution $\mathcal{D}'$ is invariant with respect to $\bar{\mathcal{J}}$ and $\mathcal{R}ad(T\mathcal{M})$ is a distribution on $\mathcal{M}$ such that $\mathcal{R}ad(T\mathcal{M}) = \mathcal{D}_1 \oplus \mathcal{D}_2$, where $\bar{\mathcal{J}}\mathcal{D}_1 \subset S(T\mathcal{M})$ and $\bar{\mathcal{J}}\mathcal{D}_2 \subset S(T\mathcal{M}^{\perp})$. Now, for any $\mathcal{X} \in \Gamma(\mathcal{D}'_1)$ we have $|P\mathcal{X}| = |\bar{\mathcal{J}}\mathcal{X}| \cos\theta_1$, i.e.

$$\cos\theta_1 = \frac{|P\mathcal{X}|}{|\bar{\mathcal{J}}\mathcal{X}|}. \tag{3.19}$$

In view of (3.20), we get $\cos^2\theta_1 = \frac{|P\mathcal{X}|^2}{|\bar{\mathcal{J}}\mathcal{X}|^2} = \frac{g(P\mathcal{X}, P\mathcal{X})}{g(\bar{\mathcal{J}}\mathcal{X}, \bar{\mathcal{J}}\mathcal{X})} = \frac{g(\mathcal{X}, P^2\mathcal{X})}{g(\mathcal{X}, \bar{\mathcal{J}}^2\mathcal{X})}$, which gives

$$g(\mathcal{X}, P^2\mathcal{X}) = \cos^2\theta_1 g(\mathcal{X}, \bar{\mathcal{J}}^2\mathcal{X}). \tag{3.20}$$

Since $\mathcal{M}$ is a STCR bi-slant lightlike submanifold, $\cos^2\theta_1 = \lambda_1(\text{constant}) \in [0,1)$ and therefore from (3.21) we get $g(\mathcal{X}, P^2\mathcal{X}) = \lambda_1 g(\mathcal{X}, \bar{\mathcal{J}}^2\mathcal{X}) = g(\mathcal{X}, \lambda_1\bar{\mathcal{J}}^2\mathcal{X})$, for all $\mathcal{X} \in \Gamma(\mathcal{D}'_1)$, which implies

$$g(\mathcal{X}, (P^2 - \lambda_1\bar{\mathcal{J}}^2)\mathcal{X}) = 0 \tag{3.21}$$

Since $(P^2 - \lambda_1\bar{\mathcal{J}}^2)\mathcal{X} \in \Gamma(\mathcal{D}'_1)$ and the induced metric $g = g|_{\mathcal{D}'_1 \times \mathcal{D}'_1}$ is non-degenerate (positive definite). From (3.22) we have $(P^2 - \lambda_1\bar{\mathcal{J}}^2)\mathcal{X} = 0$, which implies

$$P^2\mathcal{X} = \lambda_1\bar{\mathcal{J}}^2\mathcal{X} = -\lambda_1\mathcal{X}, \ \forall\, \mathcal{X} \in \Gamma(\mathcal{D}'_1). \tag{3.22}$$

This proves (iii). Suppose for any $\mathcal{X} \in \Gamma(\mathcal{D}'_2)$ we have $|P\mathcal{X}| = |\bar{\mathcal{J}}\mathcal{X}| \cos\theta_2$, i.e.

$$\cos\theta_2 = \frac{|P\mathcal{X}|}{|\bar{\mathcal{J}}\mathcal{X}|}. \tag{3.23}$$

Now the proof follows by using similar steps above of proof of (iii), which gives $\cos^2\theta_2 = \lambda_2(\text{constant})$. This proves (iv).

Conversely, suppose that conditions (i), (ii), (iii) and (iv) are satisfied. From (iii), we have $P^2\mathcal{X} = \lambda_1\bar{\mathcal{J}}^2\mathcal{X}, \ \forall \mathcal{X} \in \Gamma(\mathcal{D}'_1)$, where $\lambda_1 \in [0,1)$.

Now $\cos\theta_1 = \frac{g(\bar{\mathcal{J}}\mathcal{X}, P\mathcal{X})}{|\bar{\mathcal{J}}\mathcal{X}||P\mathcal{X}|} = -\frac{g(\mathcal{X}, \bar{\mathcal{J}}P\mathcal{X})}{|\bar{\mathcal{J}}\mathcal{X}||P\mathcal{X}|} = -\frac{g(\mathcal{X}, P^2\mathcal{X})}{|\bar{\mathcal{J}}\mathcal{X}||P\mathcal{X}|} = -\lambda_1\frac{g(\mathcal{X}, \bar{\mathcal{J}}^2\mathcal{X})}{|\bar{\mathcal{J}}\mathcal{X}||P\mathcal{X}|} = \lambda_1\frac{g(\bar{\mathcal{J}}\mathcal{X}, \bar{\mathcal{J}}\mathcal{X})}{|\bar{\mathcal{J}}\mathcal{X}||P\mathcal{X}|}$.

