Palestine Journal of Mathematics

Vol 15(2)(2026) , 144–163 © Palestine Polytechnic University-PPU 2026

PYTHAGOREAN FUZZY SUBLATTICES AND IDEALS

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Communicated by: Madeleine Al Tahan

MSC2020 Classifications: Primary 03E72; Secondary 06B23, 08A72.

Keywords and phrases: Lattice, Pythagorean fuzzy sets, Pythagorean fuzzy sublattice, Pythagorean fuzzy ideals, Pythagorean fuzzy prime ideal, homomorphism.

The authors express their sincere gratitude to anonymous reviewers and the editorial board for their valuable comments and suggestions that helped to improve the quality and presentation of this paper.

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Abstract In this paper, we introduce and explore the concepts of *Pythagorean fuzzy sublattices and Pythagorean fuzzy ideals* within the framework of lattice theory. Building upon classical fuzzy and intuitionistic fuzzy structures, the Pythagorean fuzzy approach offers a richer representation of uncertainty by allowing the squared sum of membership and non-membership degrees to remain less than one. In this context, we establish the fundamental properties of Pythagorean fuzzy sublattices and ideals and develop the concepts of (ω, δ)-level sets, strong level sets, and convex sublattices. Additionally, we investigate lattice-to-lattice homomorphisms by introducing the concept of f-invariant Pythagorean fuzzy sets and analyzing the preservation of the supremum and infimum under homomorphisms. Notably, we demonstrate that the (ω, δ)-level set of a homomorphic image aligns with the homomorphic image of the corresponding level set. We also derive conditions for the invariance of Pythagorean fuzzy prime ideals, as well as for homomorphic images and preimages.

1 Introduction

The concept of fuzzy sets, first introduced by Zadeh [1], had a profound impact on mathematics and its applications, particularly in modeling uncertainty. Since then, fuzzy set theory has been extended to numerous algebraic structures, including groups, rings, semigroups and lattices, as investigated by various researchers. Within lattice theory, Rosenfeld [2] initiated the study of fuzzy sublattices, thereby opening new directions for the interaction of fuzzy set theory with algebra. To address situations where both membership and non-membership information are relevant, Atanassov [4] introduced the notion of *intuitionistic fuzzy sets* (IFS), which broadened the scope of fuzzy modeling. Building on this, several authors studied intuitionistic fuzzy sublattices and ideals [3], and their algebraic properties and potential applications.

In intuitionistic fuzzy sets, the membership and non-membership functions satisfy the condition $\mu(x) + \nu(x) \leq 1$. While this formulation provides more flexibility than classical fuzzy sets, it becomes restrictive in cases where the combined membership and non-membership degrees of an element exceed the unity. To overcome this limitation, Yager [5, 6] introduced the concept of *Pythagorean fuzzy sets* (PFS), which generalize IFS by replacing the linear constraint with a quadratic one, $\mu(x)^2 + \nu(x)^2 \leq 1$. This modification allows for a richer representation of hesitation and uncertainty, enabling PFS to capture situations that cannot be adequately described within the framework of the IFS. As a result, Pythagorean fuzzy sets have attracted considerable attention in both theoretical research and applications, including decision-making, pattern recognition, clustering, and data analysis. In recent years, significant advancements have been made in both the theory and applications of Pythagorean fuzzy sets. Some related studies includes investigations on Pythagorean fuzzy ideals in Γ-near-rings [8], and Pythagorean fuzzy ideals in lattice-based algebras [9]. For instance, Sun and Wang [10] proposed novel ranking methods for probabilistic hesitant Pythagorean fuzzy information and applied them to multi-criteria decision-making (MCDM) problems, thus addressing the need for reliable comparisons under complex fuzzy data. Bozyiğit et al. [12] introduced circular Pythagorean fuzzy sets, developed algebraic operations, similarity measures, and weighted aggregation operators, and used them to select optimal photovoltaic cells. In educational settings, Nuur and Munandar [14] employed PFS with max-min composition to determine student concentration in a study program. These studies collectively illustrate how extensions of PFS are leveraged for decision-making, aggregation, and similarity measures in diverse application domains. However, relatively little work has explored algebraic structures such as sublattices, ideals, and homomorphism properties in the context of Pythagorean fuzzy theory. This gap motivates the present study, which focuses on formal lattice-theoretic concepts (Pythagorean fuzzy sublattices and ideals), their level sets, convexity, and behavior under homomorphisms.

In this paper we introduce and study *Pythagorean fuzzy sublattices* (PFL) and *Pythagorean fuzzy ideals* (PFI), and establishing their fundamental properties. Compared to their classical and intuitionistic fuzzy counterparts, PFLs and PFIs provide a more expressive framework for modeling hierarchical structures under the uncertainty. Such generalizations are not only of intrinsic mathematical interest but also carry potential applications in areas such as multi-criteria decision-making, knowledge representation, and uncertain information systems, where lattice-based models play a key role.

The main contributions of this study are as follows.

- We introduce the notions of Pythagorean fuzzy sublattices and Pythagorean fuzzy ideals and investigated their basic properties.
- We define (ω, δ)-level sets and strong level sets in the Pythagorean fuzzy setting and analyzed the convex sublattices and their related properties.
- We study lattice-to-lattice homomorphisms, introducing the concept of f-invariant Pythagorean fuzzy sets and proving preservation results for supremum, infimum, and prime ideals.

2 Preliminaries

In this section some elementary notions such as Intuitionistic Fuzzy set(IFS), Pythagorean fuzzy set (PFS) are given

Definition 2.1. [4] Let M be a non-empty set. An Intuitionistic fuzzy set (IFS) N in M is an object of the form $\{< m, \varphi_N(m), \eta_N(m) > /m \in M\}$ where $\varphi_N(m) : M \to [0,1]$ and $\eta_N(m) : M \to [0,1]$ represents degree of the membership function and non-membership function resp, with the condition that $\varphi_N(m) + \eta_N(m) \le 1$

Definition 2.2. [4] Let $\{< m, \varphi_{N_1}(m), \eta_{N_1}(m) > /m \in M\}$ and $\{< m, \varphi_{N_2}(m), \eta_{N_2}(m) > /m \in M\}$ are two intuitionistic fuzzy sets on M, then for all $m, n \in M$

(i) $N_1 \subseteq N_2$ implies $\varphi_{N_1}(m) \le \varphi_{N_2}(m)$ and $\eta_{N_1}(m) \ge \eta_{N_2}(m)$

(ii) $N_1 = N_2$ implies $\varphi_{N_1}(m) = \varphi_{N_2}(m)$ and $\eta_{N_1}(m) = \eta_{N_2}(m)$

(iii) ${N_1}^c = \{< m, \eta_{N_1}(m), \varphi_{N_1}(m) > /m \in M\}$

(iv) $N_1 \cap N_2 = \{< m, \varphi_{N_1\cap N_2}(m), \eta_{N_1\cap N_2}(m) > /m \in M\}$ where $\varphi_{N_1\cap N_2}(m) = min\{\varphi_{N_1}(m), \varphi_{N_2}(m)\}$ and $\eta_{N_1\cap N_2}(m) = max\{\eta_{N_1}(m), \eta_{N_2}(m)\}$

(v) $N_1 \cup N_2 = \{< m, \varphi_{N_1\cup N_2}(m), \eta_{N_1\cup N_2}(m) > /m \in M\}$ where $\varphi_{N_1\cup N_2}(m) = max\{\varphi_{N_1}(m), \varphi_{N_2}(m)\}$ and $\eta_{N_1\cup N_2}(m) = min\{\eta_{N_1}(m), \eta_{N_2}(m)\}$

(vi) $[N] = \{< m, \varphi_N(m), 1 - \varphi_N(m) > m \in M\}$

(vii) $< N >= \{< m, 1 - \eta_N(m), \eta_N(m) > m \in M\}$

Definition 2.3. [3] Let T be a lattice and $A = \{< m, \varphi_A, \eta_A > /m \in T\}$ be a IFS of T. Then A is called an intuitionistic fuzzy sublattice if the following conditions are satisfied for all $m,\ n \in T$

(i) $\varphi_A(m \vee n) \ge min\{\varphi_A(m), \varphi_A(n)\}$

(ii) $\varphi_A(m \wedge n) \ge min\{\varphi_A(m), \varphi_A(n)\}$

(iii) $\eta_A(m \vee n) \leq max\{\eta_A(m), \eta_A(n)\}$

(iv) $\eta_A(m \wedge n) \leq max\{\eta_A(m), \eta_A(n)\}$

Definition 2.4. [3] Let T be a lattice and $A = \{< m, \varphi_A, \eta_A > /m \in T\}$ be a IFS of T. Then A is called an intuitionistic fuzzy ideal if the following conditions are satisfied for all $m,\ n \in T$

