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# STRONGLY $\mathcal{E}$-CONVEXITY ON INTERVAL-VALUED FUNCTIONS

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**Abstract** *This article presents some novel categories of convex functions: the strongly $LU$-$\mathcal{E}$ convex function and the pseudo strongly $LU$-$\mathcal{E}$ convex function. These functions are defined within the framework of strongly $\mathcal{E}$ convex sets. To illustrate the significance of these definitions, we provide several non-trivial examples that demonstrate the existence of such functions in mathematical analysis. Furthermore, we explore and discuss a variety of interesting and fundamental properties associated with these functions. These include characterizations that distinguish them from other convex functions and relationships that link them to other mathematical constructs. Additionally, we delve into a nonlinear programming problem where the objective function is a quasi strongly $LU$-$\mathcal{E}$ convex function, offering insights into how these new function classes can be applied in optimization theory. This work expands the theoretical understanding of convexity and its applications in mathematical programming.*

## 1 Introduction

*Convexity serves as a fundamental concept in mathematical optimization and plays a crucial role in addressing real-world problems. Over the years, generalizations of convexity have expanded its scope: [14, 20], with concepts such as $\mathcal{E}$-convexity being introduced by [25] and further developed by several authors in pure and applied mathematical analysis, see [8, 9, 19]. Among these advancements, [25, 26], introduced the strongly $\mathcal{E}$-convex functions, highlighting their unique properties and significance in optimization theory. Subsequent research, such as that by [6], introduced a semi-$\mathcal{E}$-convex function and proved some interesting properties related to these functions. Building on these foundational studies, concepts like quasi $\mathcal{E}$-convexity and pseudo $\mathcal{E}$-convexity were investigated by [2]. While [17] explored the properties of $\mathcal{E}$-quasi convex and $\mathcal{E}$-pseudo convex functions. These developments have demonstrated the importance of $\mathcal{E}$-convexity in non-linear optimization and its potential to address complex problems. Numerous properties and outcomes in non-linear optimization theory have been established with respect to $\mathcal{E}$-convex sets and $\mathcal{E}$-convex functions.*

*Interval-valued functions (IVFs) have emerged as a powerful tool for modeling uncertainty in optimization problems by representing outcomes as intervals rather than precise numbers. This approach is especially valuable when dealing with imprecise data and uncertain environments. The pioneering work of [12], who introduced interval arithmetic, set the stage for later advances by [1] through the development of numerical methods for interval computations.*

*A key challenge in this field is defining a meaningful ordering for intervals, as conventional scalar comparisons do not suffice. To address this, [21] introduced specialized order relations and solution concepts that enable systematic comparisons based on characteristics such as endpoints, midpoints, or widths. In parallel, Stefanini and Bede's work on generalized Hukuhara differentiability [18] has extended classical optimization techniques to the interval framework.*

*Recent research has built upon these foundational contributions by integrating novel order relationships, enhanced numerical methods, and robust optimization algorithms. In this context, significant strides have been made by [9] who have explored both Euclidean and non-Euclidean settings. Their contributions include the development of KKT optimality conditions for IVFs , an investigation into the generalized Hukuhara directional differentiability on Riemannian manifolds, see [8, 9] and the derivation of optimality conditions for multi-objective interval-valued problems on Hadamard manifolds. Additionally, the work by [15, 16] established optimality conditions for $\mathcal{E}$-convex interval-valued programming problems using the gH-symmetrical derivative, thereby offering a robust framework for addressing both convexity and optimality in uncertain environments.*

*Complementing these efforts, further advancements have been achieved by [3] in Riemannian settings. Their research has refined the concept of generalized Hukuhara directional differentiability for IVFs on Riemannian manifolds and established rigorous necessary and sufficient optimality conditions under a total order relation, see [4]. Extending these ideas, they have also derived first and second-order necessary conditions for multiobjective programming with IVFs on Riemannian manifolds, see [5] . Moreover, [10] introduced the notions of strongly geodesic preinvexity and strongly invariant $\eta$-monotonicity on Riemannian manifolds, further enriching the theoretical framework for optimization on such spaces.*

*Motivated by [7], this paper introduces the concept of quasi-strongly $LU$-$\mathcal{E}$ convex function, a novel generalization that integrates $LU$-$\mathcal{E}$-convexity and quasi-strongly $\mathcal{E}$-convexity. $LU$-$\mathcal{E}$ convexity represents an additional layer of generalization that incorporates the localized upper and lower bounds into the $\mathcal{E}$-convex framework. The quasi strongly $LU$-$\mathcal{E}$ convex functions extend this idea by relaxing the strong $\mathcal{E}$-convexity condition while retaining essential structural properties.*

*This work aims to:*

- *Define and characterize quasi strongly $LU$-$\mathcal{E}$ convex functions with interval-valued objective functions.*
- *Establish several interesting properties and study their connections to strongly $LU$-$\mathcal{E}$ convex functions.*
- *Offer meaningful examples that demonstrate the existence of quasi-strongly $LU$-$\mathcal{E}$ convex functions within interval-valued contexts.*
- *Explore applications of quasi-strongly $LU$-$\mathcal{E}$ convex functions in solving nonlinear optimization problems with interval-valued objectives.*

*The structure of the paper is outlined as follows: Section 2: In this section, the paper lays the groundwork by defining essential concepts and notations. The section also introduces the concept of E-convex sets and various related convexity properties such as $LU$-$\mathcal{E}$ convexity.*

*Section 3: This section is devoted to the introduction and detailed study of quasi strongly $LU$-$\mathcal{E}$ convex functions. A series of examples are provided to highlight the distinctions between quasi strongly $LU$-$\mathcal{E}C$, $LU$-$\mathcal{E}C$, and other related convex functions. Several theorems are presented to establish key properties, including conditions for the equivalence and implication among various convexity notions, thereby deepening the theoretical understanding of these functions.*

*Section 4: In this section, the theoretical framework is applied to a nonlinear programming problem where the objective function is interval-valued and exhibits quasi strongly $LU$-$\mathcal{E}C$ properties. The problem formulation involves constraints that are defined in terms of these convex functions. The authors derive optimality conditions, proving that under the $QSLU$-$\mathcal{E}C$ framework a local minimum is in fact a global minimum and, under stricter conditions, the optimal solution is unique. This section demonstrates the practical impact of the new theory on solving optimization problems that involve uncertainty*

## 2 Preliminaries

*Let $\mathcal{I}(\mathbb{R})$ denotes the set of all closed and bounded intervals on the real number line $\mathbb{R}$. For any interval $\mathcal{U} \in \mathcal{I}(\mathbb{R})$, it is defined as*

$$\mathcal{U} = [u^L, u^U], \quad \textit{where} \quad u^L, u^U \in \mathbb{R} \quad \textit{and} \quad u^L \leq u^U.$$

*Given two intervals* $\mathcal{U}_1 = [u_1^L, u_1^U]$ *and* $\mathcal{U}_2 = [u_2^L, u_2^U]$ *in* $\mathcal{I}(\mathbb{R})$*, their sum is defined as*

$$\mathcal{U}_1 + \mathcal{U}_2 = [u_1^L + u_2^L, u_1^U + u_2^U].$$

*The negation of* $\mathcal{U}_1$ *is given by*

$$-\mathcal{U}_1 = [-u_1^U, -u_1^L].$$

*Consequently, the difference between* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *is expressed as*

$$\mathcal{U}_1 - \mathcal{U}_2 = \mathcal{U}_1 + (-\mathcal{U}_2) = [u_1^L - u_2^U, u_1^U - u_2^L].$$

*Additionally, scalar multiplication of* $\mathcal{U}_1$ *by a real number* $k$ *is defined as*

$$k\mathcal{U}_1 = \begin{cases} [ku_1^L, ku_1^U] & \text{if } k \geq 0, \\ [ku_1^U, ku_1^L] & \text{if } k < 0. \end{cases}$$

*This summarizes the fundamental operations on intervals within the set* $\mathcal{I}(\mathbb{R})$*. For further details on interval analysis, we refer the reader to [12, 13] and [1].*

*The Hausdorff distance between two intervals* $\mathcal{U}_1 = [u_1^L, u_1^U]$ *and* $\mathcal{U}_2 = [u_2^L, u_2^U]$ *is defined as*

$$d_H(\mathcal{U}_1, \mathcal{U}_2) = \max\{|u_1^L - u_2^L|, |u_1^U - u_2^U|\}.$$

*A limitation of standard interval subtraction is that, for any interval* $\mathcal{U} \in \mathcal{I}(\mathbb{R})$*, the result of* $\mathcal{U} - \mathcal{U}$ *is not equal to zero. For instance, if* $\mathcal{U} = [0, 1]$*, then*

$$\mathcal{U} - \mathcal{U} = [0, 1] - [0, 1] = [-1, 1] \neq 0.$$

*To resolve this issue, the Hukuhara difference between two intervals* $\mathcal{U}_1 = [u_1^L, u_1^U]$ *and* $\mathcal{U}_2 = [u_2^L, u_2^U]$ *is introduced as*

$$\mathcal{U}_1 \ominus \mathcal{U}_2 = [u_1^L - u_2^L, u_1^U - u_2^U].$$

*With this definition, for any interval* $\mathcal{U} \in \mathcal{I}(\mathbb{R})$*,* $\mathcal{U} \ominus \mathcal{U} = 0$*. However, the Hukuhara difference is not always valid for arbitrary intervals. For example,* $[0, 4] \ominus [0, 5] = [0, -1]$*, which is not a valid interval, since the lower bound exceeds the upper bound. This highlights a constraint in the applicability of the Hukuhara difference.*