From the above equation, we obtain

$$\cos\theta_1 = \lambda_1\frac{|\bar{\mathcal{J}}\mathcal{X}|}{|P\mathcal{X}|}. \tag{3.24}$$

Therefore (3.20) and (3.21) give $\cos^2\theta_1 = \lambda_1(\text{constant})$.

Furthermore, from (iv) we have $P^2\mathcal{X} = \lambda_2\bar{\mathcal{J}}^2\mathcal{X}, \ \forall \mathcal{X} \in \Gamma(\mathcal{D}'_2)$, where $\lambda_2 \in [0,1)$. Now by using the similar steps above we get $\cos^2\theta_2 = \lambda_2(\text{constant})$. This completes the proof. Hence $\mathcal{M}$ is a STCR bi-slant lightlike submanifold. □

Theorem 3.6. *Let* $\mathcal{M}$ *be a* q*-lightlike submanifold of an indefinite Kaehler manifold* $\bar{\mathcal{M}}$ *of index* $2q$. *Then* $\mathcal{M}$ *is a STCR bi-slant lightlike submanifold if and only if*
(i) $\mathcal{R}ad(T\mathcal{M})$ *is a distribution on* $\mathcal{M}$ *such that*

$$\mathcal{R}ad(T\mathcal{M}) = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

where $\bar{\mathcal{J}}\mathcal{D}_1 \subset S(T\mathcal{M})$ *and* $\bar{\mathcal{J}}\mathcal{D}_2 \subset S(T\mathcal{M}^\perp)$. *Furthermore, we have* $ltr(T\mathcal{M}) = L_1 \oplus L_2$ *where* $\bar{\mathcal{J}}L_1 \subset S(T\mathcal{M})$ *and* $\bar{\mathcal{J}}L_2 \subset S(T\mathcal{M}^\perp)$,
(ii) the screen distribution $S(T\mathcal{M})$ *can be split as a direct sum*

$$S(T\mathcal{M}) = (\bar{\mathcal{J}}\mathcal{D}_1 \oplus \bar{\mathcal{J}}L_1) \oplus_{orth} \mathcal{D}'_1 \oplus_{orth} \mathcal{D}'_2,$$

(iii) there exists a constant $\mu_1 \in [0,1)$ *such that* $BF\mathcal{X} = -\mu_1\mathcal{X}$, $\forall \mathcal{X} \in \Gamma(\mathcal{D}'_1)$, *where* $\mu_1 = sin^2\theta_1$ *and* θ_1 *is the slant angle of* $\mathcal{D}'_1$,
(iv) there exists a constant $\mu_2 \in [0,1)$ *such that* $BF\mathcal{X} = -\mu_2\mathcal{X}$, $\forall \mathcal{X} \in \Gamma(\mathcal{D}'_2)$, *where* $\mu_2 = sin^2\theta_2$ *and* θ_2 *is the slant angle of* $\mathcal{D}'_2$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then the distribution $\mathcal{D}'$ is invariant with respect to $\bar{\mathcal{J}}$ and $\mathcal{R}ad(T\mathcal{M})$ is a distribution on $\mathcal{M}$ such that $\mathcal{R}ad(T\mathcal{M}) = \mathcal{D}_1 \oplus \mathcal{D}_2$, where $\bar{\mathcal{J}}\mathcal{D}_1 \subset S(T\mathcal{M})$ and $\bar{\mathcal{J}}\mathcal{D}_2 \subset S(T\mathcal{M}^\perp)$.

Now, for any vector field $\mathcal{X} \in \Gamma(\mathcal{D}'_1)$, we have

$$\bar{\mathcal{J}}\mathcal{X} = P\mathcal{X} + F\mathcal{X}, \tag{3.25}$$

where $P\mathcal{X}$ and $F\mathcal{X}$ are the tangential and transversal parts of $\bar{\mathcal{J}}\mathcal{X}$ respectively. Applying $\bar{\mathcal{J}}$ to (3.26) and taking the tangential component, we get

$$-\mathcal{X} = P^2\mathcal{X} + BF\mathcal{X}, \ \forall\ \mathcal{X} \in \Gamma(\mathcal{D}'_1). \tag{3.26}$$

Since $\mathcal{M}$ is a STCR bi-slant lightlike submanifold, $P^2\mathcal{X} = -\lambda_1\mathcal{X}$, $\forall \mathcal{X} \in \Gamma(\mathcal{D}'_1)$, where $\lambda_1 \in [0,1)$ and therefore from (3.27) we get

$$BF\mathcal{X} = -\mu_1\mathcal{X}, \ \forall\ \mathcal{X} \in \Gamma(\mathcal{D}'_1), \tag{3.27}$$

where $1 - \lambda_1 = \mu_1(\text{constant}) \in (0,1]$. Now, in view of Theorem 3.5, we have $\lambda_1 = \cos^2\theta_1$. This proves (iii).

Suppose for any vector field $\mathcal{X} \in \Gamma(\mathcal{D}'_2)$, we have

$$\bar{\mathcal{J}}\mathcal{X} = P\mathcal{X} + F\mathcal{X}, \tag{3.28}$$

where $P\mathcal{X}$ and $F\mathcal{X}$ are the tangential and transversal parts of $\bar{\mathcal{J}}\mathcal{X}$ respectively. Now the proof follows by using similar steps above of proof of (iii), which gives $1 - \lambda_2 = \mu_2(\text{constant}) \in [0,1)$, where $\lambda_2 = \cos^2\theta_2$. This proves (iv).