(i) $\varphi_A(m \vee n) \geq min\{\varphi_A(m), \varphi_A(n)\}$

(ii) $\varphi_A(m \wedge n) \geq max\{\varphi_A(m), \varphi_A(n)\}$

(iii) $\eta_A(m \vee n) \leq max\{\eta_A(m), \eta_A(n)\}$

(iv) $\eta_A(m \wedge n) \leq min\{\eta_A(m), \eta_A(n)\}$

Definition 2.5. [5, 6] Let M be a nonempty set. A Pythagorean fuzzy set (PFS) G in M is an object of the form $\{< m, \varphi_G(m), \eta_G(m) > /m \in M\}$ where $\varphi_G(m) : M \to [0, 1]$ and $\eta_G(m) : M \to [0, 1]$ represents degree of the membership function and non-membership function resp, with the condition that $\varphi_G^2(m) + \eta_G^2(m) \leq 1$

Definition 2.6. [5, 6, 7] Let $\{< m, \varphi_{G_1}(m), \eta_{G_1}(m) > /m \in M\}$ and $\{< m, \varphi_{G_2}(m), \eta_{G_2}(m) > /m \in M\}$ are two Pythagorean fuzzy sets on M, then for all $m, n \in M$

(i) $G_1 \subseteq G_2 \Longrightarrow \varphi_{G_1}^2(m) \leq \varphi_{G_2}^2(m)$ and $\eta_{G_1}^2(m) \geq \eta_{G_2}^2(m)$

(ii) $G_1 = G_2 \Longrightarrow \varphi_{G_1}(m) = \varphi_{G_2}(m)$ and $\eta_{G_1}(m) = \eta_{G_2}(m)$

(iii) ${G_1}^c = \{< m, \eta_G(m), \varphi_G(m) > /m \in M\}$

(iv) $G_1 \cap G_2 = \{< m, \varphi_{G_1 \cap G_2}(m), \eta_{G_1 \cap G_2}(m) > /m \in M\}$ where $\varphi_{G_1 \cap G_2}^2(m) = min\{\varphi_{G_1}^2(m), \varphi_{G_2}^2(m)\}$ and $\eta_{G_1 \cap G_2}^2(m) = max\{\eta_{G_1}^2(m), \eta_{G_2}^2(m)\}$

(v) $G_1 \cup G_2 = \{< m, \varphi_{G_1 \cup G_2}(m), \eta_{G_1 \cup G_2}(m) > /m \in M\}$ where $\varphi_{G_1 \cup G_2}^2(m) = max\{\varphi_{G_1}^2(m), \varphi_{G_2}^2(m)\}$ and $\eta_{G_1 \cup G_2}^2(m) = min\{\eta_{G_1}^2(m), \eta_{G_2}^2(m)\}$

(vi) $[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$

(vii) $< G >= \{< m, (1 - \eta_G^2(m))^{0.5}, \eta_G(m) > /m \in M\}$

3 Pythagorean fuzzy sublattices and ideals

In this section, we define Pythagorean fuzzy sublattices (PFL) and Pythagorean fuzzy ideals (PFI) and study their characterizations. Throughout this paper, T stands for the lattice $(T, \vee, \wedge)$, where $\vee$ and $\wedge$ represent join and meet, respectively.

Definition 3.1. Let T be a lattice and $G = \{< m, \varphi_G, \eta_G > /m \in T\}$ be the PFS of T. Then G is called a Pythagorean fuzzy sublattice if the following conditions are satisfied for all $m,\ n \in T$

(i) $\varphi_G^2(m \vee n) \geq min\{\varphi_G^2(m), \varphi_G^2(n)\}$

(ii) $\varphi_G^2(m \wedge n) \geq min\{\varphi_G^2(m), \varphi_G^2(n)\}$

(iii) $\eta_G^2(m \vee n) \leq max\{\eta_G^2(m), \eta_G^2(n)\}$

(iv) $\eta_G^2(m \wedge n) \leq max\{\eta_G^2(m), \eta_G^2(n)\}$

Example 3.2. Pythagorean Fuzzy Sublattice on the Lattice of Divisors of 12

Let

$$T = \{1, 2, 3, 4, 6, 12\}$$

be the lattice of positive divisors of 12 under divisibility. The join ($\vee$) is given by the least common multiple (LCM), and the meet ($\wedge$) is given by the greatest common divisor (GCD) of the two elements.

Define a Pythagorean fuzzy subset G of T as

$$G = \{\langle m, \varphi_G(m), \eta_G(m)\rangle \mid m \in T\},$$

where the membership and non-membership functions are listed in Table 1.

To verify that G is a **Pythagorean fuzzy sublattice**, we must check that for all $m, n \in T$,

m	$\varphi_G(m)$	$\eta_G(m)$	$\varphi_G^2(m)$	$\eta_G^2(m)$
1	0.9	0.3	0.81	0.09
2	0.8	0.4	0.64	0.16
3	0.9	0.3	0.81	0.09
4	0.7	0.6	0.49	0.36
6	0.8	0.4	0.64	0.16
12	0.8	0.4	0.64	0.16

Table 1. Membership and non-membership values of G on T

(i) $\varphi_G^2(m \vee n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\}$,

(ii) $\varphi_G^2(m \wedge n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\}$,

(iii) $\eta_G^2(m \vee n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\}$,

(iv) $\eta_G^2(m \wedge n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\}$.

Verification:

- Consider $m = 2$, $n = 3$. Then $m \vee n = \text{lcm}(2, 3) = 6$ and $m \wedge n = \text{gcd}(2, 3) = 1$. From Table 1:

  $$\varphi_G^2(6) = 0.64, \quad \min\{0.64, 0.81\} = 0.64,$$

  Therefore, condition (1) holds. Similarly,

  $$\varphi_G^2(1) = 0.81 \geq \min\{0.64, 0.81\} = 0.64,$$

  Hence, (2) holds. For non-membership:

  $$\eta_G^2(6) = 0.16 \leq \max\{0.16, 0.09\} = 0.16,$$

  and

  $$\eta_G^2(1) = 0.09 \leq \max\{0.16, 0.09\} = 0.16,$$

  Therefore, (3) and (4) hold.

- Consider $m = 4$, $n = 6$. Then $m \vee n = \text{lcm}(4, 6) = 12$, $m \wedge n = \text{gcd}(4, 6) = 2$. We have:

  $$\varphi_G^2(12) = 0.64, \quad \min\{0.49, 0.64\} = 0.49,$$

  Therefore, (1) is satisfied. Next,

  $$\varphi_G^2(2) = 0.64 \geq \min\{0.49, 0.64\} = 0.49,$$

  so (2) holds. For non-membership:

  $$\eta_G^2(12) = 0.16 \leq \max\{0.36, 0.16\} = 0.36,$$

  and

  $$\eta_G^2(2) = 0.16 \leq \max\{0.36, 0.16\} = 0.36.$$

- Other pairs can be checked in the same manner. In each case, the inequalities hold due to the assignments in Table 1.

Since all four conditions are verified for representative pairs (and the same reasoning applies to all others), we conclude that G forms a **Pythagorean fuzzy sublattice** of the lattice of divisors of 12.

Definition 3.3. Let T be a lattice and $G = \{< m, \varphi_G, \eta_G > / m \in T\}$ be a PFS of T. Then G is called a Pythagorean fuzzy ideal (PFI) if the following conditions are satisfied for all $m,\ n \in T$

(i) $\varphi_G^2(m \vee n) \geq min\{\varphi_G^2(m), \varphi_G^2(n)\}$

(ii) $\varphi_G^2(m \wedge n) \geq max\{\varphi_G^2(m), \varphi_G^2(n)\}$

(iii) $\eta_G^2(m \vee n) \leq max\{\eta_G^2(m), \eta_G^2(n)\}$

(iv) $\eta_G^2(m \wedge n) \leq min\{\eta_G^2(m), \eta_G^2(n)\}$

Example 3.4. Pythagorean Fuzzy Sublattice on the Lattice of Divisors of 10.

Proof. Consider the lattice of positive divisors of 10 under divisibility:

$$T = \{1, 2, 5, 10\}.$$

The join operation $(m \vee n)$ corresponds to the least common multiple (LCM), and the meet operation $(m \wedge n)$ corresponds to the greatest common divisor (GCD) operation.

Define the Pythagorean fuzzy subset G of T by

$$G = \{\langle 1, 0.9, 0.4\rangle, \langle 2, 0.8, 0.4\rangle, \langle 5, 0.7, 0.4\rangle, \langle 10, 0.7, 0.4\rangle\}.$$

The squared membership and non-membership degrees are as follows:

m	$\varphi_G(m)$	$\eta_G(m)$	$\varphi_G^2(m)$	$\eta_G^2(m)$
1	0.9	0.4	0.81	0.16
2	0.8	0.4	0.64	0.16
5	0.7	0.4	0.49	0.16
10	0.7	0.4	0.49	0.16

To show that G is a **Pythagorean fuzzy sublattice**, we must check that for all $m, n \in T$,

(i) $\varphi_G^2(m \vee n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\}$,

(ii) $\varphi_G^2(m \wedge n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\}$,

(iii) $\eta_G^2(m \vee n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\}$,

(iv) $\eta_G^2(m \wedge n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\}$.

