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# SOME ALGEBRAIC PROPERTIES OF TRANSLATIONS ON $n$-ARY SEMIHYPERGROUPS

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**Abstract** Let $n$ be a natural integer such that $n \geq 2$. The notion of $n$-ary semihypergroups arose as a generalization of $n$-ary semigroups. Moreover, their algebraic hypersructures can also be a generalization of semihypergroups and ternary semihypergroups. This article investigates translations on $n$-ary semihypergroups, utilizing the algebraic connection between $n$-ary semigroups and $n$-ary semihypergroups. First, we introduce the notion of translations on $n$-ary semihypergroups. Then, we construct an $n$-ary semigroup of all multivalued full functions together with its $n$-ary operation. Finally, we investigate some results of the $n$-ary semigroups of all multivalued full functions via translations. Furthermore, we present the characterization of scalar zero elements and conditions under which all multivalued functions act as translations.

## 1 Introduction

Kasner is credited as the first mathematician who introduced the notion of $n$-ary algebraic structures which can be considered as a generalization of classical algebraic structures [19]. Since then, over the following decades until the present day, the $n$-ary structures were developed to construct a number of $n$-ary systems, which were studied in depth. In 1963, Sioson [28] established regular $n$-ary semigroups, an extension of the notion of regular semigroups, and also investigated their algebraic properties. Moreover, the same author also investigated some results of $k$-ideals on $n$-ary semigroups where $1 \leq k \leq n$ and $n \geq 2$. In 1980, Dudek and Grozdinska [17] investigated some interesting algebraic properties of regular $n$-ary semigroups for $n \geq 3$. Dudek proved some algebraic results on $n$-ary semigroups and $n$-ary groups, and also gave many examples of them, see [14, 15, 16, 18]. Simuen et al. [27] extended the concept of $k$-ideals in $n$-ary semigroups to ordered $n$-ary semigroups. Recently, regular $n$-ary semigroups were described via idealistic soft $n$-ary semigroups by Wang et al. [26]. For the special case $n = 3$, the authors in [24] gave representations of ternary semigroups and discussed their related properties. Nowadays, there is the so-called Menger algebras of rank $n$ which can be regarded as another kind of generalization of semigroups. The structures are reduced to semigroups when $n = 1$. By using the idea of Menger algebras of rank $n$, the authors first established the concept of ternary Menger algebras of rank $n$, which can be regarded as a generalization of ternary semigroups. The structures are reduced to ternary semigroups when $n = 1$. Moreover, the authors introduced the so-called $v$-regular ternary Menger algebras and studied some results, see [25]. Menger algebras of rank $n$ and related topics have been developed in depth and in different contexts, see [13, 21, 23].

Based on the knowledge of classical algebraic structures, the French mathematician Marty is credited as the first researcher who established the so-called *hypergroups*, see [20]. In classical algebraic structures, the operation of two elements (or $n$ elements) is again an element. However, in algebraic hyperstructures, the composition of two elements (or $n$ elements) is not an element, but it is a nonempty set. As a result, the notion of algebraic hyperstructures is considered as a novel extension of classical algebraic structures. The algebraic properties of hyperstructures have been studied in both theoretical results and applications. In 2006, by using the notion of hypergroups, Davvaz and Vougiouklis [9] established the so-called *$n$-ary hypergroups* which are a generalization of hypergroups. Moreover, the mentioned hyperalgebras can be regarded as a natural generalization of $n$-ary groups. In this work, the authors also investigated their algebraic properties via the concept of relations. In 2008, Leoreanu-Fotea and Davvaz [11] published the paper in which several properties related to the ideas of rough sets on $n$-ary hypergroups were studied . In 2009, Davvaz et al. [8] presented a novel class of hyperstructures, called *$n$-ary semihypergroups*. These hyperalgebras can be represented as a generalization of semihypergroups and ternary semihypergroups. In the same year, Davvaz et al. [10] also discussed their algebraic properties, such as the quotient structures, the neutral elements and the fundamental relations. Recently, Daengsaen et al. [5] presented the idea of $k$-(0-)simple $n$-ary semihypergroups and investigated their minimal and maximal hyperideals, where $1 \leq k \leq n$. In 2023, Daengsaen and Leeratanavalee [4] extended the well-known results on semigroups, the so-called Green's relations to ordered $n$-ary semihypergroups. The authors also discussed many interesting results of ordered $n$-ary semihypergroups, see [6, 7]. Nowadays, there are the real significance or applications of $n$-ary hyperstructures. In 2018, Al-Tahanand and Davvaz [3] studied on weak chemical hyperstructures associated to electrochemical cells. In 2019, the same authors introduced a new application of $n$-ary weak hyperstructures in Chemistry, see [2]. Recently, the same group of authors studied $n$-ary (semi)hypergroups and subsets of these hypergroups. See [1].