Conversely, assume that conditions (i), (ii), (iii) and (iv) are satisfied. From (3.27) we get

$$-\mathcal{X} = P^2\mathcal{X} - \mu_1\mathcal{X}, \ \forall\ \mathcal{X} \in \Gamma(\mathcal{D}'_1), \tag{3.29}$$

which implies

$$P^2\mathcal{X} = -\lambda_1\mathcal{X}, \ \forall\ \mathcal{X} \in \Gamma(\mathcal{D}'_1), \tag{3.30}$$

where $1 - \mu_1 = \lambda_1(\text{constant}) \in [0,1)$. Furthermore, for any $\mathcal{X} \in \Gamma(\mathcal{D}'_2)$, by using the similar steps above we have $1 - \mu_2 = \lambda_2(\text{constant}) \in [0,1)$. Now the proof follows from Theorem 3.5. Therefore, $\mathcal{M}$ is a STCR bi-slant lightlike submanifold. □

Corollary 3.7. *Let* $\mathcal{M}$ *be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold* $\bar{\mathcal{M}}$. *Then for any slant distribution* $\mathcal{D}$ *of* $\mathcal{M}$ *with slant angle* θ, *we have*

$$g(P\mathcal{X}, P\mathcal{Y}) = \cos^2\theta g(\mathcal{X}, \mathcal{Y}),$$
$$g(F\mathcal{X}, F\mathcal{Y}) = \sin^2\theta g(\mathcal{X}, \mathcal{Y}),$$

for all $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$.

The proof of the above corollary follows by using similar steps as in the proof of Corollary 3.1 of [15].

Theorem 3.8. *Let* $\mathcal{M}$ *be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold* $\bar{\mathcal{M}}$. *Then the distribution* $\mathcal{D}_1 \subset \mathcal{R}ad(T\mathcal{M})$ *is integrable if and only if*
(i) $P_1(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y}) = P_1(\nabla_{\mathcal{Y}}\bar{\mathcal{J}}\mathcal{X})$ *and* $Q_1h^l(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Y}) = Q_1h^l(\mathcal{Y}, \bar{\mathcal{J}}\mathcal{X})$,
(ii) $P_5(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y}) = P_5(\nabla_{\mathcal{Y}}\bar{\mathcal{J}}\mathcal{X})$ *and* $P_6(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y}) = P_6(\nabla_{\mathcal{Y}}\bar{\mathcal{J}}\mathcal{X})$,
(iii) $R_3h^s(\mathcal{Y}, \bar{\mathcal{J}}\mathcal{X}) = R_3h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Y})$ *and* $R_4h^s(\mathcal{Y}, \bar{\mathcal{J}}\mathcal{X}) = R_4h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Y})$,
for all $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}_1)$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Suppose $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}_1)$. From (3.8), we have $P_1(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y})=\bar{\mathcal{J}}P_3\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_1(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y})-P_1(\nabla_{\mathcal{Y}}\bar{\mathcal{J}}\mathcal{X})=\bar{\mathcal{J}}P_3[\mathcal{X},\mathcal{Y}]$. Now from (3.14), $Q_1h^l(\mathcal{X},\bar{\mathcal{J}}\mathcal{Y})=\bar{\mathcal{J}}P_4\nabla_{\mathcal{X}}\mathcal{Y}$, which implies $Q_1h^l(\mathcal{X},\bar{\mathcal{J}}\mathcal{Y})-Q_1h^l(\mathcal{Y},\bar{\mathcal{J}}\mathcal{X})=\bar{\mathcal{J}}P_4[\mathcal{X},\mathcal{Y}]$.

From (3.12), $P_5(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y})=fP_5\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_5(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y})-P_5(\nabla_{\mathcal{Y}}\bar{\mathcal{J}}\mathcal{X})=fP_5[\mathcal{X},\mathcal{Y}]$. From (3.13), we have $P_6(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y})=fP_5\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_6(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y})-P_6(\nabla_{\mathcal{Y}}\bar{\mathcal{J}}\mathcal{X})=fP_6[\mathcal{X},\mathcal{Y}]$.

From (3.16), we have $R_3h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{Y})=\bar{\mathcal{J}}P_5\nabla_{\mathcal{X}}\mathcal{Y}$, which implies $R_3h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{Y})-R_3h^s(\mathcal{Y},\bar{\mathcal{J}}\mathcal{X})=\bar{\mathcal{J}}P_5[\mathcal{X},\mathcal{Y}]$. Now from (3.17), we have $R_4h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{Y})=FP_6\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $R_4h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{Y})-R_4h^s(\mathcal{Y},\bar{\mathcal{J}}\mathcal{X})=FP_6[\mathcal{X},\mathcal{Y}]$, which completes the proof. □