Verification for representative pairs:

- Take $m = 2$, $n = 5$. Then

$$m \vee n = \text{lcm}(2, 5) = 10, \quad m \wedge n = \text{gcd}(2, 5) = 1.$$

For membership:

$$\varphi_G^2(10) = 0.49, \quad \min\{0.64, 0.49\} = 0.49,$$

Hence, condition (1) is satisfied. Similarly,

$$\varphi_G^2(1) = 0.81 \geq \min\{0.64, 0.49\} = 0.49,$$

Therefore, condition (2) holds. For non-membership:

$$\eta_G^2(10) = 0.16 \leq \max\{0.16, 0.16\} = 0.16,$$

and

$$\eta_G^2(1) = 0.16 \leq \max\{0.16, 0.16\} = 0.16.$$

Thus, conditions (3) and (4) hold.

- Take $m = 2$, $n = 10$. Then

$$m \vee n = \text{lcm}(2, 10) = 10, \quad m \wedge n = \text{gcd}(2, 10) = 2.$$

From the table,

$$\varphi_G^2(10) = 0.49, \quad \min\{0.64, 0.49\} = 0.49,$$

and

$$\varphi_G^2(2) = 0.64 \geq \min\{0.64, 0.49\} = 0.49.$$

For non-membership,

$$\eta_G^2(10) = 0.16 \leq \max\{0.16, 0.16\} = 0.16, \quad \eta_G^2(2) = 0.16 \leq \max\{0.16, 0.16\} = 0.16.$$

Again, all the conditions are satisfied.

The verification of all possible pairs in this manner confirmed that each of the four conditions was satisfied. Hence, G forms a **Pythagorean fuzzy sublattice** of the lattice of divisors of 10. □

Definition 3.5. Let T be a lattice and $G = \{< m, \varphi_G, \eta_G > /m \in T\}$ be a PFS of T. Then G is called a Pythagorean fuzzy prime ideal (PFPI) if the following conditions are satisfied for all $m,\ n \in T$

(i) $\varphi_G^2(m \wedge n) = max\{\varphi_G^2(m), \varphi_G^2(n)\}$

(ii) $\eta_G^2(m \wedge n) = min\{\eta_G^2(m), \eta_G^2(n)\}$

Theorem 3.6. *If $G = \{< m, \varphi_G, \eta_G >|\ m \in T\}$ is an IFL, then it is a PFL.*

Proof. We examine five cases.

Case 1: Suppose $\varphi_G(m) > \varphi_G(n)$ and $\eta_G(m) > \eta_G(n)$ for all $m, n \in T$. Then we have

$$\varphi_G^2(m) > \varphi_G^2(n) \quad \text{and} \quad \eta_G^2(m) > \eta_G^2(n).$$

For the join:

$$\varphi_G(m \vee n) \geq \min\{\varphi_G(m), \varphi_G(n)\} = \varphi_G(n),$$

hence

$$\varphi_G^2(m \vee n) \geq \varphi_G^2(n) = \min\{\varphi_G^2(m), \varphi_G^2(n)\}.$$

For the meet:

$$\varphi_G(m \wedge n) \geq \min\{\varphi_G(m), \varphi_G(n)\} = \varphi_G(n),$$

thus

$$\varphi_G^2(m \wedge n) \geq \varphi_G^2(n) = \min\{\varphi_G^2(m), \varphi_G^2(n)\}.$$

For the join with respect to η_G, we have

$$\eta_G(m \vee n) \leq \max\{\eta_G(m), \eta_G(n)\} = \eta_G(m),$$

so

$$\eta_G^2(m \vee n) \leq \eta_G^2(m) = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

For the meet with respect to η_G, we have

$$\eta_G(m \wedge n) \leq \max\{\eta_G(m), \eta_G(n)\} = \eta_G(m),$$

and hence

$$\eta_G^2(m \wedge n) \leq \eta_G^2(m) = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

Case 2: Suppose $\varphi_G(m) < \varphi_G(n)$ and $\eta_G(m) < \eta_G(n)$ for all $m, n \in T$. Then

$$\varphi_G^2(m) < \varphi_G^2(n) \quad \text{and} \quad \eta_G^2(m) < \eta_G^2(n).$$

For the join:

$$\varphi_G(m \vee n) \geq \min\{\varphi_G(m), \varphi_G(n)\} = \varphi_G(m),$$

so

$$\varphi_G^2(m \vee n) \geq \varphi_G^2(m) = \min\{\varphi_G^2(m), \varphi_G^2(n)\}.$$

For the meet:

$$\varphi_G(m \wedge n) \geq \min\{\varphi_G(m), \varphi_G(n)\} = \varphi_G(m),$$

hence

$$\varphi_G^2(m \wedge n) \geq \varphi_G^2(m) = \min\{\varphi_G^2(m), \varphi_G^2(n)\}.$$

For the join with respect to η_G:

$$\eta_G(m \vee n) \leq \max\{\eta_G(m), \eta_G(n)\} = \eta_G(n),$$

so

$$\eta_G^2(m \vee n) \leq \eta_G^2(n) = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

For the meet with respect to η_G:

$$\eta_G(m \wedge n) \leq \max\{\eta_G(m), \eta_G(n)\} = \eta_G(n),$$

thus

$$\eta_G^2(m \wedge n) \leq \eta_G^2(n) = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

Case 3: Suppose $\varphi_G(m) = \varphi_G(n)$ and $\eta_G(m) = \eta_G(n)$ for all $m, n \in T$. Then

$$\varphi_G^2(m) = \varphi_G^2(n) \quad \text{and} \quad \eta_G^2(m) = \eta_G^2(n).$$

For the join:

$$\varphi_G(m \vee n) \geq \min\{\varphi_G(m), \varphi_G(n)\} = \varphi_G(m) = \varphi_G(n),$$

so

$$\varphi_G^2(m \vee n) \geq \varphi_G^2(m) = \varphi_G^2(n) = \min\{\varphi_G^2(m), \varphi_G^2(n)\}.$$

For the meet:

$$\varphi_G(m \wedge n) \geq \max\{\varphi_G(m), \varphi_G(n)\} = \varphi_G(m) = \varphi_G(n),$$

thus

$$\varphi_G^2(m \wedge n) \geq \varphi_G^2(m) = \varphi_G^2(n) = \max\{\varphi_G^2(m), \varphi_G^2(n)\}.$$

For the join with respect to η_G:

$$\eta_G(m \vee n) \leq \max\{\eta_G(m), \eta_G(n)\} = \eta_G(m) = \eta_G(n),$$

so

$$\eta_G^2(m \vee n) \leq \eta_G^2(m) = \eta_G^2(n) = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

For the meet with respect to η_G:

$$\eta_G(m \wedge n) \leq \max\{\eta_G(m), \eta_G(n)\} = \eta_G(m) = \eta_G(n),$$

hence

$$\eta_G^2(m \wedge n) \leq \eta_G^2(m) = \eta_G^2(n) = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

The proof of the remaining cases is done in a similar manner. □

Remark 3.7. Every IFL of T is a PFL, but the converse is not necessarily true.

Example 3.8. Consider the lattice of divisors of 12,

$$T = \{1, 2, 3, 4, 6, 12\}.$$

Define a Pythagorean fuzzy set G on T as

$$G = \{\, \langle 1, 0.9, 0.3\rangle,\ \langle 2, 0.8, 0.4\rangle,\ \langle 3, 0.9, 0.3\rangle,\ \langle 4, 0.7, 0.6\rangle,\ \langle 6, 0.8, 0.4\rangle,\ \langle 12, 0.8, 0.4\rangle \,\}.$$

It can be verified that G satisfies all four conditions required for a Pythagorean fuzzy lattice (PFL). Hence, G forms a PFL of T.

On the other hand, for the element $1 \in T$, we observe that

$$0.9 + 0.3 = 1.2 > 1,$$

This violates the condition $\mu() + \nu() \leq 1$ required for an intuitionistic fuzzy set (IFS). Therefore, G is not an IFS, and consequently, G cannot be an intuitionistic fuzzy lattice (IFL) of T.

This example shows that a PFL of T need not be an IFL of T.

Theorem 3.9. *If G_1, G_2 are two PFLs of T, then $G_1 \cap G_2$ is also a PFL (or PFI) of T.*

Proof. Let

$$G_1 = \{\langle m, \varphi_{G_1}(m), \eta_{G_1}(m)\rangle \mid m \in M\}, \quad G_2 = \{\langle m, \varphi_{G_2}(m), \eta_{G_2}(m)\rangle \mid m \in M\}$$

be two Pythagorean fuzzy sets on M. By Definition 2.2,

$$G_1 \cap G_2 = \{\langle m, \varphi_{G_1 \cap G_2}(m), \eta_{G_1 \cap G_2}(m)\rangle \mid m \in M\},$$

where

$$\varphi^2_{G_1 \cap G_2}(m) = \min\{\varphi^2_{G_1}(m), \varphi^2_{G_2}(m)\}, \quad \eta^2_{G_1 \cap G_2}(m) = \max\{\eta^2_{G_1}(m), \eta^2_{G_2}(m)\}.$$