*To overcome this limitation, Stefanini and Bede [18] proposed the generalized Hukuhara difference (gH-difference) for two intervals* $\mathcal{U}_1$ *and* $\mathcal{U}_2$*, which is defined as*

$$\mathcal{U}_1 \ominus_{gH} \mathcal{U}_2 = \mathcal{U}_3 \iff \begin{cases} (i) & \mathcal{U}_1 = \mathcal{U}_2 + \mathcal{U}_3, \quad \textit{or} \\ (ii) & \mathcal{U}_2 = \mathcal{U}_1 - \mathcal{U}_3. \end{cases}$$

*In case* $(i)$*, the gH-difference is equivalent to the Hukuhara difference (H-difference) [21].*

*For any two intervals* $\mathcal{U}_1 = [u_1^L, u_1^U]$ *and* $\mathcal{U}_2 = [u_2^L, u_2^U]$*, the gH-difference* $\mathcal{U}_1 \ominus_{gH} \mathcal{U}_2$ *always exists and is uniquely determined. Moreover, the following properties hold.*

$$\mathcal{U}_1 \ominus_{gH} \mathcal{U}_1 = [0, 0] \quad \textit{and} \quad \mathcal{U}_1 \ominus_{gH} \mathcal{U}_2 = \left[\min\{u_1^L - u_2^L, u_1^U - u_2^U\}, \max\{u_1^L - u_2^L, u_1^U - u_2^U\}\right].$$

*This generalized approach ensures that the difference between intervals is always well-defined and resolves the issues associated with the standard Hukuhara difference.*

**Definition 2.1.** [21] **Order-relation** Let $\mathcal{U}_1 = [u_1^L, u_1^U]$ and $\mathcal{U}_2 = [u_2^L, u_2^U]$ be two closed intervals in $\mathbb{R}$. We define the relation $\leq_{LU}$ as follows:

$$\mathcal{U}_1 \leq_{LU} \mathcal{U}_2 \quad \text{if and only if} \quad u_1^L \leq u_2^L \quad \text{and} \quad u_1^U \leq u_2^U.$$

It is straightforward to verify that $\leq_{LU}$ is a partial order on the set of intervals $\mathcal{I}(\mathbb{R})$.

Next, we define the strict inequality $\mathcal{U}_1 <_{LU} \mathcal{U}_2$ to mean that $\mathcal{U}_1 \leq_{LU} \mathcal{U}_2$ and $\mathcal{U}_1 \neq \mathcal{U}_2$. This can be equivalently expressed as

$$\begin{cases} u_1^L < u_2^L \\ u_1^U \leq u_2^U \end{cases} \quad \text{or} \quad \begin{cases} u_1^L \leq u_2^L \\ u_1^U < u_2^U \end{cases} \quad \text{or} \quad \begin{cases} u_1^L < u_2^L \\ u_1^U < u_2^U. \end{cases}$$

This strict inequality $<_{LU}$ provides a way to compare intervals based on their lower and upper bounds, ensuring a clear distinction between intervals that are not equal.

**Definition 2.2.** [21] A function $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is referred to as an interval-valued function (IVF) if it can be written in the form

$$\tilde{g}(s) = \left[\tilde{g}^L(s), \tilde{g}^U(s)\right],$$

where $\tilde{g}^L, \tilde{g}^U : \mathbb{R}^n \to \mathbb{R}$, are real valued fuctions satisfying $\tilde{g}^L(s) \leq \tilde{g}^U(s)$ for all $s \in \mathbb{R}^n$.

**Definition 2.3.** Let $\mathcal{M} \subseteq \mathcal{I}(\mathbb{R})$ be a finite subset. We say $\mathcal{U}_1 \in \mathcal{I}(\mathbb{R})$ is a maximum element of $\mathcal{M}$ if $\forall\, \mathcal{U}_2 \in \mathcal{M}$, the following holds

$$\mathcal{U}_2 \leq_{LU} \mathcal{U}_1.$$

In this case, we denote the maximum element as

$$\max \mathcal{M} = \mathcal{U}_1.$$

*In the setting of the $n$-dimensional Euclidean space $\mathbb{R}^n$, consider a mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$. The notion of $\mathcal{E}$-convexity, introduced by [22, 24], is defined as below.*

**Definition 2.4.** [22] A subset $\mathcal{M} \subseteq \mathbb{R}^n$ is said to be $\mathcal{E}$-convex ($\mathcal{E}C$) relative to the mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ if, for any two points $s, t \in \mathcal{M}$ and any scalar $\theta \in [0, 1]$, the combination $\theta\mathcal{E}(s) + (1-\theta)\mathcal{E}(t)$ also belongs to $\mathcal{M}$.

**Definition 2.5.** [24] A function $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathbb{R}$ with real outputs is classified as $\mathcal{E}$-quasi-convex on $\mathcal{M}$ relative to the mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ if $\mathcal{M}$ is an $\mathcal{E}$-convex set and for every pair $s, t \in \mathcal{M}$ and any $\theta_2 \in [0, 1]$, the function satisfies

$$\tilde{g}(\theta_2\mathcal{E}(s) + (1-\theta_2)\mathcal{E}(t)) \leq \max\left\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\right\}.$$

Moreover, $\tilde{g}$ is strictly $\mathcal{E}$-quasi-convex if for each $s, t \in \mathcal{M}$ with $\tilde{g}(\mathcal{E}(s)) \neq \tilde{g}(\mathcal{E}(t))$ and for each $\theta_2 \in (0, 1)$, we have

$$\tilde{g}(\theta_2\mathcal{E}(s) + (1-\theta_2)\mathcal{E}(t)) < \max\left\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\right\}.$$

*In 2007, Wu [21] proposed the concept of LU-convexity for IVF, which is given below:*

**Definition 2.6.** [21] Let $\tilde{g}(s) = [\tilde{g}^L(s), \tilde{g}^U(s)]$ be an IVF defined on a convex domain $\mathcal{M} \subseteq \mathbb{R}^n$. The function $\tilde{g}$ is termed *LU*-convex at $s_0 \in \mathcal{M}$ if $\forall\ s \in \mathcal{M}$ and any scalar $\theta_2 \in [0, 1]$, the following inequality is satisfied

$$\tilde{g}(\theta_2 s_0 + (1-\theta_2)s) \leq_{LU} \theta_2\tilde{g}(s_0) + (1-\theta_2)\tilde{g}(s).$$

*Subsequently, Sachin et al. [16] extended the framework of $\mathcal{E}$-convexity to encompass IVF, introducing the following definition.*

**Definition 2.7.** [16] Let $\mathcal{M} \subseteq \mathbb{R}^n$ be an $\mathcal{E}$-convex set relative to the mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$. An IVF $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$, where $\mathcal{I}(\mathbb{R})$ denotes the space of intervals on $\mathbb{R}$, is called *LU*-$\mathcal{E}$-convex at a point $s_0 \in \mathcal{M}$ if for every $s \in \mathcal{M}$ and scalar $\theta \in [0, 1]$, the following inequality holds

$$\tilde{g}\left(\theta\mathcal{E}(s_0) + (1-\theta)\mathcal{E}(s)\right) \leq_{LU} \theta\,\tilde{g}(\mathcal{E}(s_0)) + (1-\theta)\,\tilde{g}(\mathcal{E}(s)).$$

*We now generalize this definition to quasi LU-$\mathcal{E}$-convex for IVF defined on a subset $\mathcal{M} \subseteq \mathbb{R}^n$, utilizing the LU order relation.*

**Definition 2.8.** An IVF $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is said to be quasi $LU$-$\mathcal{E}$-convex ($QLU$-$\mathcal{E}C$) on $\mathcal{M}$ relative to the mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ if $\mathcal{M}$ is $\mathcal{E}$-convex set and $\forall\ s, t \in \mathcal{M}$ and $\theta_2 \in [0, 1]$,

$$\tilde{g}(\theta_2\mathcal{E}(s) + (1 - \theta_2)\mathcal{E}(t)) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}^*.$$

Furthermore, $\tilde{g}$ is termed strictly quasi $LU$-$\mathcal{E}C$, if for each $s, t \in \mathcal{M}$ with $\tilde{g}(\mathcal{E}(s)) \neq \tilde{g}(\mathcal{E}(t))$ and $\theta_2 \in (0, 1)$,

$$\tilde{g}(\theta_2\mathcal{E}(s) + (1 - \theta_2)\mathcal{E}(t)) <_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

*It is to be noted that every $LU$-$\mathcal{E}C$ function is $QLU$-$\mathcal{E}C$, but the converse need not true in general, we see the following given example.*

**Example 2.9.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [0, 1], & \text{if } s > 0, \\ [s - 1, s], & \text{if } s \leq 0, \end{cases}$$

and $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map defined as

$$\mathcal{E}(s) = |s| = \begin{cases} s, & \text{if } s > 0, \\ -s, & \text{if } s \leq 0. \end{cases}$$

Then the function $\tilde{g}(s)$ is $QLU$-$\mathcal{E}C$, but it is not $LU$-$\mathcal{E}C$. In particular, at $s = 0$, $t = 1$ and $\theta_2 = \frac{1}{2}$, we have

$$\tilde{g}(\theta_2\mathcal{E}(0) + (1 - \theta_2)\mathcal{E}(1)) = \tilde{g}\left(\frac{1}{2}\right) = [0, 1].$$

However,

$$\begin{aligned} \theta_2\tilde{g}(\mathcal{E}(0)) + (1 - \theta_2)\tilde{g}(\mathcal{E}(1)) &= \frac{1}{2}\tilde{g}(\mathcal{E}(0)) + \frac{1}{2}\tilde{g}(\mathcal{E}(1)) \\ &= \frac{1}{2}\tilde{g}(0) + \frac{1}{2}\tilde{g}(1) \\ &= \frac{1}{2}[-1, 0] + \frac{1}{2}[0, 1] \\ &= \left[-\frac{1}{2}, 0\right] + \left[0, \frac{1}{2}\right] \\ &= \left[-\frac{1}{2}, \frac{1}{2}\right], \end{aligned}$$

$$\Rightarrow \tilde{g}(\theta_2\mathcal{E}(s)) + (1 - \theta_2)\mathcal{E}(t))) \nleq_{LU} (\theta_2\tilde{g}(\mathcal{E}(s)) + (1 - \theta_2)\tilde{g}(\mathcal{E}(t))).$$

Hence, it is not $LU$-$\mathcal{E}C$.