Motivated by the previously important results and the recently works, in the present article, we aim to investigate some algebraic properties of translations on $n$-ary semihypergroups. Firstly, we start by introducing the algebraic structure of $n$-ary semigroups of all multivalued full functions. Then, we present the notions of translations and some special elements on $n$-ary semihypergroups. Finally, we determine some algebraic properties of $n$-ary semihypergroups via translations.

## 2 Preliminaries

On a nonempty set $S$, we denote the set of all nonempty subsets of $S$ by $P^*(S)$. A mapping $f$ from $S \times \cdots \times S$ into $P^*(S)$, where $S$ appears $n$ times and $n \geq 2$, is called an *$n$-ary hyperoperation*. A pair $(S, f)$, which consists of a nonempty set $S$ and an $n$-ary hyperoperation $f$, is called an *$n$-ary hypergroupoid*.

In this article, a sequence of elements $x_k, x_{k+1}, \ldots, x_l$ of the base set $S$ of $(S, f)$ is denoted by the abbreviated symbol $x_k^l$. In the case $l < k$, the previous symbol $x_k^l$ is the empty symbol. Next, we use the shortened symbol $f(x_1^n)$ in place of $f(x_1, x_2, \ldots, x_n)$. In addition, we use $f(x_1^k, y_{k+1}^l, z_{l+1}^n)$ in place of $f(x_1, \ldots, x_k, y_{k+1}, \ldots, y_l, z_{l+1}, \ldots, z_n)$. If $x_1 = \cdots = x_k = x$ and $z_{l+1} = \cdots = z_n = z$, then we use $f(x^k, y_{k+1}^l, z^{n-l})$ instead of the previously mentioned expression. Analogously, we also use the shortened symbols for any abbreviated symbols of all sequences of nonempty subsets of $S$.

For $T_1^n \in P^*(S)$, we define

$$f(T_1^n) = \bigcup_{x_k \in T_k, k \in \{1,2,\ldots,n\}} f(x_1^n).$$

In case $T_n = \{x\}$, the symbol $f(T_1^{n-1}, \{x\})$ is written as $f(T_1^{n-1}, x)$ and similarly in other cases.

In $n$-ary hypergroupoid $(S, f)$, an $n$-ary hyperoperation $f$ is called $(i, j)$-*associative*, where $i, j \in \{1, 2, \ldots, n\}$, if $f$ satisfies the $(i, j)$-*associative law* given as (2.1) below:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{2.1}$$

for all $x_1^{2n-1} \in S$. If the $n$-ary hyperoperation $f$ satisfies (2.1) for all $i, j \in \{1, 2, \ldots, n\}$, then it is called *associative*. An $n$-ary hypergroupoid $(S, f)$ is called an $n$-ary semihypergroup (or an *$n$-ary hypersemigroup* [12]) if an $n$-ary hyperoperation $f$ is associative on $S$.

According to the definition of $n$-ary semihypergroups, the concept of $n$-ary semihypergroups can be considered as a natural generalization of semihypergroups and ternary semihypergroups.

**Example 2.1.** Several examples of $n$-ary semihypergroups are already provided in [22].

(i) Let $\mathbb{N}$ be the set of all natural numbers. Define an $n$-ary hyperoperation $f$ on $\mathbb{N}$ by

$$f(x_1^n) = \{m \in \mathbb{N} \mid m = \max\{x_1^n\}\}$$

for all $x_1^n \in \mathbb{N}$. Then, $(\mathbb{N}, f)$ forms an $n$-ary semihypergroup.

(ii) Let $(S, f)$ be an $n$-ary semigroup. Define an $n$-ary hyperoperation $f_A$ on $S$ by

$$f_A(x_1^n) = \begin{cases} f(f(x_1, A, x_2, A, \ldots, x_{\frac{n}{2}}, A), x_{\frac{n}{2}+1}, A, \ldots, x_{n-1}, A, x_n) & \text{if } n \text{ is even,} \\ f(f(x_1, A, x_2, A, \ldots, x_{\frac{n+1}{2}}), A, x_{\frac{n+1}{2}+1}, \ldots, x_{n-1}, A, x_n) & \text{if } n \text{ is odd.} \end{cases}$$

Then, $(S, f_A)$ forms an $n$-ary semihypergroup.

