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ON THE RING OF FUNCTIONS CONTINUOUS ON A CO-COMPACT SET

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Abstract. In this paper, we introduce the ring $D(X)_C$, consisting of real-valued functions on a topological space X that are continuous on the complement of a compact subset of X. It is shown that this ring is more general than the ring $C(X)_F$ of all real-valued functions on X that are discontinuous on at most finite number of points. We also show that $D(X)_C$ is distinct from the ring $T'(X)$, where $f \in T'(X)$ if and only if there is an open dense subset D of X such that $f_{|_D} \in C(D)$. Furthermore, it is also shown that two disjoint subsets A, B of X are separated by a member of $D(X)_C$ if and only if $A \setminus K$, $B \setminus K$ are completely separated in $X \setminus K$ for some compact set $K \subseteq X$. The notion of cbd-property is introduced and its role is highlighted in determining when every function in $D(X)_C$ becomes bounded. Some more algebraic properties of $D(X)_C$ are studied.

1 Introduction

For any topological space X, $\mathbb{R}^X$ denotes the set of all real-valued functions on X. The ring of all real-valued continuous functions on a topological space X is denoted by $C(X)$. For any $f \in \mathbb{R}^X$, D_f denotes the set $\{x \in X \mid f \text{ is discontinuous at } x\}$ and $Z(f)$ denotes the set $\{x \in X \mid f(x) = 0\}$, called the zero-set of f. $C(X)_F$ denotes the collection of all real-valued functions on X which are discontinuous on at most finite number of points. Thus, D_f is at most finite for any $f \in C(X)_F$. The ring $C(X)_F$ was first introduced and studied in [3]. The symbol c_r denotes the constant real-valued function on X given by $c_r(x) = r$, for every $x \in X$. A topological space X is said to be a P-space if every zero-set in X is open. The ring of all $f \in \mathbb{R}^X$ where for each f there is a dense (respectively, open dense) subset D of X such that $f_{|_D} \in C(D)$ is denoted by $T(X)$ (respectively $T'(X)$) (see, [1]). A subset of a topological space is said to be dense-in-itself if it has no isolated point. The set of real numbers with usual topology will be denoted by $\mathbb{R}_u$. In [2], it was observed that for every ideal $\mathcal{P}$ of closed subsets of X, we can associate an overring of $C(X)$ denoted by $C(X)_{\mathcal{P}}$. This ring is a collection of real-valued functions f on X that satisfy $\overline{D_f} \in \mathcal{P}$. Two subsets A and B of a topological space X are said to be $\mathcal{F}$-completely separated in X if $\exists\ f \in C(X)_F$ such that $f(A) = 1$ and $f(B) = 0$ (see, [3]). Similarly, two subsets A and B of a topological space X are said to be $\mathcal{D}$-completely separated in X if $\exists\ f \in T'(X)$ such that $f(A) = 1$ and $f(B) = 0$ (see, [3]). A ring is said to be a reduced ring if it has no nonzero nilpotent elements. A ring Q containing a reduced ring R is said to be a ring of quotients of R iff for each $0 \neq s \in Q$, $\exists\ r \in R$ satisfying $0 \neq sr \in R$ (see, [6, Page 46, Ex. 5]). For undefined terms and references, we refer the reader to [4].

Studying the ring $D(X)_C$ is important for several reasons. This ring positions itself as a natural generalization and a useful object for working with broader classes of function rings. The ring aids in the study of separation axioms in topology as we introduce the weaker notion of “weakly-completely separated” sets, which extends classical separation results. Studying

$D(X)_C$ not only broadens the algebraic framework for function rings but also illuminates new relationships between algebraic and topological properties of spaces.

2 The ring $D(X)_C$

Let X be a topological space. The set of all real-valued functions f where for each f there is a compact subset K of X such that $f_{|_{X\setminus K}} \in C(X \setminus K)$ is denoted by $D(X)_C$. The subset of bounded members of this set is denoted by $D^*(X)_C$.

