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r-Almost Newton-Ricci-Yamabe solitons on $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold

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Abstract In this paper, we develop the concept of r-Almost Newton-Ricci-Yamabe solitons (in brief r-ANRY soliton) immersed in a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold. We deduce the minimal and totally geodesic criteria for hypersurface of a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold in terms of the r-ANRY solitons. We also discuss the triviality of the gradient r-ANRY soliton in the case of a compact $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold. Finally, we demonstrate the completeness and non-compactness of the r-Newton-Ricci-Yamabe soliton on hypersuface of $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold.

1 Background

For the first time, Ricci flow was introduced by Hamilton [20] in 1988. The Ricci soliton and Yamabe soliton, respectively, are the limit solutions of the Ricci flow and Yamabe flow. In fact, the Yamabe soliton coincides with the Ricci soliton for dimension $n = 2$, but when $n > 2$, the Yamabe and Ricci solitons are not the same, the Yamabe soliton retains the conformal class.

Many geometers over the past twenty years, the theory of geometric flows, including Ricci flow and Yamabe flow, has served as a source of inspiration. A certain group of solutions on which the metric evolves through dilation and diffeomorphisms plays a crucial role in the investigation of the singularities of the flows because they appear as acceptable singularity analogues. They are termed as solitons solutions.

The construction of Ricci-Yamabe solitons from a geometric flow that is a scalar composition of Ricci and Yamabe flow was recently discussed by Siddiqi et al. in [37]. The Ricci-Yamabe flow of the form (δ, ε) is another name for this. The semi-Riemannian multiple metric that gives rise to the Ricci-Yamabe flow is represented by

$$\partial_t g(t) = -2\delta\mathcal{R}ic(t) + \varepsilon\mathcal{R}(t)g(t), \quad g_0 = g(0), \quad t \in (a, b), \tag{1.1}$$

where the terms $\mathcal{R}ic$ and $\mathcal{R}$ refer for the Ricci tensor and scalar curvature, respectively. Additionally, the authors in [19] Guler treated the Ricci-Yamabe flow of type (δ, ε). In the Ricci-Yamabe flow, a Ricci-Yamabe soliton is one that evolves exclusively by diffeomorphism and scales by a single parameter group. A Ricci-Yamabe soliton is a data $(g, F, \Lambda, \delta, \varepsilon)$ obeying the Riemannain manifold (M, g).

$$\frac{1}{2}\mathfrak{L}_F g + \delta\mathcal{R}ic = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right) g, \tag{1.2}$$

where $\mathfrak{L}_F$ shows the Lie derivative along the vector field F, and Λ, δ and ε are real scalars. A Ricci-Yamabe soliton is called *shrinking*, *expanding* or *steady*, according to $\Lambda > 0$, $\Lambda < 0$ or $\Lambda = 0$, respectively.

Also, if (1.2) holds for $\Lambda, \delta, \varepsilon$ smooth functions then the soliton is called almost Ricci-Yamabe soliton [47, 34].

If there exists a smooth function $\gamma : M \to \mathbb{R}$ such that $F = \nabla\gamma$, then the (δ, ε)-type Ricci-Yamabe soliton is called a *gradient Ricci-Yamabe soliton* [30] of type (δ, ε), denoted by $(M, g, \gamma, \Lambda, \delta, \varepsilon)$ and in this case (1.4) takes the form

$$\hbar ess(\gamma) + \delta\mathcal{R}ic = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right) g, \tag{1.3}$$

where $\hbar ess$ is the Hessain of the function γ, and γ is called potential of the gradient Ricci-Yamabe soliton of type (δ, ε).

Example 1.1. Let us take the example of the Einstein soliton [41], which produces solutions to the Einstein flow that are self-similar in such a manner that

$$\partial_t g(t) = -2\left(\mathcal{R}ic - \frac{\mathcal{R}}{2}g\right).$$

Therefore, an Einstein soliton occurs as the limit of the Einstein flow solution [14], such that

$$\frac{1}{2}\mathfrak{L}_F g + \mathcal{R}ic = \left(\Lambda - \frac{\mathcal{R}}{2}\right) g. \tag{1.4}$$

Comparing equation (1.2) and (1.4), we find $(1, 1)$-type-Ricci-Yamabe soliton.