**Definition 2.2.** Let $(S, f)$ be an $n$-ary semihypergroup. An element $0 \in S$ is called

(i) a *left scalar zero element* if $f(0, x_2^n) = \{0\}$ for all $x_2^n \in S$;

(ii) a *right scalar zero element* if $f(x_1^{n-1}, 0) = \{0\}$ for all $x_1^{n-1} \in S$;

(iii) a *$k$-scalar zero element*, where $k \in \{1, 2, \ldots, n\}$, if $f(x_1^{k-1}, 0, x_{k+1}^n) = \{0\}$ for all $x_1^n \in S$;

(iv) a *scalar zero element* if $f(x_1^{k-1}, 0, x_{k+1}^n) = \{0\}$ for all $k \in \{1, 2, \ldots, n\}$ and $x_1^n \in S$.

Note that, on an $n$-ary semihypergroup $(S, f)$, it is easy to see that a 1-scalar zero element and an $n$-scalar zero element are a left scalar zero element and a right scalar zero element, respectively. Furthermore, if each element of $S$ can be acted as a left (resp. right, $k$-) scalar zero element on $(S, f)$, then $(S, f)$ is said to be a left *(resp. right, $k$-) scalar zero $n$-ary semihypergroup*. In addition, Definition 2.2 can be considered as a natural generalization of zero elements on semihypergroups and ternary semihypergroups.

## 3 Translations on $n$-ary semihypergroups

In this section, we introduce the idea of translations on $n$-ary semihypergroups which can be considered as a generalization of translations on semihypergroups and ternary semihypergroups. Then, we construct $n$-ary semihypergroups of all multivalued full functions. Moreover, we also investigate their interesting algebraic properties related to the concept of translations on $n$-ary semihypergroups.

Let $X$ be a nonempty set. On the set $\mathcal{T}(X, P^*(X))$ of all *multivalued full functions* (or 1*-ary hyperoperations*) $\beta : X \longrightarrow P^*(X)$, we define an $n$-ary operation $\odot : \mathcal{T}(X, P^*(X))^n \longrightarrow \mathcal{T}(X, P^*(X))$ as follows:

$$\odot(\beta_1^n)(x) = \beta_1(\beta_2(\ldots\beta_n(x)\ldots)) \quad \text{for all } \beta_1^n \in \mathcal{T}(X, P^*(X)), x \in X. \tag{3.1}$$

According to the definition of the $n$-ary hyperoperation $\odot$ defined in (3.1), the composition $\odot(\beta_1^n)$ is a right-to-left order operation. Moreover, it is also a multivalued full function from $X$ to $P^*(X)$. So, it belongs to $\mathcal{T}(X, P^*(X))$. Next, we determine some of its algebraic properties.

**Lemma 3.1.** *Let $X$ be a nonempty set. Then, the $n$-ary hyperoperation $\odot$ defined as in (3.1) is associative on the set $\mathcal{T}(X, P^*(X))$ of all multivalued full functions that is,*

$$\odot(\beta_1^{i-1}, \odot(\beta_i^{n+i-1}), \beta_{n+i}^{2n-1})(x) = \odot(\beta_1^{j-1}, \odot(\beta_j^{n+j-1}), \beta_{n+j}^{2n-1})(x)$$

*for all $\beta_1^{2n-1} \in \mathcal{T}(X, P^*(X)), x \in X$.*

*Proof.* Let $\beta_1^{2n-1} \in \mathcal{T}(X, P^*(X))$ and $x \in X$. Indeed, for each $x \in X$, we have

$$\begin{aligned}
\odot(\odot(\beta_1^n), \beta_{n+1}^{2n-1})(x) &= \odot(\beta_1^n)(\beta_{n+1}(\beta_{n+2}(\ldots\beta_n(x))\ldots)) \\
&= \bigcup_{y_{n+1}\in\beta_{n+1}(\beta_{n+2}(\ldots\beta_{2n-1}(x)\ldots))} \odot(\beta_1^n)(y_{n+1}) \\
&= \bigcup_{y_{n+1}\in\beta_{n+1}(\beta_{n+2}(\ldots\beta_{2n-1}(x)\ldots))} \beta_1(\beta_2(\ldots\beta_n(y_{n+1})\ldots)) \\
&= \bigcup_{y_2\in\beta_2(\beta_3(\ldots\beta_n(y_{n+1})\ldots)), y_{n+1}\in\beta_{n+1}(\beta_{n+2}(\ldots\beta_{2n-1}(x)\ldots))} \beta_1(y_2) \\
&= \bigcup_{y_2\in\beta_2(\beta_3(\ldots\beta_{2n-1}(x)\ldots))} \beta_1(y_2) \\
&= \bigcup_{y_2\in\odot(\beta_2^{n+1})(\beta_{n+2}(\beta_{n+3}(\ldots\beta_{2n-1}(x)\ldots)))} \beta_1(y_2) \\
&= \beta_1(\odot(\beta_2^{n+1})(\beta_{n+2}(\beta_{n+3}(\ldots\beta_{2n-1}(x)\ldots)))) \\
&= \odot(\beta_1, \odot(\beta_2^{n+1}), \beta_{n+2}^{2n-1})(x).
\end{aligned}$$