Theorem 3.9. *Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then the distribution $\mathcal{D}_2\subset\mathcal{R}ad(T\mathcal{M})$ is integrable if and only if*
(i) $P_1(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X})=P_1(A_{\bar{\mathcal{J}}P_2\mathcal{X}}\mathcal{Y})$ and $P_3(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X})=P_3(A_{\bar{\mathcal{J}}P_2\mathcal{X}}\mathcal{Y})$,
(ii) $Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_2\mathcal{Y})=Q_1\mathcal{D}^l(\mathcal{Y},\bar{\mathcal{J}}P_2\mathcal{X})$,
(iii) $R_3\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y}=R_3\nabla^s_{\mathcal{Y}}\bar{\mathcal{J}}P_2\mathcal{X}$ and $R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y}=R_4\nabla^s_{\mathcal{Y}}\bar{\mathcal{J}}P_2\mathcal{X}$,
for all $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}_2)$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Let $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}_1)$. From (3.8), we have $P_1(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X})=\bar{\mathcal{J}}P_3\nabla_{\mathcal{X}}\mathcal{Y}$, which implies $P_1(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X})-P_1(A_{\bar{\mathcal{J}}P_2\mathcal{X}}\mathcal{Y})=\bar{\mathcal{J}}P_3[\mathcal{X},\mathcal{Y}]$. From (3.10), we have $P_3(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X})=\bar{\mathcal{J}}P_1\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_3(A_{\bar{\mathcal{J}}P_2\mathcal{Y}}\mathcal{X})-P_3(A_{\bar{\mathcal{J}}P_2\mathcal{X}}\mathcal{Y})=\bar{\mathcal{J}}P_1[\mathcal{X},\mathcal{Y}]$.

From (3.14), we have $Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_2\mathcal{Y})=\bar{\mathcal{J}}P_4\nabla_{\mathcal{X}}\mathcal{Y}+\bar{\mathcal{J}}P_4h^l(\mathcal{X},\mathcal{Y})$, which implies $Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_2\mathcal{Y})-Q_1\mathcal{D}^l(\mathcal{Y},\bar{\mathcal{J}}P_2\mathcal{X})=\bar{\mathcal{J}}P_4[\mathcal{X},\mathcal{Y}]$. From (3.16), we have $R_3\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y}=\bar{\mathcal{J}}P_5\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $R_3\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y}-R_3\nabla^s_{\mathcal{Y}}\bar{\mathcal{J}}P_2\mathcal{X}=\bar{\mathcal{J}}P_5[\mathcal{X},\mathcal{Y}]$. From (3.17), we have $R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y}=FP_6\nabla_{\mathcal{X}}\mathcal{Y}$, which implies $R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_2\mathcal{Y}-R_4\nabla^s_{\mathcal{Y}}\bar{\mathcal{J}}P_2\mathcal{X}=FP_6[\mathcal{X},\mathcal{Y}]$, which completes the proof. □

Theorem 3.10. *Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then the distribution $\mathcal{D}'_1$ is integrable if and only if*
(i) $P_1(A_{\bar{\mathcal{J}}P_5\mathcal{Y}}\mathcal{X})=P_1(A_{\bar{\mathcal{J}}P_5\mathcal{X}}\mathcal{Y})$ and $P_3(A_{\bar{\mathcal{J}}P_5\mathcal{Y}}\mathcal{X})=P_3(A_{\bar{\mathcal{J}}P_5\mathcal{X}}\mathcal{Y})$,
(ii) $Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_5\mathcal{Y})=Q_1\mathcal{D}^l(\mathcal{Y},\bar{\mathcal{J}}P_5\mathcal{X})$ and $R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_5\mathcal{Y}=R_4\nabla^s_{\mathcal{Y}}\bar{\mathcal{J}}P_5\mathcal{X}$,
for all $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}'_1)$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Let $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}_1)$. From (3.8), we have $P_1(A_{\bar{\mathcal{J}}P_5\mathcal{Y}}\mathcal{X})=\bar{\mathcal{J}}P_3\nabla_{\mathcal{X}}\mathcal{Y}$, which implies $P_1(A_{\bar{\mathcal{J}}P_5\mathcal{Y}}\mathcal{X})-P_1(A_{\bar{\mathcal{J}}P_5\mathcal{X}}\mathcal{Y})=\bar{\mathcal{J}}P_3[\mathcal{X},\mathcal{Y}]$. From (3.10), we have $P_3(A_{\bar{\mathcal{J}}P_5\mathcal{Y}}\mathcal{X})=\bar{\mathcal{J}}P_1\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_3(A_{\bar{\mathcal{J}}P_5\mathcal{Y}}\mathcal{X})-P_3(A_{\bar{\mathcal{J}}P_5\mathcal{X}}\mathcal{Y})=\bar{\mathcal{J}}P_1[\mathcal{X},\mathcal{Y}]$.