Theorem 2.1. *$D(X)_C$ is a subring and a sub-lattice of $\mathbb{R}^X$ under point-wise addition and multiplication.*

Proof. It is enough to prove that $D(X)_C$ is closed under addition and multiplication. Let $f, g \in D(X)_C$. So, there exists compact subsets A, B of X such that $f_{|_{X\setminus A}} \in C(X \setminus A)$ and $g_{|_{X\setminus B}} \in C(X \setminus B)$. Let $K = A \cup B$, which is compact. Then $(f + g)_{|_{X\setminus K}} \in C(X \setminus K)$; i.e., $f + g \in D(X)_C$. Similarly, fg is also a member of $D(X)_C$. Next, observe that if $h \in D(X)_C$, then $|h| \in D(X)_C$. Thus, from the identity $f \vee g = \frac{(f+g)+|f-g|}{2}$, it follows that $f \vee g \in D(X)_C$ whenever $f, g \in D(X)_C$. Also, we know that $f \wedge g = -(-f \vee -g)$, from which it follows that $f \wedge g \in D(X)_C$. This completes the proof. □

Remark 2.2. Like any subring of $\mathbb{R}^X$, the ring $D(X)_C$ is a reduced ring. It is not difficult to verify that an element $f \in D(X)_C$ is a unit iff $Z(f) = \phi$. Also, if an element of $D(X)_C$ is not a unit, then it is a zero divisor.

Remark 2.3. For any $f \in C(X)_F$, we have $f_{|_{X\setminus D_f}} \in C(X \setminus D_f)$, and since any finite subset is compact, we find that $C(X)_F$ is a subring of $D(X)_C$. It is clear that for a discrete space X, $C(X)_F = D(X)_C$. In fact, when X is discrete, all rings in the chain $C(X) \subseteq C(X)_F \subseteq D(X)_C \subseteq \mathbb{R}^X$ coincide with each other. However, the two rings $C(X)_F$ and $D(X)_C$ may also coincide for non-discrete space as justified by Theorem 2.6 below.

Assuming X to be a completely regular topological space, we have the following result from [4, 4K.3].

Proposition 2.4. *If X is a P-space and every function in $C(X)$ is bounded on a subset S, then S is finite.*

Proof. Let X be a P-space and S be an infinite subset of X. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in S. Let U_1 be a neighbourhood of x_1 that does not contain x_2, U_2 be a neighbourhood of x_1 that does not contain x_3, U_3 be a neighbourhood of x_1 that does not contain x_4, and so on. Let $V_1 = U_1 \cap U_2 \cap U_3 \cap \ldots$. Here, V_1 is open, being a G_δ set. So, V_1 is a neighbourhood of x_1 that does not contain any $x_i \ \forall\, i \neq 1$. But X is completely regular, so there exists a co-zero set W_1 such that $x_1 \in W_1 \subset V_1$. Note that W_1 is closed, since every zero-set is open in a P-space. Therefore, $x_1 \in W_1$, which is clopen, and $x_i \notin W_1 \ \forall\, i \neq 1$. Similarly, we obtain $W_2, W_3, W_4, \ldots$. Now, define $f\colon X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, \text{if } x \notin \{x_i\}_{i\in\mathbb{N}} \\ n, \text{if } x = x_n \end{cases} .$$

Here, f is continuous on X and unbounded on S. Hence, the result.

□

So, based on the above Proposition, we have the following lemma and also the consequent theorem.

Lemma 2.5. *Every compact subset of a P-space is finite.*

Proof. Let K be a compact subset of a P-space X, and $f \in C(X)$. Then, $f(K)$ is a compact subset of $\mathbb{R}$, hence bounded. Since f is arbitrary, by Proposition 2.4 we can conclude that K is finite. □

Theorem 2.6. *If X is a P-space, then $D(X)_C = C(X)_F$.*

Proof. Let $f \in D(X)_C$. Then $f_{|_{X\setminus K}} \in C(X \setminus K)$, for some compact K in X. Now, by Lemma 2.5, K is finite. Thus D_f is finite and so $f \in C(X)_F$. □

Remark 2.7. There are other subrings of $\mathbb{R}^X$ that contain $C(X)_F$. For instance, $T'(X)$ introduced by Zand in [1] is one such ring. For any topological space X, $T'(X)$ is a regular subring of $\mathbb{R}^X$ [1, Proposition 2.2]. It is therefore natural to investigate the relationship between $D(X)_C$ and $T'(X)$. We first present a few examples.