It implies that $\odot$ is $(1,2)$-associative. Continuing in this way with a similar argument, we can show the rest. Then, $\odot$ is $(i,j)$-associative for all $i, j \in \{1, 2, \ldots, n\}$ and hence, $\odot$ is associative on the set $\mathcal{T}(X, P^*(X))$. □

**Theorem 3.2.** *Let $(S, f)$ be an $n$-ary semihypergroup. Then, the $n$-ary algebraic structure $\mathcal{T}(S, P^*(S))$ together with the $n$-ary operation $\odot$ defined in (3.1) forms an $n$-ary semigroup.*
.

*Proof.* The proof follows from Lemma 3.1. □

The $n$-ary semigroup $(\mathcal{T}(S, P^*(S)), \odot)$ constructed in Theorem 3.2 is called *an $n$-ary semigroup of all multivalued full functions*. Next, we extend the concept of translations on semihypergroups to $n$-ary semigroups.

**Definition 3.3.** Let $(S, f)$ be an $n$-ary semihypergroup. A multivalued full function $\mu : S \longrightarrow P^*(S)$ is said to be:

(i) a left translation if

$$\mu(f(x_1^n)) = f(\mu(x_1), x_2^n) \quad \text{for all } x_1^n \in S;$$

(ii) a right translation if

$$\mu(f(x_1^n)) = f(x_1^{n-1}, \mu(x_n)) \quad \text{for all } x_1^n \in S;$$

(iii) a $k$-translation, where $k \in \{1, 2, \ldots, n\}$, if

$$\mu(f(x_1^n)) = f(x_1^{k-1}, \mu(x_k), x_{k+1}^n) \quad \text{for all } x_1^n \in S.$$

**Example 3.4.** Let $(S, f)$ be an $n$-ary semihypergroup. Define a multivalued full function $\mu : S \longrightarrow P^*(S)$ by $\mu(x) = \{x\}$ for all $x \in S$. Indeed, for each $x_1^n \in S$, we have

$$\begin{aligned}
\mu(f(x_1^n)) &= f(x_1^n) \\
&= \bigcup_{y\in\{x_k\}} f(x_1^{k-1}, y, x_{k+1}^n) \\
&= \bigcup_{y\in\mu(x_k)} f(x_1^{k-1}, y, x_{k+1}^n) \\
&= f(x_1^{k-1}, \mu(x_k), x_{k+1}^n).
\end{aligned}$$

Therefore, $\mu$ is a $k$-translation on $S$.

**Proposition 3.5.** *Let $(S, f)$ be an $n$-ary semihypergroup. A $k$-translation $\mu$, where $k \in \{1, 2, \ldots, n-1\}$, map a right scalar zero element 0 (if it exists) to itself, that is, $\mu(0) = \{0\}$.*

*Proof.* Let $\mu$ be a $k$-translation on the base set $S$ of an $n$-ary semihypergroup $(S, f)$. Let 0 be a right scalar zero element. Then, $f(x_1^{n-1}, 0) = \{0\}$ for all $x_1^{n-1} \in S$. Consider,

$$\begin{aligned}\mu(0) &= \mu(\{0\}) \\ &= \mu(f(x_1^{n-1}, 0)) \\ &= f(x_1^{k-1}, \mu(x_k), x_{k+1}^{n-1}, 0) \\ &= \bigcup_{y \in \mu(x_k)} f(x_1^{k-1}, y, x_{k+1}^{n-1}, 0) \\ &= \{0\}\end{aligned}$$

for all $k \in \{1, 2, \ldots, n-1\}$. □

**Proposition 3.6.** *Let $(S, f)$ be an $n$-ary semihypergroup. Then, a right translation $\mu$ maps a left scalar zero element 0 (if it exists) to itself, that is, $\mu(0) = \{0\}$.*

*Proof.* The proof is similar to the proof of Proposition 3.5. □

**Theorem 3.7.** *If $(S, f)$ is an $n$-ary semihypergroup together with an $n$-ary hyperoperation $f$ defined by*

$$f(x_1^n) = \{y\} \quad \text{for all } x_1^n \in S,$$

*where $y \in S$ is a fixed element, then each multivalued full function $\mu : S \longrightarrow P^*(S)$ satisfying $\mu(y) = \{y\}$ is a $k$-translation for all $k \in \{1, 2, \ldots, n\}$. Moreover, the converse is also true.*

*Proof.* Firstly, we assume that the hypothesis holds. By the assumption, $f(x_1^n) = \{y\}$ for all $x_1^n \in S$, where $y \in S$ is a fixed element. Let $\mu$ be a multivalued full function such that $\mu(y) = \{y\}$. Indeed, for each $x_1^n \in S$, we obtain that

$$\begin{aligned}\mu(f(x_1^n)) &= \mu(\{y\}) \\ &= \mu(y) \\ &= \{y\} \\ &= \bigcup_{z \in \mu(x_k)} f(x_1^{k-1}, z, x_{k+1}^n)\end{aligned}$$

holds for all $k \in \{1, 2, \ldots, n\}$. This yields that $\mu$ is a $k$-translation on $S$ for all $k \in \{1, 2, \ldots, n\}$.