**Definition 2.10.** An IVF $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is termed pseudo $LU$-$\mathcal{E}$ convex ($PLU$-$\mathcal{E}C$) relative to the mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathcal{M}$ if $\exists$ a strictly positive IVF $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ such that whenever $\tilde{g}(\mathcal{E}(s)) <_{LU} \tilde{g}(\mathcal{E}(t))$, the following inequality holds

$$\tilde{g}(\theta_2\mathcal{E}(s) + (1 - \theta_2)\mathcal{E}(t)) \leq_{LU} \tilde{g}(\mathcal{E}(t)) + \theta_2(\theta_2 - 1)P(\mathcal{E}(s), \mathcal{E}(t)),$$

$\forall\ s, t \in \mathcal{M}$ and $\theta_2 \in [0, 1]$.

**Example 2.11.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [0, 1], & \text{if } s > 0, \\ [s, s + 1], & \text{if } s \leq 0, \end{cases}$$

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[*] The notion of maximum has been introduced in Definition 2.3, which assumes that $\tilde{g}(\mathcal{E}(s))$ and $\tilde{g}(\mathcal{E}(t))$ are always comparable.

and let $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map given by

$$\mathcal{E}(s) = |s| = \begin{cases} s, & \text{if } s > 0, \\ -s, & \text{if } s \leq 0. \end{cases}$$

A strictly positive function $P(\mathcal{E}(s), \mathcal{E}(t))$ is defined as

$$P(\mathcal{E}(s), \mathcal{E}(t)) = \tilde{g}(\mathcal{E}(t)) \ominus_{gH} \tilde{g}(\mathcal{E}(s)) >_{LU} 0.$$

Thus, IVF $\tilde{g}(s)$ is considered $PLU$-$\mathcal{E}C$.
To show that this IVF is $PLU$-$\mathcal{E}C$, we use the definition

$$\tilde{g}(\mathcal{E}(s)) <_{LU} \tilde{g}(\mathcal{E}(t)) \implies \tilde{g}(\theta_2\mathcal{E}(s) + (1-\theta_2)\mathcal{E}(t)) \leq_{LU} \tilde{g}(\mathcal{E}(t)) + \theta_2(\theta_2 - 1)P(\mathcal{E}(s), \mathcal{E}(t)).$$

**Case I:** If $s > 0$ and $t > 0$ then,

$$\begin{aligned} \tilde{g}(\theta_2 s + (1-\theta_2)t) &\leq_{LU} \tilde{g}(t) + \theta_2(\theta_2 - 1)\left(\tilde{g}(\mathcal{E}(t)) \ominus_{gH} \tilde{g}(\mathcal{E}(s))\right) \\ \tilde{g}(\theta_2 s + (1-\theta_2)t) &\leq_{LU} [0,1] + \theta_2(\theta_2 - 1)\left([0,1] \ominus_{gH} [0,1]\right) \\ \tilde{g}(\theta_2 s + (1-\theta_2)t) &\leq_{LU} [0,1] + \theta_2(\theta_2 - 1)[0,0] \\ [0,1] &\leq_{LU} [0,1]. \end{aligned}$$

**Case II:** If $s > 0$ and $t < 0$ then,

$$\begin{aligned} \tilde{g}(\theta_2 s + (1-\theta_2)(-t)) &\leq_{LU} \tilde{g}(-t) + \theta_2(\theta_2 - 1)\left(\tilde{g}(-t) \ominus_{gH} \tilde{g}(s)\right) \\ [0,1] &\leq_{LU} [0,1] + \theta_2(\theta_2 - 1)\left([0,1] \ominus_{gH} [0,1]\right) \\ [0,1] &\leq_{LU} [0,1]. \end{aligned}$$

**Case III:** If $s < 0$ and $t > 0$ then,

$$\begin{aligned} \tilde{g}(-\theta_2 s + (1-\theta_2)t) &\leq_{LU} \tilde{g}(t) + \theta_2(\theta_2 - 1)\left(\tilde{g}(t) \ominus_{gH} \tilde{g}(-s)\right) \\ [0,1] &\leq_{LU} [0,1]. \end{aligned}$$

**Case IV:** If $s < 0$ and $t < 0$ then,

$$\begin{aligned} \tilde{g}(-\theta_2 s + (1-\theta_2)(-t)) &\leq_{LU} \tilde{g}(-t) + \theta_2(\theta_2 - 1)\left(\tilde{g}(-t) \ominus_{gH} \tilde{g}(-s)\right) \\ [0,1] &\leq_{LU} [0,1]. \end{aligned}$$

**Definition 2.12.** [25] A non-empty set $\mathcal{M} \subseteq \mathbb{R}^n$ is termed strongly $\mathcal{E}$-convex ($S\mathcal{E}C$) relative to a mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$, if for every pair of elements $s, t \in \mathcal{M}$ and $\forall\, \theta_1, \theta_2 \in [0,1]$, the following inclusion holds

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}.$$

*In the following definition, we introduce strongly LU-$\mathcal{E}$-convexity for IVFs as follows.*

**Definition 2.13.** Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $S\mathcal{E}C$ set. An IVF $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ is termed to be strongly $LU$-$\mathcal{E}$ convex ($SLU$-$\mathcal{E}C$) relative to a mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathcal{M}$, if $\forall\, s, t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0,1]$, we have

$$\tilde{g}\big(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\big) \leq_{LU} \theta_2\tilde{g}(\mathcal{E}(s)) + (1-\theta_2)\tilde{g}(\mathcal{E}(t)).$$

If the above inequality becomes strict and $\theta_1 s + \mathcal{E}(s) \neq \theta_1 t + \mathcal{E}(t)$, $\forall\, s, t \in \mathcal{M}$ and for $\theta_1 \in [0,1]$, $\theta_2 \in (0,1)$, then the function $\tilde{g}$ is referred to as strictly $SLU$-$\mathcal{E}C$.

**Example 2.14.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [0,1], & \text{if } s > 0, \\ [s, s+1], & \text{if } s \leq 0, \end{cases}$$

and let $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map given by

$$\mathcal{E}(s) = |s| = \begin{cases} s, & \text{if } s > 0, \\ -s, & \text{if } s \leq 0. \end{cases}$$

To show that this IVF is $SLU$-$\mathcal{E}C$, we use the definition:
**Case I**: If $s > 0$ and $t > 0$, then

$$\begin{aligned} \tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) &\leq_{LU} \theta_2 \tilde{g}(\mathcal{E}(s)) + (1-\theta_2)\tilde{g}(\mathcal{E}(t)) \\ \tilde{g}\left(\theta_2(\theta_1 s + s) + (1-\theta_2)(\theta_1 t + t)\right) &\leq_{LU} \theta_2 \tilde{g}(s) + (1-\theta_2)\tilde{g}(t) \\ [0,1] &\leq_{LU} \theta_2[0,1] + (1-\theta_2)[0,1] \\ [0,1] &\leq_{LU} [0,1] \end{aligned}$$

**Case II**: If $s > 0$ and $t < 0$, then

$$\begin{aligned} \tilde{g}\left(\theta_2(\theta_1 s + s) + (1-\theta_2)(\theta_1 t - t)\right) &\leq_{LU} \theta_2 \tilde{g}(s) + (1-\theta_2)\tilde{g}(t) \\ \tilde{g}\left(\theta_2(\theta_1 + 1)s - (1-\theta_2)(1-\theta_1)t\right) &\leq_{LU} \theta_2 \tilde{g}(s) + (1-\theta_2)\tilde{g}(t) \\ [0,1] &\leq_{LU} [0,1] \end{aligned}$$

**Case III**: If $s < 0$ and $t > 0$, then

$$\begin{aligned} \tilde{g}\left(\theta_2(\theta_1 s - s) + (1-\theta_2)(\theta_1 t + t)\right) &\leq_{LU} \theta_2 \tilde{g}(-s) + (1-\theta_2)\tilde{g}(t) \\ [0,1] &\leq_{LU} [0,1] \end{aligned}$$

**Case IV**: If $s < 0$ and $t < 0$, then

$$\begin{aligned} \tilde{g}\left(\theta_2(\theta_1 s - s) + (1-\theta_2)(\theta_1 t - t)\right) &\leq_{LU} \theta_2 \tilde{g}(-s) + (1-\theta_2)\tilde{g}(-t) \\ [0,1] &\leq_{LU} [0,1] \end{aligned}$$

**Remark 2.15.** Every $SLU$-$\mathcal{E}C$ function relative to a map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ becomes $LU$-$\mathcal{E}C$ function, when $\theta_1 = 0$.

**Definition 2.16.** Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $S\mathcal{E}C$ set. An IVF $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ is termed as a semi-strongly $LU$-$\mathcal{E}$ convex ($SSLU$-$\mathcal{E}C$) function relative to the map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$, if for every $s, t \in \mathcal{M}$ and $\forall\, \theta_1, \theta_2 \in [0,1]$, we have

$$\tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) \leq_{LU} \theta_2 \tilde{g}(s) + (1-\theta_2)g(t)\}.$$

If the above inequality becomes strict and $\theta_1 s + \mathcal{E}(s) \neq \theta_1 t + \mathcal{E}(t)$, $\forall\, s, t \in \mathcal{M}$ with $\theta_1 \in [0,1]$ and $\theta_2 \in (0,1)$, then $\tilde{g}$ is called a strictly $SSLU$-$\mathcal{E}C$ function.