For the membership function φ:

$$\begin{aligned}
\varphi^2_{G_1 \cap G_2}(m \vee n) &= \min\{\varphi^2_{G_1}(m \vee n),\ \varphi^2_{G_2}(m \vee n)\} \\
&\geq \min\big\{ \min\{\varphi^2_{G_1}(m), \varphi^2_{G_1}(n)\},\ \min\{\varphi^2_{G_2}(m), \varphi^2_{G_2}(n)\}\big\} \\
&= \min\big\{ \min\{\varphi^2_{G_1}(m), \varphi^2_{G_2}(m)\},\ \min\{\varphi^2_{G_1}(n), \varphi^2_{G_2}(n)\}\big\} \\
&= \min\{\varphi^2_{G_1 \cap G_2}(m),\ \varphi^2_{G_1 \cap G_2}(n)\}.
\end{aligned}$$

Hence,

$$\varphi^2_{G_1 \cap G_2}(m \vee n) \ \geq\ \min\{\varphi^2_{G_1 \cap G_2}(m), \varphi^2_{G_1 \cap G_2}(n)\}.$$

Similarly, we can prove that

$$\varphi^2_{G_1 \cap G_2}(m \wedge n) \ \geq\ \min\{\varphi^2_{G_1 \cap G_2}(m), \varphi^2_{G_1 \cap G_2}(n)\}.$$

For the non-membership function η:

$$\begin{aligned}
\eta^2_{G_1 \cap G_2}(m \vee n) &= \max\{\eta^2_{G_1}(m \vee n),\ \eta^2_{G_2}(m \vee n)\} \\
&\leq \max\big\{ \max\{\eta^2_{G_1}(m), \eta^2_{G_1}(n)\},\ \max\{\eta^2_{G_2}(m), \eta^2_{G_2}(n)\}\big\} \\
&= \max\big\{ \max\{\eta^2_{G_1}(m), \eta^2_{G_2}(m)\},\ \max\{\eta^2_{G_1}(n), \eta^2_{G_2}(n)\}\big\} \\
&= \max\{\eta^2_{G_1 \cap G_2}(m),\ \eta^2_{G_1 \cap G_2}(n)\}.
\end{aligned}$$

Thus,

$$\eta^2_{G_1 \cap G_2}(m \vee n) \ \leq\ \max\{\eta^2_{G_1 \cap G_2}(m), \eta^2_{G_1 \cap G_2}(n)\}.$$

Similarly, we can prove that

$$\eta^2_{G_1 \cap G_2}(m \wedge n) \ \leq\ \max\{\eta^2_{G_1 \cap G_2}(m), \eta^2_{G_1 \cap G_2}(n)\}.$$

Since both conditions for φ and η are satisfied, it follows that $G_1 \cap G_2$ is a PFL of T. □

Remark 3.10. The union of two PFLs need not be a PFL. Consider

$$G_1 = \{\langle 1, 0.9, 0.3\rangle,\ \langle 2, 0.8, 0.4\rangle,\ \langle 3, 0.9, 0.3\rangle,\ \langle 4, 0.7, 0.6\rangle,\ \langle 6, 0.8, 0.4\rangle,\ \langle 12, 0.8, 0.4\rangle\},$$

$$G_2 = \{\langle 1, 0.8, 0.4\rangle,\ \langle 2, 0.9, 0.3\rangle,\ \langle 3, 0.8, 0.4\rangle,\ \langle 4, 0.9, 0.3\rangle,\ \langle 6, 0.9, 0.3\rangle,\ \langle 12, 0.8, 0.4\rangle\}.$$

Then their union is

$$G_1 \cup G_2 = \{\langle 1, 0.9, 0.3\rangle,\ \langle 2, 0.9, 0.3\rangle,\ \langle 3, 0.9, 0.3\rangle,\ \langle 4, 0.9, 0.3\rangle,\ \langle 6, 0.9, 0.3\rangle,\ \langle 12, 0.8, 0.4\rangle\}.$$

Now, observe that

$$\varphi^2(4 \vee 3) = \varphi^2(12) = (0.8)^2 = 0.64,$$

while

$$\min\{\varphi^2(4),\ \varphi^2(3)\} = \min\{0.81,\ 0.81\} = 0.81.$$

Since

$$\varphi^2(4 \vee 3) = 0.64 \le 0.81 = \min\{\varphi^2(4), \varphi^2(3)\},$$

It follows that $G_1 \cup G_2$ is not a PFL.

Remark 3.11. Every PFI of T is a PFL, but the converse is not necessarily true.

Let

$$G = \{\langle m, \varphi_G, \eta_G\rangle \mid m \in T\}$$

be a PFI of T. Then, the following holds:

(i) $\varphi_G^2(m \vee n) \ \ge\ \min\{\varphi_G^2(m), \varphi_G^2(n)\}$,

(ii) $\varphi_G^2(m \wedge n) \ \ge\ \max\{\varphi_G^2(m), \varphi_G^2(n)\} \ \ge\ \min\{\varphi_G^2(m), \varphi_G^2(n)\}$,

(iii) $\eta_G^2(m \vee n) \ \le\ \max\{\eta_G^2(m), \eta_G^2(n)\}$,

(iv) $\eta_G^2(m \wedge n) \ \le\ \min\{\eta_G^2(m), \eta_G^2(n)\} \ \le\ \max\{\eta_G^2(m), \eta_G^2(n)\}$.

Hence G is a PFL.

However, the converse is not necessarily true. For example, consider the lattice of divisors of 12:

$$T = \{1, 2, 3, 4, 6, 12\},$$

and let

$$G = \{\langle 1, 0.9, 0.3\rangle,\ \langle 2, 0.8, 0.4\rangle,\ \langle 3, 0.9, 0.3\rangle,\ \langle 4, 0.7, 0.6\rangle,\ \langle 6, 0.8, 0.4\rangle,\ \langle 12, 0.8, 0.4\rangle\}.$$

Now,

$$\varphi_G^2(4 \wedge 12) = \varphi_G^2(4) = (0.7)^2 = 0.49,$$

while

$$\max\left(\varphi_G^2(4), \varphi_G^2(12)\right) = \max(0.49, 0.64) = 0.64.$$

Thus,

$$\varphi_G^2(4 \wedge 12) = 0.49 \ \le\ 0.64 = \max\{\varphi_G^2(4), \varphi_G^2(12)\},$$

This shows that G is not a PFI.

Remark 3.12. The union of two PFIs of T need not be a PFI of T. For example, consider

$$G_1 = \{\langle 1, 0.9, 0.4\rangle,\ \langle 2, 0.8, 0.4\rangle,\ \langle 5, 0.8, 0.5\rangle,\ \langle 10, 0.7, 0.4\rangle\},$$

$$G_2 = \{\langle 1, 0.8, 0.4\rangle,\ \langle 2, 0.7, 0.4\rangle,\ \langle 5, 0.7, 0.4\rangle,\ \langle 10, 0.6, 0.5\rangle\}.$$

Then

$$G_1 \cup G_2 = \{\langle 1, 0.9, 0.4\rangle,\ \langle 2, 0.8, 0.4\rangle,\ \langle 5, 0.8, 0.4\rangle,\ \langle 10, 0.7, 0.4\rangle\},$$

which is not a PFI, since

$$\varphi_{G_1\cup G_2}^2(10) = \varphi_{G_1\cup G_2}^2(2 \vee 5) = (0.7)^2 = 0.49$$

while

$$\min\{\varphi_{G_1\cup G_2}^2(2),\ \varphi_{G_1\cup G_2}^2(5)\} = \min\{0.64,\ 0.64\} = 0.64.$$

Thus,

$$\varphi_{G_1\cup G_2}^2(2 \vee 5) = 0.49 \ <\ 0.64 = \min\{\varphi_{G_1\cup G_2}^2(2),\ \varphi_{G_1\cup G_2}^2(5)\},$$

Hence, $G_1 \cup G_2$ is not a PFI.

Proposition 3.13. *G is a PFL(PFI) of T if and only if $[G]$ and $< G >$ are PFL(PFI) of T*

Proof. :Assume that G is a PFL of T.
$[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$

$$\varphi_G^2(m \vee n) \geq \min\{(\varphi_G^2(m), \varphi_G^2(n)\}$$
$$\varphi_G^2(m \wedge n) \geq \min\{(\varphi_G^2(m), \varphi_G^2(n)\}$$
$$1 - \varphi_G^2(m \vee n) \leq 1 - \min\{(\varphi_G^2(m), \varphi_G^2(n)\}$$
$$= max(1 - \varphi_G^2(m), 1 - \varphi_G^2(n)$$
$$\Longrightarrow 1 - \varphi_G^2(m \vee n) \leq \max\left\{\left(1 - \varphi_G^2(m)\right), 1 - \varphi_G^2(n)\right)\right\}$$

Similarly, we can prove that

$$1 - \varphi_G^2(m \wedge n) \leq \max\left\{\left(1 - \varphi_G^2(m)\right), \left(1 - \varphi_G^2(n)\right)\right\}$$

Hence $[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$ is a PFL.
$< G >= \{< m, (1 - \eta_G^2(m))^{0.5}, \eta_G(m) > /m \in M\}$

$$\eta_G^2(m \vee n) \leq \max\{(\eta_G^2(m), \eta_G^2(n)\}$$
$$\eta_G^2(m \wedge n) \leq \max\{(\eta_G^2(m), \eta_G^2(n)\}$$
$$1 - \eta_G^2(m \vee n) \geq 1 - \max\{(\eta_G^2(m), \eta_G^2(n)\}$$
$$= \min\{1 - \eta_G^2(m), 1 - \eta_G^2(n)\}$$
$$\Longrightarrow (1 - \eta_G^2(m \vee n) \geq \min\{(1 - \eta_G^2(m), (1 - \eta_G^2(n)\}$$