Conversely, we assume that each multivalued full function $\mu$ on $S$ such that $\mu(y) = \{y\}$ is a $k$-translation for all $k \in \{1, 2, \ldots, n\}$. Next, suppose that there are $x_1^n, z \in S$ such that $y \neq z$ and $f(x_1^n) = \{z\}$. Choose a multivalued full function $\delta$ on $S$ satisfying the following conditions:

$$\delta(y) = \{y\}, \delta(z) = \{y\} \text{ and } \delta(x_1) = \{x_1\}.$$

Then by assumption, $\delta$ is a $k$-translation. So, $\delta$ is also a left translation on $S$. Consider,

$$\begin{aligned}\{y\} &= \delta(z) \\ &= \delta(\{z\}) \\ &= \delta(f(x_1^n)) \\ &= f(\delta(x_1), x_2^n) \\ &= f(x_1^n).\end{aligned}$$

It follows that $f(x_1^n) = \{y\}$, which contradicts the assumption that $f(x_1^n) = \{y\} \neq \{z\}$. Consequently, $(S, f)$ is an $n$-ary semihypergroup such that an $n$-ary hyperoperation $f$ satisfying $f(x_1^n) = \{y\}$ for all $x_1^n \in S$. □

On an $n$-ary semihypergroup $(S, f)$, we construct a set of all left (resp. right, $k$-) translations defined on the base set $S$, and denote by $\Lambda(S)$ (resp. $\Gamma(S), \Delta_k(S)$), that is,

$$\Lambda(S) = \{\lambda \in \mathcal{T}(S, P^*(S)) \mid \lambda \text{ is a left translation on } S\},$$
$$\Gamma(S) = \{\gamma \in \mathcal{T}(S, P^*(S)) \mid \gamma \text{ is a right translation on } S\},$$
$$\Delta_k(S) = \{\delta \in \mathcal{T}(S, P^*(S)) \mid \delta \text{ is a } k\text{-translation on } S\}.$$

Now, we investigate some algebraic connections between the sets $\Lambda(S)$ (resp. $\Gamma(S), \Delta_k(S)$) and $\mathcal{T}(S, P^*(S))$.

**Theorem 3.8.** *Let $(S, f)$ be an $n$-ary semihypergroup. Then, $\Delta_k(S) = \mathcal{T}(S, P^*(S))$ if and only if $(S, f)$ is a scalar zero $n$-ary semihypergroup.*

*Proof.* ($\Longrightarrow$) Firstly, we assume that $\Delta_k(S) = \mathcal{T}(S, P^*(S))$. Suppose that $(S, f)$ is not a scalar zero $n$-ary semihypergroup. This means that there are $m \in \{1, 2, \ldots, n\}$ and an element $z \in S$ such that

$$f(x_1^{m-1}, z, x_{m+1}^n) \neq \{z\} \quad \text{for some } x_1^{m-1}, x_{m+1}^n \in S.$$

Secondly, we choose a multivalued full function $\delta \in \mathcal{T}(S, P^*(S))$ satisfying the following conditions:

$$\delta(z) = \{z\} \text{ and } \delta(a) = \{z\} \quad \text{for all } a \in f(x_1^{m-1}, z, x_{m+1}^n).$$

By assumption, $\delta$ is also an $m$-translation on $S$. Consider,

$$\begin{aligned} \{z\} &= \bigcup_{a \in f(x_1^{m-1}, z, x_{m+1}^n)} \delta(a) \\ &= \delta(f(x_1^{m-1}, z, x_{m+1}^n)) \\ &= f(x_1^{m-1}, \delta(z), x_{m+1}^n) \\ &= f(x_1^{m-1}, z, x_{m+1}^n). \end{aligned}$$

This implies that $f(x_1^{m-1}, z, x_{m+1}^n) = \{z\}$ which is a contradiction with $f(x_1^{m-1}, z, x_{m+1}^n) \neq \{z\}$. Therefore, $(S, f)$ forms a scalar zero $n$-ary semihypergroup.