Example 2.8. Let $X = \mathbb{R}_u$, $A = \mathbb{Q} \cap [0, 1]$, $B = \mathbb{Q}^c \cap [0, 1]$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, \text{if } x \in A \\ 1, \text{if } x \in B \\ 2, \text{if } x \notin [0,1] \end{cases} .$$

Here, $D_f = [0, 1]$ which is compact. Put $K = D_f$. Then $f_{|_{X\setminus K}} \in C(X \setminus K)$ and so $f \in D(X)_C$. But $f \notin T'(X)$. Next, consider the function g defined on X by

$$g(x) = \begin{cases} 0, \text{if } x \in \mathbb{Z} \\ 1, \text{if } x \notin \mathbb{Z} \end{cases} .$$

Then g is continuous on $\mathbb{R} \setminus \mathbb{Z}$, which is dense and open in X; i.e., $g \in T'(X)$. But $D_g = \mathbb{Z}$, and we know that there is no compact subset in $\mathbb{R}$ containing $\mathbb{Z}$. In other words, there does not exist any compact subset K such that g will be continuous on $X \setminus K$, showing $g \notin D(X)_C$. Therefore, it is clear from this example that the rings $D(X)_C$ and $T'(X)$ are, in general, distinct from each other.

The next example shows a function that is contained in both $D(X)_C$ and $T'(X)$.

Example 2.9. Let $X = \mathbb{R}^2$, $K = [0, 1] \times \{0\}$ and $B = \mathbb{R}^2 \setminus K$. Then B is an open and dense subset of $\mathbb{R}^2$, and the characteristic function χ_B in B is continuous on B. So $\chi_B \in T'(X)$. Further, K is compact, and $\chi_{B_{|_{X\setminus K}}} \in C(X \setminus K)$, and so $\chi_B \in D(X)_C$. Thus, the non-continuous function χ_B lies at the intersection of $D(X)_C$ and $T'(X)$.

Remark 2.10. If X is a discrete space, then $C(X) = \mathbb{R}^X$, and hence $C(X)_F = D(X)_C = T'(X)$. However, $D(X)_C = C(X)_F$ or $D(X)_C = T'(X)$ need not imply that X is discrete. This can be justified by considering the following examples.

Example 2.11. Consider the non-discrete P-space $\mathbb{R}$ with topology $\mathcal{T}$ as described in [4, 4N], taking s to be the point 0. Here, in this topological space, all points are isolated except for the point 0, a neighbourhood of 0 being any set containing 0 whose complement is countable. We denote this topological space by $\mathbb{R}_{\mathcal{T}}$. Then, by Theorem 2.6, we get $D(X)_C = C(X)_F$.

Example 2.12. Consider the non-discrete T_0-space $X = \{a, b\}$ with the topology τ given by $\tau = \{\emptyset, \{a\}, X\}$. Since X is compact, we get $D(X)_C = \mathbb{R}^X$ which is vacuously true. Also, for any $f \in \mathbb{R}^X$, f is continuous at a. But $\{a\}$ is dense in X. Therefore $f \in T'(X)$, which means $\mathbb{R}^X = T'(X)$. Hence, $D(X)_C = T'(X)$.

Remark 2.13. It is clear that in spaces for which every compact set is finite, we will always get $D(X)_C = C(X)_F$. For example, X can be a dense-in-itself Hausdorff space in which each subset of X containing a dense open set is open (see, [7]). Another class of spaces are those spaces which possesses no isolated points and for which each dense subset is open (see, [5]).

Remark 2.14. If X is a compact space, then $T'(X) \subseteq D(X)_C$ since $D(X)_C = \mathbb{R}^X$ in this case. Also, it is straightforward to verify that if, for every open dense subset D of X, $X \setminus D$ is compact, then $T'(X) \subseteq D(X)_C$.

We present here a few results that state the conditions under which the ring $D(X)_C$ can be realized as $C(X)_{\mathcal{P}}$ for some ideal of closed sets $\mathcal{P}$.

Theorem 2.15. *Let X be a Hausdorff topological space, and let*

$$\mathcal{P} = \{A \subseteq X \mid A \text{ is a closed subset of a compact set in } X\},$$

then $C(X)_{\mathcal{P}} = D(X)_C$.