Similarly, we can show that

$$1 - \eta_G^2(m \wedge n) \geq \min\left\{\left(1 - \eta_G^2(m)\right), \left(1 - \eta_G^2(n)\right)\right\}$$

Therefore $< G >= \{< m, (1 - \eta_G^2(m))^{0.5}, \eta_G(m) > /m \in M\}$ is a PFL of T.
Conversely assume $[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$ is a PFL of T, then

$$\varphi_G^2(m \vee n) \geq \min\{(\varphi_G^2(m), \varphi_G^2(n)\}$$
$$\varphi_G^2(m \wedge n) \geq \min\{(\varphi_G^2(m), \varphi_G^2(n)\}$$
$$\eta_G^2(m \vee n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\}$$
$$\eta_G^2(m \wedge n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\}$$

Hence G is a PFL(PFI) □

Definition 3.14. Let G is PFS of any set M, then (ω, δ) level subset of G is defined as

$$G^{[\omega,\delta]} = \{m \in M/ \varphi_G^2(m) \geq \omega^2, \eta_G^2(m) \leq \delta^2\}$$

where $(\omega, \delta) \in [0, 1] \times [0, 1]$ and $\omega^2 + \delta^2 \leq 1$ and (ω, δ) strong level set of G is defined as

$$G^{(\omega,\delta)} = \{m \in M/ \varphi_G^2(m) > \omega^2, \eta_G^2(m) < \delta^2\}$$

where $(\omega, \delta) \in [0, 1] \times [0, 1]$ and $\omega^2 + \delta^2 \leq 1$
Upper level subset of G is defined as

$$U(\varphi_G, \omega) = \{m \in M / \varphi_G^2(m) \geq \omega^2\}$$

and lower level subset of G is defined as

$$T(\eta_G, \delta) = \{m \in M/ \eta_G^2(m) \leq \delta^2\}$$

It is clear from the definition that $G^{[\omega,\delta]} = U(\varphi_G, \eta) \cap T(\eta_G, \delta)$

Theorem 3.15. *A PFS G of a lattice T is a PFL of T (respectively, a PFI of T) if and only if each non-empty (ω, δ)-level subset $G^{[\omega,\delta]}$ is a sublattice (respectively, an ideal) of T.*

Proof. Assume that G is a PFL of T. Let $m, n \in G^{[\omega,\delta]}$. Then

$$\varphi_G^2(m) \geq \omega^2, \quad \eta_G^2(m) \leq \delta^2.$$

Hence,

$$\varphi_G^2(m \vee n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\} \geq \omega^2,$$

and

$$\eta_G^2(m \vee n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\} \leq \delta^2.$$

Similarly,

$$\varphi_G^2(m \wedge n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\} \geq \omega^2,$$
$$\eta_G^2(m \wedge n) \leq \max\{\eta_G^2(m), \eta_G^2(n)\} \leq \delta^2.$$

Thus, $m \vee n, m \wedge n \in G^{[\omega,\delta]}$, so $G^{[\omega,\delta]}$ is a sublattice of T.

Now, assume that G is a PFI of T. Let $m, n \in G^{[\omega,\delta]}$ and $l \in T$. Then as above,

$$\varphi_G^2(m) \geq \omega^2, \quad \eta_G^2(m) \leq \delta^2.$$

Therefore,

$$m \vee l \in G^{[\omega,\delta]}, \quad \varphi_G^2(m \wedge l) \geq \min\{\varphi_G^2(m), \varphi_G^2(l)\} \geq \omega^2,$$
$$\eta_G^2(m \wedge l) \leq \max\{\eta_G^2(m), \eta_G^2(l)\} \leq \delta^2.$$

Hence, $m \wedge l \in G^{[\omega,\delta]}$, and so $G^{[\omega,\delta]}$ is an ideal of T.

Conversely, assume that each nonempty (ω, δ)-level subset $G^{[\omega,\delta]}$ is a sublattice of T. Let $m, n \in G^{[\omega,\delta]}$. Set

$$\varphi_G^2(m) = \omega_1^2, \quad \varphi_G^2(n) = \omega_2^2, \quad \eta_G^2(m) = \delta_1^2, \quad \eta_G^2(n) = \delta_2^2.$$

Without loss of generality, assume $\omega_2^2 \leq \omega_1^2$ and $\delta_2^2 \geq \delta_1^2$. Then

$$m \in G^{[\omega_2,\delta_2]}, \quad n \in G^{[\omega_2,\delta_2]}.$$

Since $G^{[\omega_2,\delta_2]}$ is a sublattice, both $m \vee n$ and $m \wedge n$ belong to $G^{[\omega_2,\delta_2]}$. Hence

$$\varphi_G^2(m \vee n) \geq \omega_2^2 = \min\{\varphi_G^2(m), \varphi_G^2(n)\},$$
$$\varphi_G^2(m \wedge n) \geq \omega_2^2 = \min\{\varphi_G^2(m), \varphi_G^2(n)\},$$
$$\eta_G^2(m \vee n) \leq \delta_2^2 = \max\{\eta_G^2(m), \eta_G^2(n)\},$$
$$\eta_G^2(m \wedge n) \leq \delta_2^2 = \max\{\eta_G^2(m), \eta_G^2(n)\}.$$

Thus, G is a PFL of T. The proof for the PFI case is similar. □

Proposition 3.16. *A PFS G of a lattice T is a PFL of T if and only if all the nonempty upper and lower level subsets $U(\varphi_G, \omega)$ and $L(\eta_G, \delta)$ are sublattices of T, for each $(\omega, \delta) \in [0, 1] \times [0, 1]$ with $\omega^2 + \delta^2 \leq 1$.*

Theorem 3.17. *Let T be a lattice and $G = (\varphi_G, \eta_G)$ a Pythagorean fuzzy ideal of T. Then G is a PFPI if and only if every non-empty upper level set $U(\varphi_G, \eta_G)$ and every nonempty lower level set $L((\varphi_G, \eta_G)$ is the prime ideal of T.*

Definition 3.18. Let G be a Pythagorean fuzzy sublattice of T. Then G is said to be a **Pythagorean fuzzy convex sublattice** if for every interval $[a, b] \subseteq T$ and for all $m \in [a, b]$,

$$\varphi_G^2(m) \geq \min\{\varphi_G^2(a), \varphi_G^2(b)\}, \qquad \eta_G^2(m) \leq \max\{\eta_G^2(a), \eta_G^2(b)\}.$$

Theorem 3.19. *Let G be a Pythagorean fuzzy lattice of T. Then, G is a Pythagorean fuzzy convex sublattice of T if and only if each level set $G^{[\omega,\delta]}$ for $\omega \in \text{Im}(\varphi_G)$ and $\delta \in \text{Im}(\eta_G)$ is a convex sublattice of T.*

Proof. Assume that G is a Pythagorean fuzzy convex sublattice of T. By Theorem 3.15, $G^{[\omega,\delta]}$ is a sublattice of T. To show that $G^{[\omega,\delta]}$ is convex, let $\omega^2 \in \text{Im}(\varphi_G^2)$ and $\delta^2 \in \text{Im}(\eta_G^2)$.

For any $[a, b] \subseteq G^{[\omega,\delta]}$ we have

$$\varphi_G^2(a) \geq \omega^2, \quad \varphi_G^2(b) \geq \omega^2, \quad \eta_G^2(a) \leq \delta^2, \quad \eta_G^2(b) \leq \delta^2.$$

Thus

$$\min\{\varphi_G^2(a), \varphi_G^2(b)\} \geq \omega^2, \qquad \max\{\eta_G^2(a), \eta_G^2(b)\} \leq \delta^2.$$

Since G is a Pythagorean fuzzy convex sublattice, for all $m \in [a, b]$,

$$\varphi_G^2(m) \geq \min\{\varphi_G^2(a), \varphi_G^2(b)\}, \qquad \eta_G^2(m) \leq \max\{\eta_G^2(a), \eta_G^2(b)\}.$$

Hence by definition 3.18

$$\varphi_G^2(m) \geq \omega^2, \qquad \eta_G^2(m) \leq \delta^2,$$

so $m \in G^{[\omega,\delta]}$. Therefore, $G^{[\omega,\delta]}$ is convex in T.

Conversely, assume that $G^{[\omega,\delta]}$ is a convex sublattice of T, for $\omega \in \text{Im}(\varphi_G)$ and $\delta \in \text{Im}(\eta_G)$. Let $[a, b]$ be an interval in T. Set

$$\min\{\varphi_G^2(a), \varphi_G^2(b)\} = \omega^2, \qquad \max\{\eta_G^2(a), \eta_G^2(b)\} = \delta^2.$$

Then

$$a, b \in G^{[\omega,\delta]}.$$

Since $G^{[\omega,\delta]}$ is convex, it follows that $m \in G^{[\omega,\delta]}$ for all $m \in [a, b]$. Equivalently,

$$\varphi_G^2(m) \geq \min\{\varphi_G^2(a), \varphi_G^2(b)\}, \qquad \eta_G^2(m) \leq \max\{\eta_G^2(a), \eta_G^2(b)\}, \quad \forall m \in [a, b].$$

Thus, G is a Pythagorean fuzzy convex sublattice of T. □

4 Pythagorean fuzzy Ideals and homomorphisms

We study the properties of Pythagorean fuzzy ideals and Pythagorean fuzzy prime ideals under a lattice homomorphism.