**Remark 2.17.** Every $SSLU$-$\mathcal{E}C$ function relative to a map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ becomes semi $LU$-$\mathcal{E}C$ function, when $\theta_1 = 0$.

**Remark 2.18.** A function that is $SSLU$-$\mathcal{E}C$ concerning an operator $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ is not necessarily $SLU$-$\mathcal{E}C$.

**Example 2.19.** Let us consider an IVF $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ defined as

$$\tilde{g}(s) = [s, s+1]$$

and let $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map given by

$$\mathcal{E}(s) = -s.$$

This function $\tilde{g}$ is $SSLU$-$\mathcal{E}C$ as well as a semi $LU$-$\mathcal{E}C$, but it is not a $SLU$-$\mathcal{E}C$. A function $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ is $SSLU$-$\mathcal{E}C$ if it satisfies the following inequality,

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \theta_2\tilde{g}(s) + (1-\theta_2)\tilde{g}(t).$$

Now substituting $\mathcal{E}(s) = -s$ in the above inequality, we get

$$\tilde{g}(\theta_2(\theta_1 s - s) + (1-\theta_2)(\theta_1 t - t)) \leq_{LU} \theta_2\tilde{g}(s) + (1-\theta_2)\tilde{g}(t),$$

which simplifies to

$$\tilde{g}((\theta_1 - 1)(\theta_2 s + (1-\theta_2)t)) \leq_{LU} \theta_2[s, s+1] + (1-\theta_2)[t, t+1].$$

Expanding this expression yields

$$[(\theta_1-1)(\theta_2 s+(1-\theta_2)t), (\theta_1-1)(\theta_2 s+(1-\theta_2)t+1)] \leq_{LU} [\theta_2 s+(1-\theta_2)t, \theta_2 s+(1-\theta_2)t+1].$$

It is clear that

$$(\theta_1 - 1)(\theta_2 s + (1-\theta_2)t) \leq \theta_2 s + (1-\theta_2)t,$$

and

$$(\theta_1 - 1)(\theta_2 s + (1-\theta_2)t + 1) \leq \theta_2 s + (1-\theta_2)t + 1.$$

This verifies the conditions for $SSLU$-$\mathcal{E}$ convexity. It is semi $LU$-$\mathcal{E}C$, when $\theta_1 = 0$.

In particular, at points $\theta_1 = \theta_2 = \frac{1}{2}$, $s = 0$, and $t = 1$, we have

$$\begin{aligned}\tilde{g}\left(\frac{1}{2}\left(\frac{1}{2}\cdot 0 + \mathcal{E}(0)\right) + \left(1-\frac{1}{2}\right)\left(\frac{1}{2}\cdot 1 + \mathcal{E}(1)\right)\right) &= \tilde{g}\left(0 + \frac{1}{2}\left(\frac{1}{2} - 1\right)\right)\\ &= \left[\frac{-1}{4}, \frac{3}{4}\right].\end{aligned}$$

However,

$$\begin{aligned}\frac{1}{2}\tilde{g}(\mathcal{E}(0)) + \left(1-\frac{1}{2}\right)\tilde{g}(\mathcal{E}(1)) &= \frac{1}{2}\tilde{g}(0) + \frac{1}{2}\tilde{g}(-1),\\ &= \frac{1}{2}[0,1] + \frac{1}{2}[-1,0]\\ &= \left[0, \frac{1}{2}\right] + \left[-\frac{1}{2}, 0\right]\\ &= \left[-\frac{1}{2}, \frac{1}{2}\right].\end{aligned}$$

$$\implies \tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \nleq_{LU} (\theta_2\tilde{g}(\mathcal{E}(s)) + (1-\theta_2)\tilde{g}(\mathcal{E}(t))).$$

Hence, it is not $SLU$-$\mathcal{E}C$.

**Remark 2.20.** Not every $SLU$-$\mathcal{E}C$ concerning an operator $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ is $SSLU$-$\mathcal{E}C$.

**Example 2.21.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [0,1], & \text{if } s > 0,\\ [s, s+1], & \text{if } s \leq 0,\end{cases}$$

and let $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map given by

$$\mathcal{E}(s) = |s| = \begin{cases} s, & \text{if } s > 0,\\ -s, & \text{if } s \leq 0.\end{cases}$$

Given that IVF $\tilde{g}$ is $SLU$-$\mathcal{E}C$ but not $SSLU$-$\mathcal{E}C$, we demonstrate this as follows. In particular, at points $\theta_1 = \frac{1}{2}, \theta_2 = 0, s = 0, t = -1$. We have

$$\tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right)$$

$$= \tilde{g}\left(0 + 1\left(-\frac{1}{2} + \mathcal{E}(-1)\right)\right) = \tilde{g}\left(\frac{1}{2}\right) = [0,1].$$

Also

$$\theta_2\tilde{g}(s) + (1-\theta_2)\tilde{g}(t) = 0\tilde{g}(0) + (1-0)\tilde{g}(-1) = \tilde{g}(-1) = [-1,0].$$

$$\implies [0,1] \not\leq_{LU} [-1,0].$$

Hence, it is not $SSLU$-$\mathcal{E}C$.

**Theorem 2.22.** *Let $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ be a semi-$\mathcal{E}C$ function defined on an $\mathcal{E}C$ set $\mathcal{M}$. Then $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s)$ for every $s \in \mathcal{M}$.*

***Proof.*** *Since $\tilde{g}$ is semi-$\mathcal{E}C$ on an $\mathcal{E}C$ set $\mathcal{M} \subseteq \mathbb{R}^n$, then for any $s, t \in \mathcal{M}$ and $0 \leq \theta_2 \leq 1$, we have*

$$\theta_2\mathcal{E}(s) + (1-\theta_2)\mathcal{E}(t) \in \mathcal{M}$$

*and*

$$\tilde{g}(\theta_2\mathcal{E}(s) + (1-\theta_2)\mathcal{E}(t)) \leq_{LU} \theta_2\tilde{g}(s) + (1-\theta_2)\tilde{g}(t).$$

*Thus, for $\theta_2 = 1$,*

$$\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s).$$

**Remark 2.23.** An IVF $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ that is $LU$-$\mathcal{E}C$ on an $\mathcal{E}C$ set is not necessarily semi-$LU$-$\mathcal{E}C$.

**Example 2.24.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [0,1], & \text{if } s > 0, \\ [s, s+2], & \text{if } s \leq 0, \end{cases}$$

and let $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map given by

$$\mathcal{E}(s) = |s| = \begin{cases} s, & \text{if } s > 0, \\ -s, & \text{if } s \leq 0. \end{cases}$$

It is $LU$-$\mathcal{E}C$ but not semi $LU$-$\mathcal{E}C$ as well as not $SLU$-$\mathcal{E}C$. In particular, at points $s = -1, t = 1, \theta_2 = \frac{1}{2}$, we get

$$\tilde{g}\left(\frac{1}{2}\mathcal{E}(-1) + (1-\frac{1}{2})\mathcal{E}(1)\right) = \tilde{g}(1) = [0,1]$$

and

$$\frac{1}{2}\tilde{g}(-1) + \frac{1}{2}\tilde{g}(1) = \frac{1}{2}[-1,1] + \frac{1}{2}[0,1]$$

$$= \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, \frac{1}{2}]$$

$$= \left[-\frac{1}{2}, 1\right]$$

$$\Rightarrow [0,1] \not\leq_{LU} \left[-\frac{1}{2}, 1\right].$$

Thus, the function is not semi $LU$-$\mathcal{E}C$.

For a $SLU$-$\mathcal{E}C$ function, in particular points $\theta_1 = 1, \theta_2 = 0, s = 1, t = -1$

$$\begin{aligned}\tilde{g}\left(\theta_2(\theta_1 s+\mathcal{E}(s))+(1-\theta_2)(\theta_1 t+\mathcal{E}(t))\right) &= \tilde{g}\left(0+(1-0)(-1+1)\right)\\ &= \tilde{g}(0)\\ &= [0,2]\end{aligned}$$

and

$$\begin{aligned}\theta_2\tilde{g}(\mathcal{E}(s))+(1-\theta_2)\tilde{g}(\mathcal{E}(t)) &= 0\cdot\tilde{g}(\mathcal{E}(s))+(1-0)\tilde{g}(\mathcal{E}(t))\\ &= 0+\tilde{g}(1)\\ &= [0,1]\\ \Rightarrow [0,2] &\not\leq_{LU} [0,1].\end{aligned}$$

Thus it is not $SLU$-$\mathcal{E}C$.

## 3 Quasi strongly $LU$-$\mathcal{E}C$ functions

*In this section, we present the generalized category of SLU-$\mathcal{E}C$ functions, referred to as quasi strongly LU-$\mathcal{E}$ convex (QSLU-$\mathcal{E}C$) functions which are defined on S$\mathcal{E}C$ sets $\mathcal{M}\subseteq\mathbb{R}^n$. Additionally, we examine some of their key properties.*

**Definition 3.1.** Let $\mathcal{M}\subseteq\mathbb{R}^n$ be a $S\mathcal{E}C$ set. An IVF $\tilde{g}:\mathcal{M}\subseteq\mathbb{R}^n\to\mathcal{I}(\mathbb{R})$ is termed a quasi strongly $LU$-$\mathcal{E}$ convex ($QSLU$-$\mathcal{E}C$) function relative to the mapping $\mathcal{E}:\mathbb{R}^n\to\mathbb{R}^n$ on $\mathcal{M}$ if $\forall$ $s,t\in\mathcal{M}$ and $\theta_1,\theta_2\in[0,1]$, we have

$$\tilde{g}\left(\theta_2(\theta_1 s+\mathcal{E}(s))+(1-\theta_2)(\theta_1 t+\mathcal{E}(t))\right)\leq_{LU}\max\{\tilde{g}(\mathcal{E}(s)),\tilde{g}(\mathcal{E}(t))\}.$$

If this inequality is strict and $\tilde{g}(\mathcal{E}(s))\neq\tilde{g}(\mathcal{E}(t))$, $s,t\in\mathcal{M}$ and $\theta_1\in[0,1]$, $\theta_2\in(0,1)$, then $\tilde{g}$ is called a strictly $QSLU$-$\mathcal{E}C$ function.