Proof. It is clear that $\mathcal{P}$ is an ideal of closed subsets of X. If $f \in C(X)_{\mathcal{P}}$, then $\overline{D_f} \in \mathcal{P}$; i.e., $\overline{D_f}$ is contained in a compact set in X. This means D_f, being a subset of $\overline{D_f}$, is contained in a compact set, say K, in X. This shows that $f \in D(X)_C$. Conversely, if $f \in D(X)_C$, then $D_f \subseteq K$ for some compact subset K of X. So $\overline{D_f} \subseteq \overline{K} = K$, as K is closed. Thus, $\overline{D_f} \in \mathcal{P}$, so that $f \in C(X)_{\mathcal{P}}$. □

Remark 2.16. More generally, for the same ideal of closed subsets as in the previous theorem, if X is a space in which every compact set has compact closure, then $C(X)_{\mathcal{P}} = D(X)_C$.

Remark 2.17. Let X be any arbitrary topological space, and consider the ideals of closed sets below:

$$\mathcal{P}_1 = \{A \subseteq X : A \text{ is closed; } A \subseteq K \text{where } K \text{ is compact or } \overline{K} \text{ is compact}\},$$
$$\mathcal{P}_2 = \{A \subseteq X : A \text{ is closed; } A \subseteq \overline{K} \text{ for some compact set } K \text{ in } X\}.$$

Then we have the following relation

$$C(X)_{\mathcal{P}_1} \subseteq D(X)_C \subseteq C(X)_{\mathcal{P}_2}.$$

3 On weakly-completely separation in X

In this section, the idea of weakly-completely separated subsets is introduced and its basic properties are investigated.

Definition 3.1. Two subsets A and B are said to be weakly-completely separated if there exists $f \in D(X)_C$ such that $f(A) = 0$ and $f(B) = 1$.

Remark 3.2. It is clear that if A and B are $\mathcal{F}$-completely separated, then they are also weakly-completely separated. However, the converse does not hold, as can be seen by taking $X = \mathbb{R}_u$ and A, B, f as defined in Example 2.8. Then, A and B are weakly-completely separated by the function f. But A and B are not $\mathcal{F}$-completely separated.

Now, we investigate the relationship between $\mathcal{D}$-completely separated and weakly-completely separated subsets in X. These two separation axioms are independent of each other, as justified by the next two examples.

Example 3.3. Let $X = \mathbb{R}$ with topology in which all rational points are isolated, and the neighborhoods of irrational points are usual open intervals in $\mathbb{R}$. In this topology, $\mathbb{Q}$ is dense and open in $\mathbb{R}$. As such, the characteristic function $\chi_{\mathbb{Q}}$ of $\mathbb{Q}$ is continuous on $\mathbb{Q}$. Thus, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are $\mathcal{D}$-completely separated by $\chi_{\mathbb{Q}}$. But $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not weakly-completely separated because any function g that separates them, say, $g(\mathbb{Q}) = \{1\}$ and $g(\mathbb{R} \setminus \mathbb{Q}) = \{0\}$, will be discontinuous on $\mathbb{R} \setminus \mathbb{Q}$, as $\mathbb{R} \setminus \mathbb{Q}$ is not open. It is also easy to see that no set containing $\mathbb{R} \setminus \mathbb{Q}$ can be compact, so that such a function g cannot be in $D(X)_C$. Thus, two sets which are $\mathcal{D}$-completely separated need not be weakly-completely separated.

Example 3.4. Let $X = \mathbb{R}_u$, and let A, B, f be defined as in Example 2.8. Then A and B are weakly-completely separated by f. But clearly, they are not $\mathcal{D}$-completely separated.

Remark 3.5. The collection of the zero-sets of all the functions in $D(X)_C$ is denoted by $Z(D(X)_C)$. By considering the space X and the function f as defined in Example 2.8, we see that $Z(f) = A$, but $A \notin Z(C(X))$. So, in general, we have $Z(C(X)) \subsetneq Z(D(X)_C)$.

The next two results are the counterparts of (see, [4, Theorem 1.15]).