($\Longleftarrow$) Assume that $(S, f)$ is a scalar zero $n$-ary semihypergroup. Then, for every $x_k \in S$ such that $k \in \{1, 2, \ldots, n\}$, we have the following property:

$$f(x_1^{k-1}, x_k, x_{k+1}^n) = \{x_k\} \quad \text{for all } x_1^{k-1}, x_k, x_{k+1}^n \in S.$$

Since $\Delta_k(S) \subseteq \mathcal{T}(S, P^*(S))$, we only show that $\mathcal{T}(S, P^*(S)) \subseteq \Delta_k(S)$. Let $\delta \in \mathcal{T}(S, P^*(S))$. Indeed, for each $x_1^n \in S$, we have

$$\begin{aligned} \delta(f(x_1^{k-1}, x_k, x_{k+1}^n)) &= \delta(\{x_k\}) \\ &= \delta(x_k) \\ &= \bigcup_{y \in \delta(x_k)} \{y\} \\ &= \bigcup_{y \in \delta(x_k)} f(x_1^{k-1}, y, x_{k+1}^n) \\ &= f(x_1^{k-1}, \delta(x_k), x_{k+1}^n). \end{aligned}$$

This implies that $\delta$ is a $k$-translation on $S$, and hence, $\delta \in \Delta_k(S)$. Therefore, $\mathcal{T}(S, P^*(S)) \subseteq \Delta_k(S)$. Consequently, we conclude that $\Delta_k(S) = \mathcal{T}(S, P^*(S))$. $\square$

**Corollary 3.9.** *Let $(S, f)$ be an $n$-ary semihypergroup. Then,*

*(i)* $\Lambda(S) = \mathcal{T}(S, P^*(S))$ *if and only if $(S, f)$ is a left scalar zero $n$-ary semihypergroup;*

*(ii)* $\Gamma(S) = \mathcal{T}(S, P^*(S))$ *if and only if $(S, f)$ is a right scalar zero $n$-ary semihypergroup.*

Let $(S, f)$ be an $n$-ary semihypergroup and $\delta$ be a $k$-translation on $S$, where $k \in \{1, 2, \ldots, n\}$. Define an $n$-ary hyperoperation $f_\delta$ on $S$ as follows:

$$f_\delta(x_1^n) = f(\delta(x_1), \delta(x_2), \ldots, \delta(x_{k-1}), x_k, \delta(x_{k+1}), \ldots, \delta(x_n)) \quad \text{for all } x_1^n \in S. \tag{3.2}$$

In particular, the right-hand side means that

$$\begin{aligned} f(\delta(x_1), \delta(x_2), \ldots, \delta(x_{k-1}),&x_k, \delta(x_{k+1}), \ldots, \delta(x_n)) \\ &= \bigcup_{y_m \in \delta(x_m), m \in \{1,2,\ldots,k-1,k+1,\ldots,n\}} f(y_1^{k-1}, x_k, y_{k+1}^n). \end{aligned}$$

Now, the $n$-ary hyperoperation $f_\delta$ on $S$ is called an *$n$-ary hyperoperation via a $k$-translation* $\delta$.

Next, let $\lambda$ and $\gamma$ be a left translation and a right translation on $S$, respectively. By using the definition of the $n$-ary hyperoperation $f_\delta$ via a $k$-translation $\delta$, we immediately have other $n$-ary hyperoperations $f_\lambda$ and $f_\gamma$ on $S$, which are defined by

$$f_\lambda(x_1^n) = f(x_1, \lambda(x_2), \lambda(x_3), \ldots, \lambda(x_n)) \quad \text{for all } x_1^n \in S;$$
$$f_\gamma(x_1^n) = f(\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_{n-1}), x_n) \quad \text{for all } x_1^n \in S.$$

**Lemma 3.10.** *Let $(S, f)$ be an $n$-ary semihypergroup and $\delta$ be a $k$-translation on $S$, where $k \in \{1, 2, \ldots, n\}$. Then,*

$$\delta(f(\delta(x_1), \delta(x_2), \ldots, \delta(x_{k-1}), x_k, \delta(x_{k+1}), \ldots, \delta(x_n))) = f(\delta(x_1), \delta(x_2), \ldots, \delta(x_n))$$

*for all $x_1^n \in S$.*

*Proof.* Indeed, for each $x_1^n \in S$, we get

$$\begin{aligned} \delta(f(\delta(x_1),&\delta(x_2), \ldots, \delta(x_{k-1}), x_k, \delta(x_{k+1}), \ldots, \delta(x_n))) \\ &= \delta\left( \bigcup_{y_m \in \delta(x_m), m \in \{1,2,\ldots,k-1,k+1,\ldots,n\}} f(y_1^{k-1}, x_k, y_{k+1}^n) \right) \\ &= \bigcup_{y_m \in \delta(x_m), m \in \{1,2,\ldots,k-1,k+1,\ldots,n\}} \delta(f(y_1^{k-1}, x_k, y_{k+1}^n)) \\ &= \bigcup_{y_m \in \delta(x_m), m \in \{1,2,\ldots,n\}} f(y_1^n) \\ &= f(\delta(x_1), \delta(x_2), \ldots, \delta(x_n)). \end{aligned}$$