From (3.14), we have $Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_5\mathcal{Y})=\bar{\mathcal{J}}P_4\nabla_{\mathcal{X}}\mathcal{Y}+\bar{\mathcal{J}}P_4h^l(\mathcal{Y},\mathcal{X})$, which implies $Q_1\mathcal{D}^l(\mathcal{X},\bar{\mathcal{J}}P_5\mathcal{Y})-Q_1\mathcal{D}^l(\mathcal{Y},\bar{\mathcal{J}}P_5\mathcal{X})=\bar{\mathcal{J}}P_4[\mathcal{X},\mathcal{Y}]$. From (3.17), we have $R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_5\mathcal{Y}=FP_6\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $R_4\nabla^s_{\mathcal{X}}\bar{\mathcal{J}}P_5\mathcal{Y}-R_4\nabla^s_{\mathcal{Y}}\bar{\mathcal{J}}P_5\mathcal{X}=FP_6[\mathcal{X},\mathcal{Y}]$, which completes the proof. □

Theorem 3.11. *Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then the distribution $\mathcal{D}'_2$ is integrable if and only if*
(i) $P_1(\nabla_{\mathcal{X}}fP_6\mathcal{Y})-P_1(\nabla_{\mathcal{Y}}fP_6\mathcal{X})=P_1(A_{FP_6\mathcal{Y}}\mathcal{X})-P_1(A_{FP_6\mathcal{X}}\mathcal{Y})$ and $P_3(\nabla_{\mathcal{X}}fP_6\mathcal{Y})-P_3(\nabla_{\mathcal{Y}}fP_6\mathcal{X})=P_3(A_{FP_6\mathcal{Y}}\mathcal{X})-P_3(A_{FP_6\mathcal{X}}\mathcal{Y})$,
(ii) $P_5(\nabla_{\mathcal{X}}fP_6\mathcal{Y})-P_5(\nabla_{\mathcal{Y}}fP_6\mathcal{X})=P_5(A_{FP_6\mathcal{Y}}\mathcal{X})-P_5(A_{FP_6\mathcal{X}}\mathcal{Y})$ and $Q_1\mathcal{D}^l(\mathcal{X},fP_6\mathcal{Y})-Q_1\mathcal{D}^l(\mathcal{Y},fP_6\mathcal{X})=Q_1\mathcal{D}^l(\mathcal{Y},FP_6\mathcal{X})-Q_1\mathcal{D}^l(\mathcal{X},FP_6\mathcal{Y})$,
(iii) $R_3h^s(\mathcal{X},fP_6\mathcal{Y})-R_3h^s(\mathcal{Y},fP_6\mathcal{X})=R_3\nabla^s_{\mathcal{Y}}FP_6\mathcal{X}-R_3\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}$ and $R_4h^s(\mathcal{X},fP_6\mathcal{Y})-R_4h^s(\mathcal{Y},fP_6\mathcal{X})=R_4\nabla^s_{\mathcal{Y}}FP_6\mathcal{X}-R_4\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}$,
for all $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}'_2)$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Suppose $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}_1)$. From (3.8), we have $P_1(\nabla_{\mathcal{X}}fP_6\mathcal{Y})=P_1(A_{FP_6\mathcal{Y}}\mathcal{X})+\bar{\mathcal{J}}P_3\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_1(\nabla_{\mathcal{X}}fP_6\mathcal{Y})-P_1(\nabla_{\mathcal{Y}}fP_6\mathcal{X})=P_1(A_{FP_6\mathcal{Y}}\mathcal{X})-P_1(A_{FP_6\mathcal{X}}\mathcal{Y})+\bar{\mathcal{J}}P_3[\mathcal{X},\mathcal{Y}]$. From (3.10), we have $P_3(\nabla_{\mathcal{X}}fP_6\mathcal{Y})=P_3(A_{FP_6\mathcal{Y}}\mathcal{X})+\bar{\mathcal{J}}P_1\nabla_{\mathcal{X}}\mathcal{Y}$, which gives $P_3(\nabla_{\mathcal{X}}fP_6\mathcal{Y})-P_3(\nabla_{\mathcal{Y}}fP_6\mathcal{X})=P_3(A_{FP_6\mathcal{Y}}\mathcal{X})-P_3(A_{FP_6\mathcal{X}}\mathcal{Y})+\bar{\mathcal{J}}P_1[\mathcal{X},\mathcal{Y}]$.

From (3.12), we have $P_5(\nabla_{\mathcal{X}} fP_6\mathcal{Y}) = fP_6\nabla_{\mathcal{X}}\mathcal{Y} + P_5(A_{FP_6\mathcal{Y}}\mathcal{X}) + \bar{\mathcal{J}}R_3h^s(\mathcal{X},\mathcal{Y})$, which gives $P_5(\nabla_{\mathcal{X}} fP_6\mathcal{Y}) - P_5(\nabla_{\mathcal{Y}} fP_6\mathcal{X}) = fP_6[\mathcal{X},\mathcal{Y}] + P_5(A_{FP_6\mathcal{Y}}\mathcal{X}) - P_5(A_{FP_6\mathcal{X}}\mathcal{Y})$. Now from (3.14), we have $Q_1\mathcal{D}^l(\mathcal{X}, fP_6\mathcal{Y}) = \bar{\mathcal{J}}P_4\nabla_{\mathcal{X}}\mathcal{Y} - Q_1\mathcal{D}^l(\mathcal{X}, FP_6\mathcal{Y}) + \bar{\mathcal{J}}P_4h^l(\mathcal{X},\mathcal{Y})$, which shows that $Q_1\mathcal{D}^l(\mathcal{X}, fP_6\mathcal{Y}) - Q_1\mathcal{D}^l(\mathcal{Y}, fP_6\mathcal{X}) = \bar{\mathcal{J}}P_4[\mathcal{X},\mathcal{Y}] - Q_1\mathcal{D}^l(\mathcal{X}, FP_6\mathcal{Y}) + Q_1\mathcal{D}^l(\mathcal{Y}, FP_6\mathcal{X})$.