Theorem 3.6. *Two subsets A and B of X are weakly-completely separated iff they are contained in disjoint members of $Z(D(X)_C)$.*

Proof. For the necessity part, since A and B are weakly-completely separated, there exists a function $f \in D(X)_C$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Now consider the disjoint zero-sets $Z_1 = \{x \in X | f(x) \leq \frac{1}{4}\}$ and $Z_2 = \{x \in X | f(x) \geq \frac{3}{4}\}$. It is clear that $A \subseteq Z_1$ and $B \subseteq Z_2$. For the sufficiency part, let $Z(f)$ and $Z(g)$ be two disjoint zero-sets containing A and B, respectively. Since $Z(f) \cap Z(g) = Z(|f| + |g|)$, it follows that $|f| + |g|$ has no zeros. Now we define, for $x \in X$,

$$h(x) = \frac{|g(x)|}{|f(x)|+|g(x)|},$$

i.e., $h = g.(|f| + |g|)^{-1}$. Then $h \in D(X)_C$, $h[Z(f)] = \{1\}$, and $h[Z(g)] = \{0\}$. □

Theorem 3.7. *If A and B are weakly-completely separated, then there exist zero-sets J and K in $Z(D(X)_C)$ such that $A \subseteq X \setminus K \subseteq J \subseteq X \setminus B$.*

Proof. Since A and B are weakly-completely separated, there exists $f \in D(X)_C$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Now, define $J = \{x \in X | f(x) \leq \frac{1}{4}\}$ and $K = \{x \in X | f(x) \geq \frac{1}{4}\}$. Then $J, K \in Z(D(X)_C)$, and $A \subseteq X \setminus K \subseteq J \subseteq X \setminus B$. □

It is known (see, [3]) that two disjoint subsets A and B of X are $\mathcal{F}$-completely separated if and only if there is a finite subset F of X such that $A \setminus F$ and $B \setminus F$ are completely separated in $X \setminus F$. We provide a similar result for the ring $D(X)_C$.

Theorem 3.8. *Two disjoint subsets A and B of X are weakly-completely separated if and only if there is a compact subset K of X such that $A \setminus K$ and $B \setminus K$ are completely separated in $X \setminus K$.*

Proof. For the necessary part, given that A and B are weakly-completely separated, by Theorem 3.6 there exist disjoint zero-sets $Z(f_1)$ and $Z(f_2)$ in $Z(D(X)_C)$ such that $A \subseteq Z(f_1)$ and $B \subseteq Z(f_2)$. Let K_1, K_2 be compact subsets of X such that $D_{f_1} \subseteq K_1$, $D_{f_2} \subseteq K_2$. Let $K = K_1 \cup K_2$, which is compact. Therefore, $A \setminus K \subseteq Z(f_1) \setminus K$ and $B \setminus K \subseteq Z(f_2) \setminus K$. Note that $Z(f_1) \setminus K$, $Z(f_2) \setminus K$ are zero-sets in $X \setminus K$ (in fact, $Z(f_1) \setminus K$ is nothing but $Z({f_1}_{|_{X \setminus K}})$). Also, since f_1 and f_2 are continuous on $X \setminus K$, by [4, Theorem 1.15], $A \setminus K$ and $B \setminus K$ are completely separated in $X \setminus K$.

For the sufficiency part, let A and B be disjoint subsets of X, and let K be a compact subset of X such that $A \setminus K$ and $B \setminus K$ are completely separated in $X \setminus K$. Then $\exists\ f \in C(X \setminus K)$ such that $f(A \setminus K) = 0$ and $f(B \setminus K) = 1$. Now, extend f to a function $g \in \mathbb{R}^X$ such that $g(K \cap A) = 0$ (provided $K \cap A \neq \phi$), $g(K \cap B) = 1$ (provided $K \cap B \neq \phi$), and g assumes the value 2 elsewhere. Then $g \in D(X)_C$, and A, B are weakly-completely separated by g. □

4 Some algebraic characterizations of $D(X)_C$

Definition 4.1. A function f in $D(X)_C$ is said to have the compactly bounded discontinuities property (*cbd*-property) if it is bounded on some compact set containing D_f. $D(X)_C$ is said to have *cbd*-property if every element of $D(X)_C$ has the *cbd*-property.