The proof is complete. □

**Corollary 3.11.** *Let $(S, f)$ be an $n$-ary semihypergroup. Let $\lambda$ and $\gamma$ be a left translation and a right translation on $S$, respectively. Then, the following assertions hold:*

*(i)* $\lambda(f(x_1, \lambda(x_2), \lambda(x_3), \ldots, \lambda(x_n))) = f(\lambda(x_1), \lambda(x_2), \ldots, \lambda(x_n))$ *for all $x_1^n \in S$;*

*(ii)* $\gamma(f(\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_{n-1}), x_n)) = f(\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_n))$ *for all $x_1^n \in S$.*

**Theorem 3.12.** *Let $(S, f)$ be an $n$-ary semihypergroup and $\lambda$ be a left translation on $S$. Then, an $n$-ary algebraic hyperstructure $(S, f_\lambda)$ forms an $n$-ary semihypergroup under the $n$-ary hyperoperation $f_\lambda$ via the left translation $\lambda$ on $S$.*

*Proof.* The base set $S$ is clearly closed under the $n$-ary hyperoperation $f_\lambda$ via a left translation $\lambda$ on $S$. By Corollary 3.11 (i), and the $(1, 2)$-associativity of $f$ on $S$, we get

$$\begin{aligned} f_\lambda&(f_\lambda(x_1^n), x_{n+1}^{2n+1}) \\ &= f_\lambda(f(x_1, \lambda(x_2), \lambda(x_3), \ldots, \lambda(x_n)), x_{n+1}^{2n+1}) \\ &= f(f(x_1, \lambda(x_2), \lambda(x_3), \ldots, \lambda(x_n)), \lambda(x_{n+1}), \lambda(x_{n+2}), \ldots, \lambda(x_{2n-1})) \\ &= f(x_1, f(\lambda(x_2), \lambda(x_3), \ldots, \lambda(x_{n+1})), \lambda(x_{n+2}), \ldots, \lambda(x_{2n-1})) \\ &= f(x_1, \lambda(f_\lambda(x_2^{n+1})), \lambda(x_{n+2}), \ldots, \lambda(x_{2n-1})) \\ &= f_\lambda(x_1, f_\lambda(x_2^{n+1}), x_{n+2}^{2n-1}) \end{aligned}$$

for all $x_1^{2n-1} \in S$. This means that the $n$-ary hyperoperation $f_\lambda$ is $(1,2)$-associative on $S$. Continuing in this way by using Corollary 3.11 (i), and the $(i,j)$-associativity of $f$ on $S$, where $i,j \in \{1,2,\ldots,n\}$, we can show the rest. Then, $f_\lambda$ is associative. Consequently, $(S, f_\lambda)$ forms an $n$-ary semihypergroups under the $n$-ary hyperoperation $f_\lambda$ via the left translation $\lambda$ on $S$. □

**Theorem 3.13.** *Let $(S, f)$ be an $n$-ary semihypergroup and $\gamma$ be a right translation on $S$. Then, an $n$-ary algebraic hyperstructure $(S, f_\gamma)$ forms an $n$-ary semihypergroup under the $n$-ary hyperoperation $f_\gamma$ via the right translation $\gamma$ on $S$.*

*Proof.* The proof is similar to the proof of Theorem 3.12 which follows from Corollary 3.11 (ii) and the associativity of the $n$-ary hyperoperation $f$ on $S$. □

## 4 Concluding remarks

In this work, we introduced the concept of left (resp. right, $k$-) translations on $n$-ary semihypergroups. In order to get our main results, we constructed the $n$-ary semihypergroups of all multivalued full functions (or 1-ary hyperoperations) and their $n$-ary hyperoperations. Finally, we used the concept of left (resp. right, $k$-) scalar zero elements to show that the set of all left (resp. right, $k$-) translations and the set of all multivalued functions are the same. Based on the results of left (right, $k$-) translations on $n$-ary semihypergroups in this article, there is an interesting research question for the future work. Can we extend the significance results on semigroup theory, the so-called Cayley's theorem to $n$-ary semihypergroups.

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## References


[1] M. Al-Tahan and B. Davvaz, *A Study on ordered $n$-ary (semi)hypergroups and their subsets*, Indian J Pure Appl Math(2025). https://doi.org/10.1007/s13226-025-00854-x

[2] M. Al-Tahan and B. Davvaz, *Electrochemical cells as experimental verifications of $n$-ary hyperstructures*, MATEMATIKA, **35**, 13–14, (2019).