Furthermore, Definition 3.1 is referred to as a quasi semi strongly $LU$-$\mathcal{E}$ convex ($QSSLU$-$\mathcal{E}C$) function when $\tilde{g}(\mathcal{E}(s))$ and $\tilde{g}(\mathcal{E}(t))$ in the above inequality are replaced by $\tilde{g}(s)$ and $g(t)$, respectively.

**Remark 3.2.** Every $QSLU$-$\mathcal{E}C$ function relative to a map $\mathcal{E}:\mathbb{R}^n\to\mathbb{R}^n$ becomes $QLU$-$\mathcal{E}C$ when $\theta_1=0$.

**Example 3.3.** The function $\tilde{g}(s)$, as defined in Example 2.9, is both $QSLU$-$\mathcal{E}C$ and $QLU$-$\mathcal{E}C$, but it is not a $SLU$-$\mathcal{E}C$.

In particular, at points $s=0$, $t=1$, $\theta_1=\frac{1}{2}$, and $\theta_2=\frac{1}{2}$, we have

$$\tilde{g}(\theta_2(\theta_1 0+\mathcal{E}(0))+(1-\theta_2)(\theta_1 1+\mathcal{E}(1)))=g\left(\frac{1}{2}\left(\frac{1}{2}+1\right)\right)=\tilde{g}\left(\frac{3}{4}\right)=[0,1].$$

However,

$$\begin{aligned}\theta_2\tilde{g}(\mathcal{E}(0))+(1-\theta_2)\tilde{g}(\mathcal{E}(1)) &= \frac{1}{2}\tilde{g}(\mathcal{E}(0))+\frac{1}{2}\tilde{g}(\mathcal{E}(1))\\ &= \frac{1}{2}\tilde{g}(0)+\frac{1}{2}\tilde{g}(1)\\ &= \frac{1}{2}[-1,0]+\frac{1}{2}[0,1]\\ &= \left[-\frac{1}{2},0\right]+\left[0,\frac{1}{2}\right]\\ &= \left[-\frac{1}{2},\frac{1}{2}\right],\end{aligned}$$

$$\Rightarrow \tilde{g}(\theta_2(\theta_1 s+\mathcal{E}(s))+(1-\theta_2)(\theta_1 t+\mathcal{E}(t)))\not\leq_{LU}\left(\theta_2\tilde{g}(\mathcal{E}(s))+(1-\theta_2)\tilde{g}(\mathcal{E}(t))\right).$$

Hence, it is not $SLU$-$\mathcal{E}C$ .

*$QLU$-$\mathcal{E}C$ function relative to a map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ does not necessarily have to be a $QSLU$-$\mathcal{E}C$ function, as demonstrated in the following example.*

**Example 3.4.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [s^2, s^2 + 2], & \text{if } s \geq 0, \\ [0, 1], & \text{if } s < 0, \end{cases}$$

and $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map defined as

$$\mathcal{E}(s) = -s^2.$$

The function $\tilde{g}(s)$ is $QLU$-$\mathcal{E}C$ relative to the mapping $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ defined by $\mathcal{E}(s) = -s^2$. However, it is not a $QSLU$-$\mathcal{E}C$ function. Specifically, at points $s = -1$, $t = 1$, $\theta_1 = 1$, and $\theta_2 = 0$, we have

$$\tilde{g}(0 + (1 - 0)(1 + \mathcal{E}(1))) = \tilde{g}(1 - 1) = \tilde{g}(0) = [0, 2],$$

and

$$\max\{\tilde{g}(\mathcal{E}(-1)), \tilde{g}(\mathcal{E}(1))\} = \max\{\tilde{g}(-1), \tilde{g}(-1)\} = [0, 1].$$

Thus,

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t))) \nleq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Therefore, $\tilde{g}(s)$ is not a $QSLU$-$\mathcal{E}C$ function.

**Example 3.5.** Let $\tilde{g} : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ be defined as

$$\tilde{g}(s) = \begin{cases} [0, 1], & \text{if } s > 0, \\ [s - 1, s], & \text{if } s \leq 0, \end{cases}$$

and $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ be a map defined as

$$\mathcal{E}(s) = |s| = \begin{cases} s, & \text{if } s > 0, \\ -s, & \text{if } s \leq 0. \end{cases}$$

Then, the function $\tilde{g}(s)$ is $QSLU$-$\mathcal{E}C$, but it is not $QSSLU$-$\mathcal{E}C$. In particular, at points $s = 0$, $t = -1$ and $\theta_1 = \frac{1}{2}$, $\theta_2 = 0$ we have

$$\tilde{g}\left(0 + 1\left(\frac{-1}{2} + 1\right)\right) = \tilde{g}\left(\frac{1}{2}\right) = [0, 1]$$

However,

$$\begin{aligned} \max\{\tilde{g}(0), \tilde{g}(-1)\} &= \max\{[-1, 0], [-2, -1]\} \\ &= [-1, 0] \\ \Rightarrow [0, 1] &\nleq_{LU} [-1, 0]. \end{aligned}$$

**Theorem 3.6.** *If $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is a $QSLU$-$\mathcal{E}C$ function defined on a $S\mathcal{E}C$ set $\mathcal{M}$, then for any $t \in \mathcal{M}$ and $\theta_1 \in [0, 1]$, the following holds*

$$\tilde{g}(\theta_1 t + \mathcal{E}(t)) \leq_{LU} \tilde{g}(\mathcal{E}(t)).$$

*Proof.* Since $\tilde{g}$ is a $QSLU$-$\mathcal{E}C$ function defined on a $S\mathcal{E}C$ set $\mathcal{M}$, for every $s, t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0, 1]$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}.$$

Furthermore,

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

If $s = t$, then we obtain

$$\tilde{g}(\theta_1 t + \mathcal{E}(t)) \leq_{LU} \tilde{g}(\mathcal{E}(t)),$$

for any $t \in \mathcal{M}$ and $\theta_1 \in [0, 1]$.

□

**Theorem 3.7.** *Let $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ be a QSSLU-$\mathcal{E}$C function defined on S$\mathcal{E}$C set $\mathcal{M} \subseteq \mathbb{R}^n$. Then, $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s)$, for every $s \in \mathcal{M}$.*

***Proof.*** *Since $\tilde{g}$ is QSSLU-$\mathcal{E}$C function, then*

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(s), \tilde{g}(t)\},$$

*for every $s, t \in \mathcal{M}$, $\theta_1, \theta_2 \in [0, 1]$.*

*By setting $s = t$ and $\theta_1 = 0$ in the above inequality, we obtain $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s)$, for each $s \in \mathcal{M}$.*

**Theorem 3.8.** *Let $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ be a QSLU-$\mathcal{E}$C function defined on a S$\mathcal{E}$C set $\mathcal{M} \subseteq \mathbb{R}^n$. Then, $\tilde{g}$ is QSSLU-$\mathcal{E}$C on $\mathcal{M}$ if $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s)$, for every $s \in \mathcal{M}$.*

*Proof.* Suppose that $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s)$ for every $s \in \mathcal{M}$. From the *QSLU-$\mathcal{E}$* convexity of $\tilde{g}$, it follows that

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Given $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(s)$

$$\Rightarrow \tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(s), \tilde{g}(t)\},$$

for every $s, t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0, 1]$ .

□

*A significant relationship between SLU-$\mathcal{E}$C functions and QSLU-$\mathcal{E}$C as follows:*

**Theorem 3.9.** *A function $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ defined on a S$\mathcal{E}$C set $\mathcal{M}$ is SLU-$\mathcal{E}$C and also QSLU-$\mathcal{E}$C if the condition $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(\mathcal{E}(t))$ holds, $\forall\, s, t \in \mathcal{M}$.*

*Proof.* Suppose that the function $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is *SLU-$\mathcal{E}$C* , and $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(\mathcal{E}(t))$ $\forall\, s, t \in \mathcal{M}$. $\theta_1, \theta_2 \in [0, 1]$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M},$$

and

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \theta_2 \tilde{g}(\mathcal{E}(s)) + (1-\theta_2)\tilde{g}(\mathcal{E}(t)),$$

or

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \tilde{g}(\mathcal{E}(t)) = \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Thus,

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Therefore, $\tilde{g}$ is a *QSLU-$\mathcal{E}$C* on $\mathcal{M}$. □

**Theorem 3.10.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a S$\mathcal{E}$C set, and let $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ be a QSLU-$\mathcal{E}$C function defined on $\mathcal{M}$. If $\mathcal{E}(\mathcal{M}) \subseteq \mathcal{M}$ is also a S$\mathcal{E}$C set, then the restriction of $\tilde{g}$ to $\mathcal{E}(\mathcal{M})$, denoted by $\tilde{g}_0$, is a QSLU-$\mathcal{E}$C function on $\mathcal{E}(\mathcal{M})$.*

*Proof.* Let $s, t \in \mathcal{M}$ and suppose that $\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}$, for any $\theta_1, \theta_2 \in [0, 1]$.