The following are examples of different topological spaces in which $D(X)_C$ may or may not have *cbd*-**property**.

Example 4.2. Let X be any discrete space. Then every compact subset of X is finite. So, any function f is bounded on any compact subset of X (since every function on a finite set is bounded). Therefore, $D(X)_C$ has the *cbd*-property.

Example 4.3. Let $X = \mathbb{R}_u$. Consider the function $f\colon X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x}, & x \in (0,1) \cap \mathbb{Q} \\ 1, & \text{elsewhere} \end{cases}.$$

Here, f is continuous on $X \setminus [0, 1]$, so $f \in D(X)_C$. But $f([0, 1])$ is not bounded. So $D(X)_C$ doesn't have the *cbd*-property.

Theorem 4.4. *The collection of members of $D(X)_C$ that have cbd-property is a subring of $D(X)_C$.*

Proof. Let f_1, f_2 be members of $D(X)_C$ that have *cbd*-property. Then f_1 is bounded on some compact set K_1 with $K_1 \supseteq D_{f_1}$, and f_2 is bounded on some compact set K_2 with $K_2 \supseteq D_{f_2}$. Now, $f_1 + f_2$ is bounded on the compact set $K_1 \cup K_2 \supseteq D_{f_1} \cup D_{f_2} \supseteq D_{f_1+f_2}$. So $f_1 + f_2$ also has the *cbd*-property. Similarly, we can show that $f_1 f_2$ also has the *cbd*-property. □

Theorem 4.5. *If $D(X)_C = D^*(X)_C$, then for any compact subset K of X, $X \setminus K$ is pseudocompact. Conversely, suppose $D(X)_C$ has the cbd-property, and if $X \setminus K$ is pseudocompact for any compact subset K of X, then $D(X)_C = D^*(X)_C$.*

Proof. Let K be a compact subset of X. We need to show that $X \setminus K$ is pseudocompact; i.e., $C(X \setminus K) = C^*(X \setminus K)$. Let $f \in C(X \setminus K)$. Define $g : X \to \mathbb{R}$ such that $g(K) = \{0\}$ and $g(X \setminus K) = f$. So $g \in D(X)_C$. But $D(X)_C = D^*(X)_C$, so $g \in D^*(X)_C$. This implies that g is bounded on X, and thus it is bounded on $X \setminus K$. So f is bounded and continuous on $X \setminus K$; i.e., $f \in C^*(X \setminus K)$. Hence, $C(X \setminus K) = C^*(X \setminus K)$.

Conversely, let $f \in D(X)_C$. Then f has the *cbd*-property. This means that there exists a compact subset K of X containing D_f such that $f(K)$ is bounded. Note that $f_{|_{X \setminus K}} \in C(X \setminus K)$. But $X \setminus K$ is pseudocompact, so $f_{|_{X \setminus K}} \in C^*(X \setminus K)$. Thus, f is bounded on $X \setminus K$ and on K. Hence, f is bounded on X; i.e., $f \in D^*(X)_C$. □

Remark 4.6. In the converse of the above theorem, the *cbd*-property of $D(X)_C$ cannot be dropped. This can be seen from the following example.

Example 4.7. Consider the space X of real numbers with the co-finite topology. In this space, every continuous function is constant, and every subspace of X also has the co-finite topology. So, for any compact subset K in X, we have $C(X \setminus K) = C^*(X \setminus K)$. However, consider the function f in $\mathbb{R}^X$ defined by

$$f(x) = \begin{cases} 1, & x \in (-\infty, 0] \\ x, & x \in (0, \infty) \end{cases} .$$

We observe that $f \in D(X)_C$ because $(0, \infty)$ is compact and f is continuous on $X \setminus (0, \infty)$. Also, $f \notin D^*(X)_C$. Notice that f does not have *cbd*-property, as $D_f = X$ and f is not bounded.

Remark 4.8. If X compact, this does not guarantee that $D(X)_C$ has the *cbd*-property.

The following example supports the above remark.