[3] M. Al-Tahan and B. Davvaz, *Weak chemical hyperstructures associated to electrochemical cells*, Iranian J. Math. Chem., **9**, 65–75, (2018).

[4] J. Daengsaen and S. Leeratanavalee, *Green's relations on ordered $n$-ary semihypergroups*, Categories Gen. Algebraic Struct. with Appl., **19**, 143–167, (2023).

[5] J. Daengsaen, S. Leeratanavalee, and B. Davvaz, *On minimal and maximal hyperideals in $n$-ary semihypergroups*, Mathematics, **8**, Article ID: 1656, (2023).

[6] J. Daengsaen and S. Leeratanavalee, *Regularities in ordered $n$-ary semihypergroups*, Mathematics, **9**, Article ID: 1857, (2021).

[7] J. Daengsaen and S. Leeratanavalee, *Semilattice strongly regular relations on ordered $n$-ary semihypergroups*, AIMS Mathematics, **7**, 478–498, (2022).

[8] B. Davvaz, W.A. Dudek and T. Vougiouklis, *A generalization of $n$-ary algebraic systems*, Commun. Algebra, **37**, 1248–1263, (2009).

[9] B. Davvaz and T. Vougiouklis, *$n$-Ary hypergroups*, Iran. J. Sci. Technol. Trans. A: Sci., **30**, 165–174, (2006) .

[10] B. Davvaz, W.A. Dudek and S. Mirvakili, *Neutral elements, fundamental relations and $n$-ary hypersemigroups*, Internat. J. Algebra Comput., **19**, 567–583, (2009).

[11] V. Leoreanu-Fotea and B. Davvaz, *Roughness in $n$-ary hypergroups*, Inf. Sci., **178**, 4114–4124, (2008).

[12] B. Davvaz and T. Vougiouklis, *$n$-Ary hypergroups*, Iran. J. Sci. Technol., **30**, 165–174, (2006).

[13] K. Denecke, *Menger algebras and clones of terms*, East-West J. Math., **5**, 179–193, (2003).

[14] W.A. Dudek, *Remarks on $n$-semigroups*, Demonstration Math., **13**, 165–181, (1980).

[15] W.A. Dudek, *Autodistributive $n$-groups*, Comment. Math., **23**, 1–11, (1983) .

[16] W.A. Dudek, *On $(i,j)$-associative $n$-groupoids with the non-empty center*, Ricer. Math. (Napoli), **35**, 105–111, (1986).

[17] W.A. Dudek, I. Grozdinska, *On ideals in regular $n$-semigroups*, Math. Bilten Skopje, **4**, 25–44, (1980).

[18] W.A. Dudek, V.V. Mukhin, *On $n$-ary semigroups with adjoint neutral element*, Quasigroups Relat. Syst., **14**, 163–168, (2006).

[19] E. Kasner, *An extension of the group concepts (reported by L.G.Weld)*, Bull. Amer. Math. Soc., **10**, 290–291, (1904).

[20] F. Marty, *Sur une generalization de la notion de groups* In 8th congress Math. Scandinaves, Stockholm, (1934).

[21] K. Menger, *The algebra of functions: past, present, future*, Rend. Math., **20**, 409–490, (1964).

[22] A. Nongmanee, S. Leeratanavalee, *A study on regularity and reduction for $n$-ary semihypergroups*, Asian-Eur. J. Math., **17**, Article ID: 2450031, (2024).

[23] A. Nongmanee, S. Leeratanavalee, *Algebraic connections between Mernger algebras and Menger hyperalgebras via regularity*, Algebra Discret. Math., **36**, 61–73, (2023).

[24] A. Nongmanee, S. Leeratanavalee, *Ternary Menger algebras: a generalization of ternary semigroups*, Mathematics, **9**, Article ID: 553, (2021).

[25] A. Nongmanee, S. Leeratanavalee, *$v$-Regular ternary Menger algebras and left translations of ternary Menger algebras*, Mathematics, **9**, Article ID: 2691, (2021).

[26] Q. Wang, X. Zhou and J. Zhan, *A novel study of soft sets over $n$-ary semigroups*, Ital. J. Pure Appl. Math., **37**, 583–594, (2017).

[27] A. Simuen, P. Petchkaew, R. Chinramy, *Minimal and maximal ordered $n$-ideals in ordered $n$-ary semigroups*, Thai J. Math.,, 117–125, (2021).

[28] F.M. Siosan, *On regular algebraic systems*, Proc. Japan Acad., **39**, 283–286, (1963).

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