Assume,

$$\forall s^*, t^* \in \mathcal{E}(\mathcal{M})) \quad \exists\, s, t \in \mathcal{M} \text{ s.t.}$$

$$s^* = \mathcal{E}(s) \quad \& \quad t^* = \mathcal{E}(t)$$

Now suppose $\tilde{g}(s) = \tilde{g}_0(s)\ \forall\ s^* \in \mathcal{E}(\mathcal{M})$.
Since $\mathcal{E}(\mathcal{M})$ is a $S\mathcal{E}C$ set, then we have

$$\theta_2(\theta_1 s^* + \mathcal{E}(s^*)) + (1-\theta_2)(\theta_1 t^* + \mathcal{E}(t^*)) \in \mathcal{E}(\mathcal{M}),$$

for any $\theta_1, \theta_2 \in [0, 1]$. Therefore, it follows that

$$\begin{aligned} &\tilde{g}_0(\theta_2(\theta_1 s^* + \mathcal{E}(s^*)) + (1-\theta_2)(\theta_1 t^* + \mathcal{E}(t^*))) \\ &\qquad = \tilde{g}(\theta_2(\theta_1 s^* + \mathcal{E}(s^*)) + (1-\theta_2)(\theta_1 t^* + \mathcal{E}(t^*))) \\ &\qquad \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s^*)), \tilde{g}(\mathcal{E}(t^*))\} \\ &\qquad = \max\{\tilde{g}_0(\mathcal{E}(s^*)), \tilde{g}_0(\mathcal{E}(t^*))\}. \end{aligned}$$

Thus,

$$\tilde{g}_0(\theta_2(\theta_1 s^* + \mathcal{E}(s^*)) + (1-\theta_2)(\theta_1 t^* + \mathcal{E}(t^*))) \leq_{LU} \max\{\tilde{g}_0(\mathcal{E}(s^*)), \tilde{g}_0(\mathcal{E}(t^*))\}.$$

This concludes the proof. □

**Theorem 3.11.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $S\mathcal{E}C$ set. Assume that the functions $\tilde{g}_i : \mathcal{M} \to \mathcal{I}(\mathbb{R})$, $i = 1, 2, \ldots, r$, are non-negative and $QSLU$-$\mathcal{E}C$ relative to a mapping $\mathcal{E}$ on $\mathcal{M}$. Then, the linear combination of these functions remains $QSLU$-$\mathcal{E}C$. Specifically, for coefficients $c_i \geq 0$, where $i = 1, 2, \ldots, r$, the function*

$$f(s) = \sum_{i=1}^{r} c_i \tilde{g}_i(s)$$

*is $QSLU$-$\mathcal{E}C$ on $\mathcal{M}$.*

*Proof.* Given that $\tilde{g}_i(s)$, $i = 1, 2, \ldots, r$, are $QSLU$-$\mathcal{E}C$ functions defined on $\mathcal{M}$, for any $s, t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0, 1]$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M},$$

and

$$\begin{aligned} &f(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \\ &\qquad = \sum_{i=1}^{r} c_i g_i(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \\ &\qquad \leq_{LU} \max\left\{\sum_{i=1}^{r} c_i \tilde{g}_i(\mathcal{E}(s)), \sum_{i=1}^{r} c_i \tilde{g}_i(\mathcal{E}(t))\right\} \\ &\qquad = \max\{f(\mathcal{E}(s)), f(\mathcal{E}(t))\}. \end{aligned}$$

Hence, the function $f(s)$ is also $QSLU$-$\mathcal{E}C$ on $\mathcal{M}$. □

**Theorem 3.12.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $S\mathcal{E}C$ set and consider a collection of IVF $\{\tilde{g}_j\}_{j \in J}$ defined on $\mathcal{M}$, where $\sup_{j \in J} \tilde{g}_j(s)$ exists in $\mathcal{I}(\mathbb{R})$ for every $s \in \mathcal{M}$. Define the IVF $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ as*

$$\tilde{g}(s) = \sup_{j \in J} \tilde{g}_j(s), \quad \textit{for every } s \in \mathcal{M}.$$

*If each function $\tilde{g}_j : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ for $j \in J$ is $QSLU$-$\mathcal{E}C$ on $\mathcal{M}$, then the function $\tilde{g}$ is also $QSLU$-$\mathcal{E}C$ on $\mathcal{M}$.*

*Proof.* Assume that $\tilde{g}_j : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ are $QSLU\text{-}\mathcal{E}C$ functions for each $j \in J$. Then, for every $s, t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0, 1]$, we have

$$\tilde{g}_j\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) \leq_{LU} \max\{\tilde{g}_j(\mathcal{E}(s)), \tilde{g}_j(\mathcal{E}(t))\}$$

and

$$\begin{aligned}\tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) &= \sup_{j\in J} \tilde{g}_j\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right)\\ &\leq_{LU} \sup_{j\in J} \max\{\tilde{g}_j(\mathcal{E}(s)), \tilde{g}_j(\mathcal{E}(t))\}\\ &= \max\left\{\sup_{j\in J} \tilde{g}_j(\mathcal{E}(s)), \sup_{j\in J} \tilde{g}_j(\mathcal{E}(t))\right\}\\ &= \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}\end{aligned}$$

$$\Rightarrow \tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Thus, $\tilde{g}$ is a $QSLU\text{-}\mathcal{E}C$ function on $\mathcal{M}$.

□

**Definition 3.13.** An IVF $\psi : \mathcal{I}(\mathbb{R}) \to \mathcal{I}(\mathbb{R})$ is said to be *LU non-decreasing* if the following holds:

$$\psi(\mathcal{U}_1) \leq_{LU} \psi(\mathcal{U}_2), \quad \text{whenever } \mathcal{U}_1 \leq_{LU} \mathcal{U}_2.$$

**Theorem 3.14.** *Let $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ be a QSLU-$\mathcal{E}C$ function defined on a $S\mathcal{E}C$ set $\mathcal{M} \subseteq \mathbb{R}^n$, and let $\psi : \mathcal{I}(\mathbb{R}) \to \mathcal{I}(\mathbb{R})$ be a LU non-decreasing function. Then, the composition $\psi \circ \tilde{g}$ is also $QSLU\text{-}\mathcal{E}C$ on $\mathcal{M}$.*

*Proof.* Since $\mathcal{M}$ is a $S\mathcal{E}C$ set, then for any $s, t \in \mathcal{M}$, $\theta_1, \theta_2 \in [0, 1]$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}.$$

Given $\tilde{g}$ is a $QSLU\text{-}\mathcal{E}C$ function on $\mathcal{M}$, we obtain

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Since $\psi$ is an $LU$ non-decreasing function, we have

$$\begin{aligned}(\psi \circ \tilde{g})(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) &\leq_{LU} \psi \circ (\max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\})\\ &\leq_{LU} \max\{(\psi \circ \tilde{g})(\mathcal{E}(s)), (\psi \circ \tilde{g})(\mathcal{E}(t))\}.\end{aligned}$$

Hence, $\psi \circ \tilde{g}$ is a $QSLU\text{-}\mathcal{E}C$ function on $\mathcal{M}$.

□

**Theorem 3.15.** *Let $h_i : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$, where $i = 1, 2, \ldots, r$, be $QSLU\text{-}\mathcal{E}C$ functions on $\mathbb{R}^n$ relative to a mapping $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$. If $\mathcal{E}(\mathcal{M}) \subseteq \mathcal{M}$, then the set*

$$\mathcal{M} = \{s \in \mathbb{R}^n : h_i(s) \leq_{LU} 0,\ i = 1, 2, \ldots, r\}$$

*is $S\mathcal{E}C$.*

*Proof.* Since $h_i(s), i = 1, 2, \ldots, r$, are $QSLU\text{-}\mathcal{E}C$ functions, then for each $s, t \in \mathcal{M}$, and $\theta_1, \theta_2 \in [0, 1]$, we have

$$\begin{aligned}h_i\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) &\leq_{LU} \max\left\{(h_i \circ \mathcal{E})(s), (h_i \circ \mathcal{E})(t)\right\}\\ &= \max\left\{h_i(\mathcal{E}(s)), h_i(\mathcal{E}(t))\right\}\\ &= \max\{0, 0\}\\ &\leq_{LU} 0.\end{aligned}$$

Given that $\mathcal{E}(\mathcal{M}) \subseteq \mathcal{M}$.
Thus, we get

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}.$$

Therefore, $\mathcal{M}$ is a $S\mathcal{EC}$ set.

□

**Definition 3.16.** Let $\mathcal{M} \subseteq \mathbb{R}^n$ be an $S\mathcal{EC}$ set, and consider an IVF $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$. The lower-level set of $\tilde{g}$ is given by

$$K_\gamma = \{s \in \mathcal{M} : \tilde{g}(s) \leq_{LU} \gamma, \ \forall \, \gamma \in \mathcal{I}(\mathbb{R})\}.$$

*The subsequent theorem explores the relationship between the lower level set and $QSLU$-$\mathcal{EC}$ function.*

**Theorem 3.17.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $S\mathcal{EC}$ set. If the lower level set $K_\gamma$ is $S\mathcal{EC}$ for every $\gamma \in \mathcal{I}(\mathbb{R})$, then the function $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ is $QSLU$-$\mathcal{EC}$ on $\mathcal{M}$.*

*Proof.* Suppose that $\mathcal{M}$ is a $S\mathcal{EC}$ set and that the set $K_\gamma$ is $S\mathcal{EC}$, for each $\gamma \in \mathbb{R}$. For every $s, t \in \mathcal{M}$, and $\theta_1, \theta_2 \in [0, 1]$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}.$$

By using Theorem 1 see in [25], we obtain $\mathcal{E}(t) \in \mathcal{M}$, for all $t \in \mathcal{M}$.
Let $\gamma = \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}$, and $s, t \in K_\gamma$.
Since $K_\gamma$ is $S\mathcal{EC}$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t)) \in K_\gamma,$$

$$\Rightarrow \tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t))\right) \leq_{LU} \gamma = \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Therefore, the function $\tilde{g}$ is $QSLU$-$\mathcal{EC}$ on $\mathcal{M}$.