Now from (3.16), we have $R_3h^s(\mathcal{X}, fP_6\mathcal{Y}) = \bar{\mathcal{J}}P_5\nabla_{\mathcal{X}}\mathcal{Y} - R_3\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}$, which gives $R_3h^s(\mathcal{X}, fP_6\mathcal{Y}) - R_3h^s(\mathcal{Y}, fP_6\mathcal{X}) = \bar{\mathcal{J}}P_5[\mathcal{X},\mathcal{Y}] - R_3\nabla^s_{\mathcal{X}}FP_6\mathcal{Y} + R_3\nabla^s_{\mathcal{Y}}FP_6\mathcal{X}$. Further from (3.17), we have $R_4h^s(\mathcal{X}, fP_6\mathcal{Y}) = FP_6\nabla_{\mathcal{X}}\mathcal{Y} - R_4\nabla^s_{\mathcal{X}}FP_6\mathcal{Y}$, which gives $R_4h^s(\mathcal{X}, fP_6\mathcal{Y}) - R_4h^s(\mathcal{Y}, fP_6\mathcal{X}) = FP_6[\mathcal{X},\mathcal{Y}] - R_4\nabla^s_{\mathcal{X}}FP_6\mathcal{Y} + R_4\nabla^s_{\mathcal{Y}}FP_6\mathcal{X}$, which completes the proof. □

Theorem 3.12. *There exist no totally umbilical proper STCR bi-slant lightlike submanifold of an indefinite complex space form* $\bar{\mathcal{M}}(c), c \neq 0$.

Proof. Let assume that $\mathcal{M}$ be a totally umbilial proper STCR bi-slant lightlike submanifold of an indefinite complex space form $\bar{\mathcal{M}}(c), c \neq 0$. Then from (2.23) and (2.24) we obtain

$$\bar{R}(\mathcal{X}, \bar{\mathcal{J}}\mathcal{X})\mathcal{Z} = -\frac{c}{2}g(\mathcal{X},\mathcal{X})\bar{\mathcal{J}}\mathcal{Z} \tag{3.31}$$

and

$$\begin{aligned}
\bar{R}(\mathcal{X}, \bar{\mathcal{J}}\mathcal{X})\mathcal{Z} &= (\nabla_{\mathcal{X}}h^s)(\bar{\mathcal{J}}\mathcal{X},\mathcal{Z})(\nabla_{\bar{\mathcal{J}}\mathcal{X}}h^s)(\mathcal{X},\mathcal{Z})\\
&= \mathcal{X}h^s(\bar{\mathcal{J}}\mathcal{X},\mathcal{Z})h^s(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{X},\mathcal{Z})h^s(\bar{\mathcal{J}}\mathcal{X},\nabla_{\mathcal{X}}\mathcal{Z})\\
&- \bar{\mathcal{J}}\mathcal{X}h^s(\mathcal{X},\mathcal{Z}) + h^s(\nabla_{\bar{\mathcal{J}}\mathcal{X}}\mathcal{X},\mathcal{Z}) + h^s(\mathcal{X},\nabla_{\bar{\mathcal{J}}\mathcal{X}}\mathcal{Z})\\
&= \mathcal{X}g(\bar{\mathcal{J}}\mathcal{X},\mathcal{Z})H^s g(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{X},\mathcal{Z})H^s g(\bar{\mathcal{J}}\mathcal{X},\nabla_{\mathcal{X}}\mathcal{Z})H^s\\
&- \bar{\mathcal{J}}\mathcal{X}g(\mathcal{X},\mathcal{Z})H^s + g(\nabla_{\bar{\mathcal{J}}\mathcal{X}}\mathcal{X},\mathcal{Z})H^s + g(\mathcal{X},\nabla_{\bar{\mathcal{J}}\mathcal{X}}\mathcal{Z})H^s\\
&= 0
\end{aligned}$$

for $\mathcal{X} \in \Gamma(\mathcal{D}_0), \mathcal{Z} \in \Gamma(\bar{\mathcal{J}}S)$, respectively. Thus, from (3.32) and above equation we have $c = 0$, which is a contraction and the proof is completed. □

4 Foliations Determined By Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a screen transversal Cauchy-Riemann bi-slant lightlike submanifolds lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

Definition 4.1. ([5]) A STCR bi-slant lightlike submanifold $\mathcal{M}$ of an indefinite Kaehler manifold $\bar{\mathcal{M}}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(\mathcal{X},\mathcal{Y}) = 0$, for all $\mathcal{X} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Y} \in \Gamma(\mathcal{D}_2)$. Thus $\mathcal{M}$ is a mixed geodesic STCR bi-slant lightlike submanifold if $h^l(\mathcal{X},\mathcal{Y}) = 0$ and $h^s(\mathcal{X},\mathcal{Y}) = 0$, $\forall\ \mathcal{X} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Y} \in \Gamma(\mathcal{D}_2)$.