Definition 4.1. Let $f : L \to L'$ be a mapping between two lattices L and L', and let

$$P = \{<m, \varphi_P(m), \eta_P(x)> | m \in L\}$$

Then the **image** of P under f is defined as

$$f(P) = \{\langle y, f(\varphi_P)(y), f(\eta_P)(y)\rangle \mid y \in L'\},$$

where

$$f(\varphi_P^2)(y) = \begin{cases} \sup\{\varphi_P^2(m) \mid m \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

and

$$f(\eta_P^2)(y) = \begin{cases} \inf\{\eta_P^2(m) \mid m \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{if } f^{-1}(y) = . \end{cases}$$

Conversely, if

$$P' = \{\langle y, \varphi_{P'}^2(y), \eta_{P'}^2(y)\rangle \mid y \in L'\} \in PFS(L'),$$

then the **inverse image** of P' under f is given by

$$f^{-1}(P') = \{\langle m, f^{-1}(\varphi_{P'}^2)(m), f^{-1}(\eta_{P'}^2)(m) \mid m \in L\},$$

where

$$f^{-1}(\varphi_{P'}^2)(m) = \varphi_{P'}^2(f(m)), \quad f^{-1}(\eta_{P'}^2)(m) = \eta_{P'}^2(f(m)).$$

Theorem 4.2. *Let L_1 and L_2 be two lattices. Let $f : L \to L'$ be an epimorphism, and let $P = \{< m, \varphi_P, \eta_P > /m \in L_1\}$ be a Pythagorean fuzzy ideal of L_1 Then, $f(P)$ is a Pythagorean fuzzy ideal of L_2.*

Proof. Since f is an epimorphism, we have $f(L_1) = L_2$, and hence $Let y_1, y_2 \in L_2$. Suppose $y_1 = f(m_1)$ and $y_2 = f(m_2)$ for some $m_1, m_2 \in L_1$.

The image of P under f is given by

$$f(P) = \{\langle y, f(\varphi_P)(y), f(\eta_P)(y)\rangle \mid y \in L_2\},$$

where

$$f(\varphi_P^2)(y) = \begin{cases} \sup\{\varphi_P^2(m) \mid m \in f^{-1}(y)\}, & f^{-1}(y) \neq \emptyset, \\ 0, & f^{-1}(y) = \emptyset, \end{cases}$$

$$f(\eta_P^2)(y) = \begin{cases} \inf\{\eta_P^2(m) \mid m \in f^{-1}(y)\}, & f^{-1}(y) \neq \emptyset, \\ 1, and f^{-1}(y) = \emptyset. \end{cases}$$

Verification of conditions:

$$\begin{aligned} f(\varphi_P^2)(y \vee z) &= \sup\{\varphi_P^2(m) \mid m \in f^{-1}(y \vee z)\} \\ &= \sup\{\varphi_P^2(u \vee v) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &\geq \sup\{\min(\varphi_P^2(u), \varphi_P^2(v)) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &= \min\Big\{\sup\{\varphi_P^2(u) \mid u \in f^{-1}(y)\}, \sup\{\varphi_P^2(v) \mid v \in f^{-1}(z)\}\Big\} \\ &= \min\{f(\varphi_P^2)(y), f(\varphi_P^2)(z)\}. \end{aligned}$$

$$\begin{aligned} f(\varphi_P^2)(y \wedge z) &= \sup\{\varphi_P^2(m) \mid m \in f^{-1}(y \wedge z)\} \\ &= \sup\{\varphi_P^2(u \wedge v) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &\geq \sup\{\max(\varphi_P^2(u), \varphi_P^2(v)) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &= \max\Big\{\sup\{\varphi_P^2(u) \mid u \in f^{-1}(y)\}, \sup\{\varphi_P^2(v) \mid v \in f^{-1}(z)\}\Big\} \\ &= \max\{f(\varphi_P^2)(y), f(\varphi_P^2)(z)\}. \end{aligned}$$

$$\begin{aligned} f(\eta_P^2)(y \vee z) &= \inf\{\eta_P^2(m) \mid m \in f^{-1}(y \vee z)\} \\ &= \inf\{\eta_P^2(u \vee v) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &\leq \inf\{\max(\eta_P^2(u), \eta_P^2(v)) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &= \max\Big\{\inf\{\eta_P^2(u) \mid u \in f^{-1}(y)\}, \inf\{\eta_P^2(v) \mid v \in f^{-1}(z)\}\Big\} \\ &= \max\{f(\eta_P^2)(y), f(\eta_P^2)(z)\}. \end{aligned}$$

$$\begin{aligned} f(\eta_P^2)(y \wedge z) &= \inf\{\eta_P^2(m) \mid m \in f^{-1}(y \wedge z)\} \\ &= \inf\{\eta_P^2(u \wedge v) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &\leq \inf\{\min(\eta_P^2(u), \eta_P^2(v)) \mid u \in f^{-1}(y),\, v \in f^{-1}(z)\} \\ &= \min\Big\{\inf\{\eta_P^2(u) \mid u \in f^{-1}(y)\}, \inf\{\eta_P^2(v) \mid v \in f^{-1}(z)\}\Big\} \\ &= \min\{f(\eta_P^2)(y), f(\eta_P^2)(z)\}. \end{aligned}$$

Since all required conditions are satisfied, we conclude that $f(P)$ is a **Pythagorean fuzzy ideal** of L_2. □

Theorem 4.3. *If $f : L \to L'$ is a lattice homomorphism and P' is a PFI of L', then $f^{-1}(P')$ is a PFI of L.*

Proof. Let

$$P' = \{\langle y, \varphi^2_{P'}(y), \eta^2_{P'}(y)\rangle \mid y \in L'\}$$

be a PFI of L'. Then

$$f^{-1}(P') = \{\langle m, f^{-1}(\varphi^2_{P'})(m), f^{-1}(\eta^2_{P'})(m)\rangle \mid m \in L\},$$

where

$$f^{-1}\left(\varphi^2_{P'}\right)(m) = \varphi^2_{P'}(f(m)) \quad \text{and} \quad f^{-1}\left(\eta^2_{P'}\right)(m) = \eta^2_{P'}(f(m)).$$

Since f is a lattice homomorphism and P' is a PFI of L', we have

$$\begin{aligned} f^{-1}(\varphi^2_{P'})(m \vee y) &= \varphi^2_{P'}(f(m \vee y)) \\ &= \varphi^2_{P'}(f(m) \vee f(y)) \\ &\geq \text{min}\{\varphi^2_{P'}(f(m)), \varphi^2_{P'}(f(y))\} \\ &= \text{min}\{f^{-1}(\varphi^2_{P'})(m), f^{-1}(\varphi^2_{P'})(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\varphi^2_{P'})(m \wedge y) &= \varphi^2_{P'}(f(m \wedge y)) \\ &= \varphi^2_{P'}(f(m) \wedge f(y)) \\ &\geq \text{max}\{\varphi^2_{P'}(f(x)), \varphi^2_{P'}(f(y))\} \\ &= \text{max}\{f^{-1}(\varphi^2_{P'})(m), f^{-1}(\varphi^2_{P'})(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\eta^2_{P'})(m \vee y) &= \eta^2_{P'}(f(m \vee y)) \\ &= \eta^2_{P'}(f(m) \vee f(y)) \\ &\leq \text{max}\{\eta^2_{P'}(f(m)), \eta^2_{P'}(f(y))\} \\ &= \text{max}\{f^{-1}(\eta^2_{P'})(m), f^{-1}(\eta^2_{P'})(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\eta^2_{P'})(m \wedge y) &= \eta^2_{P'}(f(m \wedge y)) \\ &= \eta^2_{P'}(f(m) \wedge f(y)) \\ &\leq \text{min}\{\eta^2_{P'}(f(m)), \eta^2_{P'}(f(y))\} \\ &= \text{min}\{f^{-1}(\eta^2_{P'})(m), f^{-1}(\eta^2_{P'})(y)\}. \end{aligned}$$

Therefore, $f^{-1}(P')$ is the PFI of L. □

Theorem 4.4. *Let $f : L \longrightarrow L'$ be an onto mapping, and let P and P' be Pythagorean fuzzy sets of lattices L and L', respectively. Then:*

(i) $f(f^{-1}(P')) = P'$,

(ii) $P \subseteq f^{-1}(f(P))$.

Proof. (i) We have

$$\begin{aligned} f\big(f^{-1}(\varphi^2_{P'})\big)(y) &= \sup\{f^{-1}(\varphi^2_{P'})(m) \mid m \in f^{-1}(y)\} \\ &= \sup\{\varphi^2_{P'}(f(m)) \mid m \in L,\ f(m) = y\} \\ &= \varphi^2_{P'}(y). \end{aligned}$$

Similarly,

$$f\big(f^{-1}(\eta^2_{P'})\big)(y) = \eta^2_{P'}(y).$$

Hence, $f(f^{-1}(P')) = P'$.