Example 4.9. Let $X = [0, 1]_u$. Now, define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x}, & x \in (0, 1) \cap \mathbb{Q} \\ 0, & \text{elsewhere} \end{cases} .$$

Then, clearly $f \in D(X)_C$. But $f([0, 1])$ is not bounded. Hence, $D(X)_C$ doesn't have the *cbd*-property even though X is compact.

Theorem 4.10. *X is discrete if and only if $D(X)_C$ is a ring of quotients of $C(X)$.*

Proof. For the necessity part, let $f \in D(X)_C$ where $f \neq 0$. Then we have the constant real-valued function c_1 such that $f.c_1 \in C(X)$ since $D(X)_C = C(X)$, as X is discrete. Hence, $D(X)_C$ is a ring of quotients of $C(X)$.

($\Leftarrow$) Fix $x_0 \in X$. Consider $\chi_{\{x_0\}} \in C(X)_F \subseteq D(X)_C$. Since $D(X)_C$ is a ring of quotients of $C(X)$, $\exists\, g \in C(X)$ such that $0 \neq \chi_{\{x_0\}}.g \in C(X)$. Clearly, the function g satisfies $g(x_0) \neq 0$. Now, choose an $\epsilon > 0$ such that $0 \notin (g(x_0) - \epsilon, g(x_0) + \epsilon) = U$. So, $(\chi_{\{x_0\}}.g)^{-1}(U) = \{x \in X \mid (\chi_{\{x_0\}}.g)(x) \in U\} = \{x_0\}$. For if $p \neq x_0$, then $(\chi_{\{x_0\}}.g)(p) = \chi_{\{x_0\}}(p).g(p) = 0.g(p) = 0 \notin U$. Continuity of $\chi_{\{x_0\}}.g$ shows that $\{x_0\}$ is open in X. Therefore, $\{x_0\}$ is isolated. Since x_0 was arbitrary, we conclude that X is discrete. □

Remark 4.11. More generally, an overring of $C(X)$ that contains $C(X)_F$ is a ring of quotients of $C(X)$ if and only if X is discrete.

Remark 4.12. Following the proof of Theorem 4.10, it is straightforward to check that X is discrete implies $D(X)_C$ is a ring of quotients of $C(X)_F$. But the converse of this statement does not hold, as illustrated by the following example.

Example 4.13. Consider the non-discrete P-space $X = \mathbb{R}_{\mathcal{T}}$ discussed in Example 2.11. Here, $C(X)_F = D(X)_C$ and trivially, $D(X)_C$ is the ring of quotients of $C(X)_F$.

Definition 4.14. An ideal I in $D(X)_C$ is said to be a fixed ideal if $\bigcap Z[I] \neq \emptyset$, and free if $\bigcap Z[I] = \emptyset$. Thus, an ideal I in $D(X)_C$ is free if and only if for every point $x \in X$, there exists a function in I that does not vanish at x.

In $C(X)$, it is well known that every proper ideal is fixed iff X is compact (see, [4, Theorem 4.11]). This result, however, does not hold in the case of $D(X)_C$. This is supported by the following example.

Example 4.15. Consider $X = [0,1]_u$. Let $S = \{f \in \mathbb{R}^X \mid f \text{ is discontinuous at most on } [0,1] \cap \mathbb{Q}\} \subseteq D(X)_C$. Let I be the ideal in $D(X)_C$ generated by S. Let $p \in X$. If $p \in \mathbb{Q}$, then the function f defined by

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

belongs to I and does not vanish at p. Likewise, if $p \notin \mathbb{Q}$, the function g defined by

$$g(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q}^c \\ 0, & x \in [0,1] \cap \mathbb{Q} \end{cases}$$

belongs to I and does not vanish at p. Hence, I is free.

Theorem 4.16. *Let X be a topological space in which the non-isolated points are contained in a compact subset. Then there exists a discrete space Y such that $D(X)_C \cong C(Y)$.*

Proof. Let $X = Z \cup K$, where K is a compact set containing all the non-isolated points. Now, for each $f \in \mathbb{R}^X$, we have $f_{|_Z} \in C(Z)$, since all points in Z are isolated. Therefore, $f \in D(X)_C$, i.e., $\mathbb{R}^X \subseteq D(X)_C$. So we get $D(X)_C = \mathbb{R}^X = C(Y)$, where $Y = X$ with the discrete topology. □

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