Theorem 4.2. *Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then $\mathcal{D}_1 \subset \mathcal{R}ad(T\mathcal{M})$ defines a totally geodesic foliation if and only if*

$$\bar{g}(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Z} + \nabla_{\mathcal{X}}fP_5\mathcal{Z} + \nabla_{\mathcal{X}}fP_6\mathcal{Z}, \bar{\mathcal{J}}P_1\mathcal{Y}) = \bar{g}(A_{\bar{\mathcal{J}}P_4\mathcal{Z}}\mathcal{X} + A_{FP_5\mathcal{Z}}\mathcal{X} + A_{FP_6\mathcal{Z}}\mathcal{X}, \bar{\mathcal{J}}P_1\mathcal{Y}),$$

for all $\mathcal{X} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Z} \in \Gamma(S(T\mathcal{M}))$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. To prove that $\mathcal{D}_1 \subset \mathcal{R}ad(T\mathcal{M})$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_{\mathcal{X}}\mathcal{Y} \in \mathcal{D}_1$, for all $\mathcal{X},\mathcal{Y} \in \Gamma(\mathcal{D}_1)$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $\mathcal{X},\mathcal{Y} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Z} \in \Gamma(S(T\mathcal{M}))$, we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z}) = \bar{g}((\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}})\mathcal{Z} - \bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Z}, \bar{\mathcal{J}}\mathcal{Y}). \tag{4.1}$$

Now from (2.20), (3.4) and (4.1) we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z}) = -\bar{g}(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Z} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_4\mathcal{Z} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_5\mathcal{Z} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_6\mathcal{Z}, \bar{\mathcal{J}}P_1\mathcal{Y}). \tag{4.2}$$

In view of (2.7)-(2.9) and (4.2), for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}_1)$ and $\mathcal{Z} \in \Gamma(S(T\mathcal{M}))$ we obtain

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = -\bar{g}(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Z} - A_{\bar{\mathcal{J}}P_4\mathcal{Z}}\mathcal{X} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_5\mathcal{Z} + \nabla_{\mathcal{X}}fP_5\mathcal{Z} - A_{FP_5\mathcal{Z}}\mathcal{X} + \nabla_{\mathcal{X}}fP_6\mathcal{Z} - A_{FP_6\mathcal{Z}}\mathcal{X}, \bar{\mathcal{J}}P_1\mathcal{Y}), \quad (4.3)$$

which completes the proof. □

Theorem 4.3. *Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then $\mathcal{D}_2 \subset \mathcal{R}ad(T\mathcal{M})$ defines a totally geodesic foliation if and only if*

$$\bar{g}(h^s(\mathcal{X}, \bar{\mathcal{J}}P_3\mathcal{Z}) + h^s(\mathcal{X}, fP_5\mathcal{Z}) + h^s(\mathcal{X}, fP_6\mathcal{Z}), \bar{\mathcal{J}}P_2\mathcal{Y}) = -\bar{g}(\mathcal{D}^s(\mathcal{X}, \bar{\mathcal{J}}P_4\mathcal{Z}) + \nabla^s_{\mathcal{X}}FP_5\mathcal{Z} + \nabla^s_{\mathcal{X}}FP_6\mathcal{Z}, \bar{\mathcal{J}}P_2\mathcal{Y}),$$

for all $\mathcal{X} \in \Gamma(\mathcal{D}_2)$ and $\mathcal{Z} \in \Gamma(S(T\mathcal{M}))$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. To prove that $\mathcal{D}_2 \subset \mathcal{R}ad(T\mathcal{M})$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_{\mathcal{X}}\mathcal{Y} \in \mathcal{D}_2$, for all $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}_2)$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}_2)$ and $\mathcal{Z} \in \Gamma(S(T\mathcal{M}))$, we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = \bar{g}((\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}})\mathcal{Z} - \bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Z}, \bar{\mathcal{J}}\mathcal{Y}). \quad (4.4)$$

Now from (2.20), (3.4) and (4.1) we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = -\bar{g}(\nabla_{\mathcal{X}}\bar{\mathcal{J}}P_3\mathcal{Z} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_4\mathcal{Z} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_5\mathcal{Z} + \nabla_{\mathcal{X}}\bar{\mathcal{J}}P_6\mathcal{Z}, \bar{\mathcal{J}}P_2\mathcal{Y}). \quad (4.5)$$

In view of (2.7)-(2.9) and (4.2), for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}_2)$ and $\mathcal{Z} \in \Gamma(S(T\mathcal{M}))$ we obtain