(ii) For $m \in L$, we have

$$\begin{aligned} f^{-1}(f(\varphi^2_P))(m) &= \sup\{\varphi^2_P(u) \mid u \in f^{-1}(f(m))\} \\ &\geq \varphi^2_P(m), \end{aligned}$$

and

$$\begin{aligned} f^{-1}(f(\eta^2_P))(m) &= \inf\{\eta^2_P(u) \mid u \in f^{-1}(f(m))\} \\ &\leq \eta^2_P(m). \end{aligned}$$

Thus, $P \subseteq f^{-1}(f(P))$.

□

Definition 4.5. Let $f : L \to L'$ be a mapping between two lattices L andL' and $P \in PFS\,(L)$ where $P = \{< m, \varphi_P, \eta_P > /m \in L\}$. Then P is said to be *f invarient* if $f(m) = f(y) \Rightarrow \varphi^2_P(m) = \varphi^2_P(y)$ and $\eta^2_P(m) = \eta^2_P(y)$

Proposition 4.6. *If a PFS P is f invarient, then $f^{-1}(f(P)) = P$*

Theorem 4.7. *Let $f : L \longrightarrow L'$ be a mapping and P_1, P_2 be PFSs of the lattice L, and P'_1, P'_2 be PFSs of L'. Then:*

(i) $P_1 \subseteq P_2 \;\Rightarrow\; f(P_1) \subseteq f(P_2)$.

(ii) $P'_1 \subseteq P'_2 \;\Rightarrow\; f^{-1}(P'_1) \subseteq f^{-1}(P'_2)$.

Proof. Let

$$P_1 = \{\langle m, \varphi_{P_1}(m), \eta_{P_1}(x)\rangle \mid x \in L\}, \quad P_2 = \{\langle x, \varphi_{P_2}(x), \eta_{P_2}(x)\rangle \mid x \in L\}$$

be two PFSs of L.

If $P_1 \subseteq P_2$, then

$$\varphi^2_{P_1}(x) \leq \varphi^2_{P_2}(x), \quad \eta^2_{P_1}(x) \geq \eta^2_{P_2}(x), \quad \forall x \in L.$$

Now,

$$f(P_1) = \{\langle y, f(\varphi^2_{P_1})(y), f(\eta^2_{P_1})(y)\rangle \mid y \in L'\}, \quad f(P_2) = \{\langle y, f(\varphi^2_{P_2})(y), f(\eta^2_{P_2})(y)\rangle \mid y \in L'\}.$$

For each $y \in L'$,

$$\begin{aligned} f(\varphi^2_{P_1})(y) &= \sup\{\varphi^2_{P_1}(x) \mid x \in f^{-1}(y)\} \\ &\leq \sup\{\varphi^2_{P_2}(x) \mid x \in f^{-1}(y)\} \\ &= f(\varphi^2_{P_2})(y), \end{aligned}$$

and similarly,

$$\begin{aligned} f(\eta^2_{P_1})(y) &= \inf\{\eta^2_{P_1}(x) \mid x \in f^{-1}(y)\} \\ &\geq \inf\{\eta^2_{P_2}(x) \mid x \in f^{-1}(y)\} \\ &= f(\eta^2_{P_2})(y). \end{aligned}$$

Hence $f(P_1) \subseteq f(P_2)$.

For (2), suppose $P_1' \subseteq P_2'$, i.e.,

$$\varphi^2_{P_1'}(x) \leq \varphi^2_{P_2'}(x), \quad \eta^2_{P_1'}(x) \geq \eta^2_{P_2'}(x), \quad \forall x \in L'.$$

Now,

$$f^{-1}(P_1') = \{\langle x, f^{-1}(\varphi^2_{P_1'})(x), f^{-1}(\eta^2_{P_1'})(x)\rangle \mid x \in L\}.$$

For each $x \in L$,

$$\begin{aligned} f^{-1}(\varphi^2_{P_1'})(x) &= \varphi^2_{P_1'}(f(x)) \\ &\leq \varphi^2_{P_2'}(f(x)) \\ &= f^{-1}(\varphi^2_{P_2'})(x), \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\eta^2_{P_1'})(x) &= \eta^2_{P_1'}(f(x)) \\ &\geq \eta^2_{P_2'}(f(x)) \\ &= f^{-1}(\eta^2_{P_2'})(x). \end{aligned}$$

Thus $f^{-1}(P_1') \subseteq f^{-1}(P_2')$. □

Theorem 4.8. *Let $f : L_1 \to L_2$ be an epimorphism and P_1, P_2 be PFIs of L_1. Then*

(i) $f(P_1 \cap P_2) \subseteq f(P_1) \cap f(P_2)$.

(ii) $f(P_1 \cap P_2) = f(P_1) \cap f(P_2)$ if P_1 and P_2 are f-invariant.

Proof. Since $P_1 \cap P_2 \subseteq P_1$ and $P_1 \cap P_2 \subseteq P_2$, it follows that

$$f(P_1 \cap P_2) \subseteq f(P_1), \qquad f(P_1 \cap P_2) \subseteq f(P_2),$$

hence $f(P_1 \cap P_2) \subseteq f(P_1) \cap f(P_2)$.

Now, assume P_1 and P_2 are f-invariant. Then

$$f(P_1 \cap P_2) = \{\langle y, f(\varphi^2_{P_1 \cap P_2})(y), f(\eta^2_{P_1 \cap P_2})(y)\rangle \mid y \in L_2\},$$

where

$$\begin{aligned} f(\varphi^2_{P_1 \cap P_2})(y) &= \sup_{x \in f^{-1}(y)} \min\{\varphi^2_{P_1}(x), \varphi^2_{P_2}(x)\}, \\ f(\eta^2_{P_1 \cap P_2})(y) &= \inf_{x \in f^{-1}(y)} \max\{\eta^2_{P_1}(x), \eta^2_{P_2}(x)\}. \end{aligned}$$

Since P_1, P_2 are f-invariant, the above reduce to

$$f(\varphi^2_{P_1 \cap P_2})(y) = \min\{f(\varphi^2_{P_1})(y), f(\varphi^2_{P_2})(y)\}, \quad f(\eta^2_{P_1 \cap P_2})(y) = \max\{f(\eta^2_{P_1})(y), f(\eta^2_{P_2})(y)\}.$$

Thus

$$f(P_1 \cap P_2) = f(P_1) \cap f(P_2).$$

□

4.1 Lemma

Let $f : L \to L'$ be an onto mapping. Let A be a Pythagorean fuzzy lattice with a supremum property in L. Then

$$U(f(\varphi^2_A), \eta) = f(U(\varphi^2_A, \eta)), \qquad L(f(\varphi^2_A), \eta) = f(L(\varphi^2_A, \eta)).$$

Proof. Let $y \in U(f(\varphi_A^2), \eta)$. Then

$$f(\varphi_A^2)(y) \geq \eta \quad \Longrightarrow \quad \sup_{x \in f^{-1}(y)} \varphi_A^2(x) \geq \eta.$$

Since A has the supremum property, there exists $x_0 \in f^{-1}(y)$ such that

$$\varphi_A^2(x_0) = \sup_{x \in f^{-1}(y)} \varphi_A^2(x), \quad f(x_0) = y.$$

Thus $\varphi_A^2(x_0) \geq \eta$, so $x_0 \in U(\varphi_A^2, \eta)$ and hence $y \in f(U(\varphi_A^2, \eta))$. Therefore

$$U(f(\varphi_A^2), \eta) \subseteq f(U(\varphi_A^2, \eta)).$$

Conversely, let $y \in f(U(\varphi_A^2, \eta))$. Then there exists $x_0 \in U(\varphi_A^2, \eta)$ such that $y = f(x_0)$. Now $\varphi_A^2(x_0) \geq \eta$, which implies

$$\sup_{x \in f^{-1}(y)} \varphi_A^2(x) \geq \eta \quad \Longrightarrow \quad f(\varphi_A^2)(y) \geq \eta.$$

Hence $y \in U(f(\varphi_A^2), \eta)$. Thus

$$f(U(\varphi_A^2, \eta)) \subseteq U(f(\varphi_A^2), \eta).$$

Combining both, we obtain

$$U(f(\varphi_A^2), \eta) = f(U(\varphi_A^2, \eta)).$$

Now let $y \in L(f(\varphi_A^2), \eta)$. Then

$$f(\varphi_A^2)(y) \leq \eta \quad \Longrightarrow \quad \inf_{x \in f^{-1}(y)} \varphi_A^2(x) \leq \eta.$$

By the supremum/infimum property, there exists $x_0 \in f^{-1}(y)$ such that

$$\varphi_A^2(x_0) = \inf_{x \in f^{-1}(y)} \varphi_A^2(x), \quad f(x_0) = y.$$

Thus $\varphi_A^2(x_0) \leq \eta$, so $x_0 \in L(\varphi_A^2, \eta)$ and hence $y \in f(L(\varphi_A^2, \eta))$. Therefore

$$L(f(\varphi_A^2), \eta) \subseteq f(L(\varphi_A^2, \eta)).$$

Conversely, let $y \in f(L(\varphi_A^2, \eta))$. Then there exists $x_0 \in L(\varphi_A^2, \eta)$ such that $y = f(x_0)$. Now $\varphi_A^2(x_0) \leq \eta$, which implies

$$\inf_{x \in f^{-1}(y)} \varphi_A^2(x) \leq \eta \quad \Longrightarrow \quad f(\varphi_A^2)(y) \leq \eta.$$