□

**Theorem 3.18.** *Let $h_i : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$, for $i = 1, 2, \ldots, r$, be $QSLU$-$\mathcal{EC}$ functions on $\mathbb{R}^n$ relative to a map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$. Then, the set*

$$\mathcal{M} = \bigcap_{i=1}^{r} \{s \in \mathbb{R}^n : h_i(s) \leq_{LU} 0, \ i = 1, 2, \ldots, r\}$$

*is a $S\mathcal{EC}$ set.*

*Proof.* By Theorem 3.15, it is clear that the set

$$\mathcal{M}_i = \{s \in \mathbb{R}^n : h_i(s) \leq_{LU} 0\}$$

is $S\mathcal{EC}$, for each $i = 1, 2, \ldots, r$, thus, it follows that the set

$$\mathcal{M} = \bigcap_{i=1}^{r} \mathcal{M}_i$$

is a $S\mathcal{EC}$ set.

□

**Definition 3.19.** [21] Let $\mathcal{M}$ be an open subset of $\mathbb{R}$. An IVF $\tilde{g} : \mathcal{M} \to \mathcal{I}(\mathbb{R})$ with $\tilde{g}(s) = [\tilde{g}^L(s), \tilde{g}^U(s)]$ is considered *weakly differentiable* at $s_0$ if the functions $\tilde{g}^L$ and $\tilde{g}^U$, which are real-valued, are differentiable at $s_0$ in the usual sense.

**Theorem 3.20.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $\mathcal{SEC}$ set. If $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is a weakly differentiable $QSLU$-$\mathcal{EC}$ function on a $\mathcal{SEC}$ set $\mathcal{M}$ with $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(\mathcal{E}(t))$, then*

$$(\mathcal{E}(s) - \mathcal{E}(t))\nabla\tilde{g}(\mathcal{E}(t)) \leq_{LU} 0, \quad \forall\, s, t \in \mathcal{M}.$$

*Proof.* Since $\tilde{g}$ is a $QSLU$-$\mathcal{EC}$ function on the $\mathcal{SEC}$ set $\mathcal{M}$, for any $s, t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0, 1]$, we have

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}.$$

Given that $\tilde{g}(\mathcal{E}(s)) \leq_{LU} \tilde{g}(\mathcal{E}(t))$, this implies

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \tilde{g}(\mathcal{E}(t)),$$
$$\tilde{g}(\theta_1 t + \mathcal{E}(t)) + \theta_2(\theta_1 s + \mathcal{E}(s) - \theta_1 t - \mathcal{E}(t)) \leq_{LU} \tilde{g}(\mathcal{E}(t)).$$

Thus,

$$\Big[\tilde{g}^L(\theta_1 t + \mathcal{E}(t)) + \theta_2(\theta_1 s + \mathcal{E}(s) - \theta_1 t - \mathcal{E}(t)),$$
$$\tilde{g}^U(\theta_1 t + \mathcal{E}(t)) + \theta_2(\theta_1 s + \mathcal{E}(s) - \theta_1 t - \mathcal{E}(t))\Big] \leq_{LU} \Big[\tilde{g}^L(\mathcal{E}(t)), \tilde{g}^U(\mathcal{E}(t))\Big].$$

Since $\tilde{g}$ is weakly differentiable, we have

$$\tilde{g}^L(\theta_1 t + \mathcal{E}(t)) + \theta_2[\theta_1 s + \mathcal{E}(s) - \theta_1 t - \mathcal{E}(t)]\nabla\tilde{g}^L(\theta_1 t + \mathcal{E}(t)) + O(\theta_2^2) \leq \tilde{g}^L(\mathcal{E}(t)),$$
$$\tilde{g}^U(\theta_1 t + \mathcal{E}(t)) + \theta_2[\theta_1 s + \mathcal{E}(s) - \theta_1 t - \mathcal{E}(t)]\nabla\tilde{g}^U(\theta_1 t + \mathcal{E}(t)) + O(\theta_2^2) \leq \tilde{g}^U(\mathcal{E}(t)).$$

Taking the limit as $\theta_1 \to 0$ in the first equation, we get

$$\tilde{g}^L(\mathcal{E}(t)) + \theta_2[\mathcal{E}(s) - \mathcal{E}(t)]\nabla\tilde{g}^L(\mathcal{E}(t)) + O(\theta_2^2) \leq \tilde{g}^L(\mathcal{E}(t)).$$

Dividing by $\theta_2 > 0$ and taking $\theta_2 \to 0$, we get

$$(\mathcal{E}(s) - \mathcal{E}(t))\nabla\tilde{g}^L(\mathcal{E}(t)) \leq 0, \quad \forall s, t \in \mathcal{M}.$$

Similarly, from the second equation, we obtain

$$(\mathcal{E}(s) - \mathcal{E}(t))\nabla\tilde{g}^U(\mathcal{E}(t)) \leq 0, \quad \forall s, t \in \mathcal{M}.$$

Thus, we have

$$(\mathcal{E}(s) - \mathcal{E}(t))\nabla\tilde{g}(\mathcal{E}(t)) \leq_{LU} 0, \quad \forall s, t \in \mathcal{M}.$$

□

*A direct result derived from Theorem 3.20 can be stated as follows:*

**Corollary 3.21.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a $\mathcal{SEC}$ set. If IVF $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is weakly differentiable and $QSLU$-$\mathcal{EC}$ at $t \in \mathcal{M}$, where $v$ is a fixed point of the map $\mathcal{E}$, then*

$$(\mathcal{E}(s) - t)\nabla\tilde{g}(t) \leq_{LU} 0, \quad \forall s \in \mathcal{M}.$$

*We introduce the concept of a pseudo $SLU$-$\mathcal{EC}$ function over a $\mathcal{SEC}$ set as demonstrated below:*

**Definition 3.22.** An IVF $\tilde{g} : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is called $PSLU$-$\mathcal{EC}$ relative to a map $\mathcal{E}$ on a $\mathcal{SEC}$ set $\mathcal{M} \subseteq \mathbb{R}^n$ if $\exists$ a strictly positive function $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ s.t., for any $s, t \in \mathcal{M}$, if $\tilde{g}(\mathcal{E}(s)) <_{LU} \tilde{g}(\mathcal{E}(t))$, then

$$\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1 - \theta_2)(\theta_1 t + \mathcal{E}(t))) \leq_{LU} \tilde{g}(\mathcal{E}(t)) + \theta_2(\theta_2 - 1)P(\mathcal{E}(s), \mathcal{E}(t)),$$

for all $\theta_1, \theta_2 \in [0, 1]$.

For $\theta_1 = 0$, the function $\tilde{g}$ simplifies to a $PLU$-$\mathcal{EC}$ function. Additionally, if $\theta_1 = 0$ and $\mathcal{E}$ is the identity map, then $\tilde{g}$ reduces to $PLUC$ function.

*In the subsequent theorem, we explore the relationship between $SLU$-$\mathcal{E}C$ functions and $PSLU$-$\mathcal{E}C$ functions.*

**Theorem 3.23.** *An IVF $\tilde g : \mathcal{M} \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ that is $SLU$-$\mathcal{E}C$ on a $S\mathcal{E}C$ set $\mathcal{M}$ is also a $PSLU$-$\mathcal{E}C$ function on $\mathcal{M}$.*

*Proof.* Let $\tilde g(\mathcal{E}(s)) <_{LU} \tilde g(\mathcal{E}(t))$. Since $\tilde g$ is a $SLU$-$\mathcal{E}C$ function on a $S\mathcal{E}C$ set $\mathcal{M}$, $\forall\ s,t \in \mathcal{M}$ and $\theta_1, \theta_2 \in [0,1]$, we have

$$
\begin{aligned}
\tilde g(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) &\le_{LU} \theta_2 \tilde g\big(\mathcal{E}(s)\big) + (1-\theta_2)\tilde g\big(\mathcal{E}(t)\big)\\
&= \theta_2[\,\tilde g^L(\mathcal{E}(s)),\, \tilde g^U(\mathcal{E}(s))\,] + (1-\theta_2)[\,\tilde g^L(\mathcal{E}(t)),\, \tilde g^U(\mathcal{E}(t))\,]\\
&= [\theta_2 \tilde g^L(\mathcal{E}(s)) + (1-\theta_2)\tilde g^L(\mathcal{E}(t)), \theta_2 \tilde g^U(\mathcal{E}(s)) + (1-\theta_2)\tilde g^U(\mathcal{E}(t))]\\
&= [\tilde g^L(\mathcal{E}(t)), \tilde g^U(\mathcal{E}(t))] + [\theta_2(\tilde g^L(\mathcal{E}(s)) - \tilde g^L(\mathcal{E}(t))), \theta_2(\tilde g^U(\mathcal{E}(s)) - \tilde g^U(\mathcal{E}(t)))]\\
&= \tilde g(\mathcal{E}(t)) + \theta_2[\tilde g^L(\mathcal{E}(s)) - \tilde g^L(\mathcal{E}(t)), \tilde g^U(\mathcal{E}(s)) - \tilde g^U(\mathcal{E}(t))]\\
&\le_{LU} \tilde g\big(\mathcal{E}(t)\big) + \theta_2(1-\theta_2)[\tilde g^L(\mathcal{E}(s)) - \tilde g^L(\mathcal{E}(t)), \tilde g^U(\mathcal{E}(s)) - \tilde g^U(\mathcal{E}(t))]\\
&= \tilde g\big(\mathcal{E}(t)\big) + \theta_2(\theta_2 - 1)[\tilde g^U(\mathcal{E}(t)) - \tilde g^U(\mathcal{E}(s)), \tilde g^L(\mathcal{E}(t)) - \tilde g^L(\mathcal{E}(s))]\\
&= \tilde g\big(\mathcal{E}(t)\big) + \theta_2(\theta_2 - 1)P\big(\mathcal{E}(s), \mathcal{E}(t)\big).
\end{aligned}
$$

where $P(\mathcal{E}(s), \mathcal{E}(t)) = [\tilde g^U(\mathcal{E}(t)) - \tilde g^U(\mathcal{E}(s)), \tilde g^L(\mathcal{E}(t)) - \tilde g^L(\mathcal{E}(s))] >_{LU} 0$.