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = -\bar{g}(h^s(\mathcal{X}, \bar{\mathcal{J}}P_3\mathcal{Z}) + \mathcal{D}^s(\mathcal{X}, \bar{\mathcal{J}}P_4\mathcal{Z}) + h^s(\mathcal{X}, fP_5\mathcal{Z}) + h^s(\mathcal{X}, fP_6\mathcal{Z}) + \nabla^s_{\mathcal{X}}FP_5\mathcal{Z} + \nabla^s_{\mathcal{X}}FP_6\mathcal{Z}, \bar{\mathcal{J}}P_2\mathcal{Y}), \quad (4.6)$$

which completes the proof. □

Theorem 4.4. *Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{\mathcal{M}}$. Then $\mathcal{D}'_2$ defines a totally geodesic foliation if and only if*
(i) $\bar{g}(\nabla_{\mathcal{X}}f\mathcal{Z} - A_{F\mathcal{Z}}\mathcal{X}, f\mathcal{Y}) = \bar{g}(h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Z}), F\mathcal{Y})$,
(ii) $g(f\mathcal{Y}, \nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{N}) = -\bar{g}(F\mathcal{Y}, h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{N}))$,
(iii) $g(f\mathcal{Y}, A_{\bar{\mathcal{J}}\mathcal{W}}\mathcal{X}) = \bar{g}(F\mathcal{Y}, \mathcal{D}^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{W}))$,
for all $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}'_2)$, $\mathcal{Z} \in \Gamma(\mathcal{D}'_1)$, $\mathcal{W} \in \Gamma(\bar{\mathcal{J}}ltr(T\mathcal{M}))$ and $\mathcal{N} \in \Gamma(ltr(T\mathcal{M}))$.

Proof. Let $\mathcal{M}$ be a STCR bi-slant lightlike submanifold of an indefinite Kaehler manifold. The distribution $\mathcal{D}'_2$ defines a totally geodesic foliation iff $\nabla_{\mathcal{X}}\mathcal{Y} \in \Gamma(\mathcal{D}'_2)$, $\forall\, \mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}'_2)$. Since $\bar{\nabla}$ is a metric connection for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}'_2)$ and $\mathcal{Z} \in \Gamma(\mathcal{D}'_1)$ we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = \bar{g}(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y}, \bar{\mathcal{J}}\mathcal{Z}) = -\bar{g}(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Z}, \bar{\mathcal{J}}\mathcal{Y}). \quad (4.7)$$

From (2.7), (3.1) and (4.7) we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = -\bar{g}(\nabla_{\mathcal{X}}(f\mathcal{Z} + F\mathcal{Z}) + h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Z}), f\mathcal{Y} + F\mathcal{Y}). \quad (4.8)$$

In view of (2.8) and (4.8) we obtain

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) = -\bar{g}(\nabla_{\mathcal{X}}f\mathcal{Z} - A_{F\mathcal{Z}}\mathcal{X}, f\mathcal{Y}) - \bar{g}(h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Z}), F\mathcal{Y}). \quad (4.9)$$

Now by (4.9) we get the required result

$$\bar{g}(\nabla_{\mathcal{X}}f\mathcal{Z} - A_{F\mathcal{Z}}\mathcal{X}, f\mathcal{Y}) = \bar{g}(h^s(\mathcal{X}, \bar{\mathcal{J}}\mathcal{Z}), F\mathcal{Y}). \quad (4.10)$$

Now for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}_2)$ and $\mathcal{N} \in \Gamma(ltr(T\mathcal{M}))$ we have

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{N}) = \bar{g}(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y}, \bar{\mathcal{J}}\mathcal{N}) = -\bar{g}(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{N}, \bar{\mathcal{J}}\mathcal{Y}). \quad (4.11)$$

From (2.7), (3.1) and (4.11) we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{N}) = -\bar{g}(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{N} + h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{N}), f\mathcal{Y} + F\mathcal{Y}). \tag{4.12}$$

In view of (4.12) we obtain

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{N}) = -\bar{g}(\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{N}, f\mathcal{Y}) - \bar{g}(h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{N}), F\mathcal{Y}). \tag{4.13}$$

Now from (4.13) we get the required result

$$\bar{g}(f\mathcal{Y},\nabla_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{N}) = -\bar{g}(F\mathcal{Y}, h^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{N})). \tag{4.14}$$

Now for any $\mathcal{X},\mathcal{Y} \in \Gamma(\mathcal{D}_2)$ and $\mathcal{W} \in \Gamma(\bar{\mathcal{J}}ltr(T\mathcal{M}))$ we have

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) = \bar{g}(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{Y},\bar{\mathcal{J}}\mathcal{W}) = -\bar{g}(\bar{\nabla}_{\mathcal{X}}\bar{\mathcal{J}}\mathcal{W},\bar{\mathcal{J}}\mathcal{Y}). \tag{4.15}$$

From (2.9), (3.1) and (4.15) we get

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) = -\bar{g}(-A_{\bar{\mathcal{J}}\mathcal{W}}\mathcal{X} + \mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{W}), f\mathcal{Y} + F\mathcal{Y}). \tag{4.16}$$

In view of (4.16) we obtain

$$\bar{g}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) = \bar{g}(A_{\bar{\mathcal{J}}\mathcal{W}}\mathcal{X}, f\mathcal{Y}) - \bar{g}(F\mathcal{Y},\mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{W})). \tag{4.17}$$

Now from (4.17) we get the required result

$$\bar{g}(f\mathcal{Y}, A_{\bar{\mathcal{J}}\mathcal{W}}\mathcal{X}) = \bar{g}(F\mathcal{Y},\mathcal{D}^s(\mathcal{X},\bar{\mathcal{J}}\mathcal{W})), \tag{4.18}$$

which completes the proof. □

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