Hence $y \in L(f(\varphi_A^2), \eta)$. Thus

$$f(L(\varphi_A^2, \eta)) \subseteq L(f(\varphi_A^2), \eta).$$

Combining both, we obtain

$$L(f(\varphi_A^2), \eta) = f(L(\varphi_A^2, \eta)).$$

□

Definition 4.9. A PFS P of L is said to have the *sup-property* if for every subset $S \subseteq L$ there exists $x_0 \in S$ such that

$$\sup\{\varphi_P^2(x) \mid x \in S\} = \varphi_P^2(x_0).$$

Similarly, P has the *inf-property* if for every $S \subseteq L$ there exists $x_0 \in S$ such that

$$\inf\{\eta_P^2(x) \mid x \in S\} = \eta_P^2(x_0).$$

—

Theorem 4.10. *Let $f : L \to L'$ be onto. If P is a PFL of L with supremum and infimum properties, then*
$$(f(P))^{[\omega,\delta]} = f(P^{[\omega,\delta]}).$$

Proof. ($\subseteq$) Let $y \in (f(P))^{[\omega,\delta]}$. Then by definition 4.1,
$$f(\varphi_P^2)(y) \geq \omega \quad \text{and} \quad f(\eta_P^2)(y) \leq \delta.$$
By the supremum and infimum properties of P, there exists $x_0 \in f^{-1}(y)$ such that
$$\varphi_P^2(x_0) = f(\varphi_P^2)(y), \quad \eta_P^2(x_0) = f(\eta_P^2)(y).$$
Hence, $\varphi_P^2(x_0) \geq \omega$ and $\eta_P^2(x_0) \leq \delta$, so $x_0 \in P^{[\omega,\delta]}$. Therefore, $y = f(x_0) \in f(P^{[\omega,\delta]})$. Thus,
$$(f(P))^{[\omega,\delta]} \subseteq f(P^{[\omega,\delta]}).$$

($\supseteq$) Conversely, let $y \in f(P^{[\omega,\delta]})$. Then, there exists $x \in P^{[\omega,\delta]}$ such that $y = f(x)$. Since $x \in P^{[\omega,\delta]}$, we have
$$\varphi_P^2(x) \geq \omega \quad \text{and} \quad \eta_P^2(x) \leq \delta.$$
By the definition of image under f,
$$f(\varphi_P^2)(y) \geq \varphi_P^2(x) \geq \omega, \quad f(\eta_P^2)(y) \leq \eta_P^2(x) \leq \delta.$$
Thus, $y \in (f(P))^{[\omega,\delta]}$. Therefore,
$$f(P^{[\omega,\delta]}) \subseteq (f(P))^{[\omega,\delta]}.$$

Combining both inclusions gives
$$(f(P))^{[\omega,\delta]} = f(P^{[\omega,\delta]}).$$
□

Theorem 4.11. *Let $f : L \to L'$ be a lattice epimorphism. If A is a PFPI of L with sup- and inf-properties, then $f(A)$ is a PFPI of L'.*

Proof. Since A is a PFPI of L, Theorem 3.17 ensures that $U(\varphi_A^2, \omega)$ and $L(\eta_A^2, \delta)$ are the prime ideals in L which implies that $f(U(\varphi_A^2, \omega))$ and $f(L(\eta_A^2, \delta))$ are prime ideals of $L^{'}$. Therefore, $f(A)$ is the PFPI of L'. □

Theorem 4.12. *Let L_1 and L_2 are two lattices. Let $f : L_1 \to L_2$ is an epimorphism , and $A = \{< x, \varphi_P, \eta_P > /x \in X$ is a Pythagorean fuzzy prime ideal of L_1, and let P be f invarient then $f(A)$ is a pythagorean fuzzy Prime ideal of L_2.*

Proof. Assume, to the contrary, that $f(A)$ is not a PFPI of L_2. Then there exist $a, b \in L_1$ such that
$$\varphi_{f(A)}^2(f(a) \wedge f(b)) \neq \varphi_{f(A)}^2(f(a)) \quad \text{and} \quad \varphi_{f(A)}^2(f(a) \wedge f(b)) \neq \varphi_{f(A)}^2(f(b)),$$
or
$$\eta_{f(A)}^2(f(a) \wedge f(b)) \neq \eta_{f(A)}^2(f(a)) \quad \text{and} \quad \eta_{f(A)}^2(f(a) \wedge f(b)) \neq \eta_{f(A)}^2(f(b)).$$
Since $f(A)$ is assumed to be a Pythagorean fuzzy ideal of L_2, it follows that
$$\varphi_{f(A)}^2(f(a) \wedge f(b)) > \varphi_{f(A)}^2(f(a)) \text{ or } \varphi_{f(A)}^2(f(a) \wedge f(b)) > \varphi_{f(A)}^2(f(b)),$$
or
$$\eta_{f(A)}^2(f(a) \wedge f(b)) < \eta_{f(A)}^2(f(a)) \text{ or } \eta_{f(A)}^2(f(a) \wedge f(b)) < \eta_{f(A)}^2(f(b)).$$
Since f is a lattice homomorphism, we have
$$\varphi_A^2(a \wedge b) > \varphi_A^2(a) \text{ or } \varphi_A^2(a \wedge b) > \varphi_A^2(b),$$
or
$$\eta_A^2(a \wedge b) < \eta_A^2(a) \text{ or } \eta_A^2(a \wedge b) < \eta_A^2(b).$$
However, this contradicts the assumption that A is a Pythagorean fuzzy prime ideal of L_1. Hence, our assumption is false, and $f(A)$ must be a Pythagorean fuzzy prime ideal of L_2. □

Theorem 4.13. *If $f : L \to L'$ is a lattice homomorphism, then $f^{-1}(A')$ is a PFPI of L whenever A' is a PFPI of L'.*

Let A' be a Pythagorean fuzzy prime ideal of L'. Then for $a, b \in L$ we have

$$f^{-1}\big(\varphi^2_{A'}(a \wedge b)\big) = \varphi^2_{A'}\big(f(a \wedge b)\big) = \varphi^2_{A'}\big(f(a) \wedge f(b)\big).$$

Since A' is a Pythagorean fuzzy prime ideal of L', it follows that

$$\varphi^2_{A'}(f(a) \wedge f(b)) = \varphi^2_{A'}(f(a)) \text{ or } \varphi^2_{A'}(f(b)).$$

This implies

$$f^{-1}\big(\varphi^2_{A'}(f(a) \wedge f(b))\big) = f^{-1}\big(\varphi^2_{A'}(f(a))\big) \text{ or } f^{-1}\big(\varphi^2_{A'}(f(b))\big).$$

Similarly, for the non-membership function we obtain

$$\eta^2_{A'}(f(a) \wedge f(b)) = \eta^2_{A'}(f(a)) \text{ or } \eta^2_{A'}(f(b)),$$

which implies

$$f^{-1}\big(\eta^2_{A'}(f(a) \wedge f(b))\big) = f^{-1}\big(\eta^2_{A'}(f(a))\big) \text{ or } f^{-1}\big(\eta^2_{A'}(f(b))\big).$$

Hence, $f^{-1}(A')$ is a Pythagorean fuzzy prime ideal of L.

sectionSignificance and Applications
The study of Pythagorean fuzzy sublattices and ideals is motivated by the need to handle uncertainty and hesitation, which are common in practical problems but cannot be fully captured by classical or intuitionistic fuzzy structures. The additional degree of freedom provided by the Pythagorean fuzzy framework allows for a more expressive modeling of vague information and decision-making scenarios.

These concepts are particularly useful in multi-criteria decision-making (e.g., supplier selection, medical diagnosis, and risk analysis), knowledge representation in data mining with incomplete information, and optimization problems involving uncertain constraints. Unlike traditional lattice-theoretic approaches, Pythagorean fuzzy sublattices and ideals provide a richer framework that incorporates hesitation, thereby offering both theoretical generalization and practical applicability in decision-making problems.

Conclusion

This study introduces the notions of Pythagorean fuzzy sublattices (PFL) and Pythagorean fuzzy ideals (PFI) within lattice theory, establishing their fundamental properties, level sets, convex variants, and behavior under lattice homomorphisms. Compared to intuitionistic fuzzy sublattices, PFLs offer greater flexibility by relaxing the membership–non-membership constraint to $\mu^2+\nu^2 \leq 1$, thereby capturing uncertainty more effectively. This generalization not only enriches the algebraic foundations of fuzzy lattice theory but also provides tools for modelling real-world problems in decision-making, optimization, and knowledge representation, where classical approaches fall short. Future work may extend these ideas to interval-valued Pythagorean fuzzy structures, exploring their applications in computational intelligence and hierarchical system modeling.

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Received: 2025-06-11
Accepted: 2025-11-09