Hence, $\tilde g$ is a $PSLU$-$\mathcal{E}C$ function defined on the $S\mathcal{E}C$ set $\mathcal{M}$.

□

*Figure 1 below illustrates the relationships among various types of generalized convexities.*

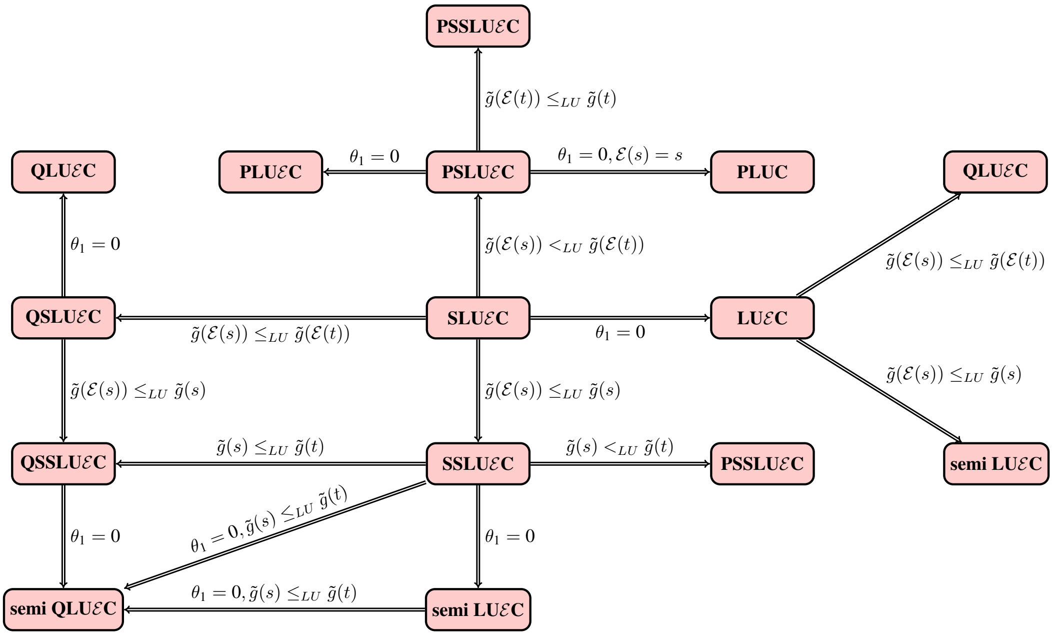

**Figure 1**

## 4 A nonlinear programming problem (NLPP) involving quasi-strongly $LU$-$\mathcal{E}C$ functions.

*In this section, we explore $QSLU$-$\mathcal{E}C$ programming problem, which extends the findings presented by [23] and [11].*

*Suppose that $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping, and let $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ and $\tilde{g}_j : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$, for $i = 1, 2, \ldots, r$ be QSLU-$\mathcal{E}$C functions on $\mathbb{R}^n$. The QSLU-$\mathcal{E}$C programming problem is defined as:*

$$(\mathcal{P}) \quad \min \tilde{g}(s)$$

*subjected to the constraints*

$$s \in \mathcal{M} = \{s \in \mathcal{E}(\mathbb{R}^n) : \tilde{g}_i(s) \leq_{LU} 0,\ i = 1, 2, \ldots, r\}.$$

**Theorem 4.1.** *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a S$\mathcal{E}$C set and $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ be a QSLU-$\mathcal{E}$C function . If $\mathcal{E}(s^*) \in \mathcal{E}(\mathcal{M})$ is a local minimum of ($\mathcal{P}$), then $\mathcal{E}(s^*)$ is a global minimum of ($\mathcal{P}$) on $\mathcal{M}$.*

*Proof.* Suppose that $\mathcal{E}(s^*) \in \mathcal{E}(\mathcal{M})$ is a nonglobal minimum of ($\mathcal{P}$) on $\mathcal{M}$. Then, $\exists\ \mathcal{E}(t) \in \mathcal{M}$ s.t. $\tilde{g}(\mathcal{E}(t)) \leq_{LU} \tilde{g}(\mathcal{E}(s^*))$. Given that the function $\tilde{g}$ is $QSLU$-$\mathcal{E}C$, then we get

$$\begin{aligned}\tilde{g}(\theta_2(\theta_1 t + \mathcal{E}(t)) + (1-\theta_2)(\theta_1 s^* + \mathcal{E}(s^*))) &\leq_{LU} \max\{\tilde{g}(\mathcal{E}(t)), \tilde{g}(\mathcal{E}(s^*))\}\\ &\leq_{LU} \tilde{g}(\mathcal{E}(s^*)).\end{aligned}$$

By setting $\theta_1 = 0$, we get

$$\tilde{g}(\theta_2\mathcal{E}(t) + (1-\theta_2)\mathcal{E}(s^*)) \leq_{LU} \tilde{g}(\mathcal{E}(s^*)).$$

For any sufficiently small $\theta_2 \in (0,1)$, contradicts the assumption that $\tilde{g}(\mathcal{E}(s^*))$ is a local minimum for ($\mathcal{P}$). Therefore, $\mathcal{E}(s^*)$ is a global minimum for ($\mathcal{P}$) over the set $\mathcal{M}$.

□

**Theorem 4.2.** *If an IVF $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is strictly QSLU-$\mathcal{E}$C function defined on S$\mathcal{E}$C set $\mathcal{M} \subseteq \mathbb{R}^n$, then ($\mathcal{P}$) has a unique global optimal solution.*

*Proof.* Suppose that $\mathcal{E}(s), \mathcal{E}(t) \in \mathcal{M}$ are two distinct global optimal solutions of ($\mathcal{P}$). Then $\tilde{g}(\mathcal{E}(s)) = \tilde{g}(\mathcal{E}(t))$. Since $\mathcal{M}$ is a $S\mathcal{E}C$ set and $\tilde{g}$ is a strictly $QSLU$-$\mathcal{E}C$ function,

$$\begin{aligned}\tilde{g}(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))) &<_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\}\\ &= \tilde{g}(\mathcal{E}(s)).\end{aligned}$$

For every $\theta_2 \in (0,1)$, conflicts the assumption that $\mathcal{E}(s)$ is an optimal solution for ($\mathcal{P}$). Consequently, it can be concluded that the global optimal solution to ($\mathcal{P}$) is unique, implying that there are no other solutions that achieve the same objective value within the feasible set. □

**Theorem 4.3.** *Suppose that $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is a QSLU-$\mathcal{E}$C function defined on a S$\mathcal{E}$C set $\mathcal{M} \subseteq \mathbb{R}^n$, and let $\Psi = \min\limits_{s\in\mathcal{M}} \tilde{g}(\mathcal{E}(s))$. Then, the set of optimal solutions to ($\mathcal{P}$), defined as $S = \{\mathcal{E}(s) \in \mathcal{M} : \tilde{g}(\mathcal{E}(s)) = \Psi\}$, is S$\mathcal{E}$C. Moreover, if $\tilde{g}$ is strictly QSLU-$\mathcal{E}$C function on the S$\mathcal{E}$C set $\mathcal{M}$, then the set $S$ contains one element.*

*Proof.* Let $\mathcal{E}(s), \mathcal{E}(t) \in \mathcal{M}$ be two distinct global optimal solutions to $\mathcal{P}$. Then, $\tilde{g}(\mathcal{E}(s)) = \Psi$ and $\tilde{g}(\mathcal{E}(t)) = \Psi$. Since $\tilde{g} : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is a $QSLU$-$\mathcal{E}C$ function defined on a $S\mathcal{E}C$ set $\mathcal{M} \subseteq \mathbb{R}^n$, we have

$$\tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) \leq_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\} = \Psi,$$

which implies that

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in S.$$

Therefore, $S$ is $S\mathcal{E}C$.

For the second part, suppose, for the sake of contradiction, that $\mathcal{E}(s), \mathcal{E}(t) \in S$ with $\mathcal{E}(s) \neq \mathcal{E}(t)$. For any $\theta_1 \in [0,1]$ and $\theta_2 \in (0,1)$, we have

$$\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t)) \in \mathcal{M}.$$

Since $\tilde{g}$ is strictly $QSLU$-$\mathcal{E}C$ function on $\mathcal{M}$, it follows that

$$\tilde{g}\left(\theta_2(\theta_1 s + \mathcal{E}(s)) + (1-\theta_2)(\theta_1 t + \mathcal{E}(t))\right) <_{LU} \max\{\tilde{g}(\mathcal{E}(s)), \tilde{g}(\mathcal{E}(t))\} = \Psi.$$

This contradicts the assumption that $\Psi = \min\limits_{s\in\mathcal{M}} \tilde{g}(\mathcal{E}(s))$. Hence, the desired result follows. □

## 5 Conclusion

*The concept of $SLU$-$\mathcal{E}$ convexity and $QSLU$-$\mathcal{E}$ convexity have been introduced and supported by various examples to clarify and demonstrate their applicability. This study delves into the characterization of such functions and examines their interrelationships in detail. Furthermore, NLPP involving $QSLU$-$\mathcal{E}C$ functions have been thoroughly investigated, with a focus on analyzing the existence and uniqueness of its global optimal solution. The findings presented in this work not only generalize and extend previously established results but also open new avenues for further research. In particular, the exploration of these concepts in the context of Riemannian manifolds holds significant promise for future studies, offering a broader scope for theoretical and practical advancements in optimization and related fields.*

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