Based on ideas of Cunha et al. ([10, 11]), consider a connected and oriented hypersurface called $\mathcal{M}^n$ that is submerged in a $(n + 1)$-dimensional Riemannian manifold called $\mathcal{N}^{n+1}$. For some $0 \leq r \leq n$, we declare that $\mathcal{M}^n$ is a gradient r-Almost Newton-Ricci-Yamabe soliton (gradient r-ANRY soliton) if the smooth function $\gamma : \mathcal{M}^n \to \mathbb{R}$ exists and the following equation retains:

$$\mathcal{P}_{\mathrm{r}} \circ \hbar\mathrm{ess}(\gamma) + \delta\mathcal{R}\mathrm{ic} = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right) g, \tag{1.5}$$

where g denotes the Riemannian metric brought about by immersion, Λ indicates a smooth function on $\mathcal{M}$, $\mathcal{R}$ symbolizes the scalar curvature of $\mathcal{M}$ with respect to g. In addition, $\mathcal{P}_r \circ \hbar ess(\gamma)$ illustrates the tensor generated by

$$\mathcal{P}_r \circ \hbar ess(\gamma)(U, V) = \langle \mathcal{P}_r \nabla_U \nabla\gamma, V\rangle,$$

for tangent vector fields $U, V \in \mathfrak{X}(\mathcal{M})$. Moreover, Siddiqi et al. also studied the notions r-ANR soliton [35, 36, 32], r-ANY [38, 40], and k-ANE [46] all are merely closed to this idea of r-ANRY [10, 11]. Eventually, r-ANRY is a generalization of r-ANR, r-ANY, and k-ANE.

The study of equation (1.5) is fascinating since a gradient r-ANRY soliton reduces to a gradient RY soliton when $r = 0$. Trivial refers to the gradient r-ANRY soliton [12], whenever the potential γ is constant. It is considered nontrivial if not. Additionally, we refer to the gradient r-ANRY soliton as a *gradient r-NRY soliton* when Λ is a constant. Moreover, Shaikh et al. also explored some crucial characterization of the Ricci soliton and Yamabe soliton for more details see ([2, 3, 4]).

On the other hand, $\mathcal{D}_\alpha$-homothetic deformations were introduced by Tanno [52]. The $\mathcal{D}_\alpha$-homothetic deformation of generalized (k, μ)-space forms was explored by Carriazo and Martin-Molina [17]. The $\mathcal{D}_\alpha$-homothetic deformation of nearly normal contact metric manifolds was researched by De and Ghosh [29]. This paper is essentially linked to Siddiqi et al. [28], who also determined some extrinsic and intrinsic invariant of $\mathcal{D}_\alpha$-homothetic deformed manifold. Several structures of a class of $\mathcal{K}$-contact manifolds were investigated by Cabrerizo et al. [18] and Siddiqi et al [43, 44, 45]. Tripathi and Dwivedi [53] studied $\mathcal{K}$-contact manifolds. Moreover, Siddiqi [42] also explored (k, μ)-Paracontact metric manifolds with almost conformal Ricci solitons.

In this paper we study r-almost Newton Ricci-Yamabe solitons immersed in $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold.

The structure of his manuscript is as follows. We review several fundamental details and notations that will appear throughout the work in Section 2. We approach the compact situation in Section 3 and demonstrate some triviality results. We also provide a Schur-type inequality. In Section 4, we investigate the entire case and, for some conditions on the potential function, find constant scalar curvature. Finally, in Section 5, we present some minimal r-Almost Newton-Ricci-Yamabe solitons nonexistence results. Additionally, we define it as completely geodesic, and in particular circumstances, we discover that it is isometric to the Euclidean sphere.

2 Preliminaries

Let $(\mathcal{N}, g, \zeta, \varphi, \eta)$ be a $(2n+1)$-dimensional almost contact metric manifold [5], including a Riemannian metric g, a vector field ζ, tensor field φ of type $(1,1)$, and a 1-form η. Then $(\mathcal{N}, g, \zeta, \varphi, \eta)$ satisfies

$$\varphi^2(E) = -p + \eta(E)\zeta, \quad \eta(\zeta) = 1, \quad \varphi(\zeta) = 0, \quad \eta \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F), \quad g(E, \zeta) = \eta(E), \tag{2.2}$$

$$g(E, \varphi F) = -g(\varphi E, F), \qquad g(E, \varphi E) = 0, \tag{2.3}$$

for all $E, F \in \mathfrak{X}(\mathcal{N})$. If the vector field ζ is Killing, then $(\mathcal{N}, g, \zeta, \varphi, \eta)$ is said to be a $\mathcal{K}$-contact Riemannian manifold [8]. A $\mathcal{K}$-contact Riemannian manifold is known as Sasakian [5], if the following relation holds

$$(\nabla_E \varphi)F = g(E, F)\zeta - \eta(F)E, \tag{2.4}$$

where ∇ indicates the covariant derivative with respect to g.

The following relation holds if $\mathcal{N}^{2n+1}$ is a $\mathcal{K}$-contact Riemannian manifold [27]:

$$\nabla_E \zeta = -\varphi E, \quad (\nabla_E \eta)F = -g(\varphi E, F), \tag{2.5}$$

$$\mathcal{R}ic = g(\mathcal{Q}E, \zeta) = 2n\eta(E), \tag{2.6}$$

$$\eta(\mathcal{R}(E, F)G) = g(F, G)\eta(E) - g(E, G)\eta(F), \tag{2.7}$$

$$\mathcal{R}(E, F)\zeta = \eta(F)E - \eta(E)F, \qquad \mathcal{R}(\zeta, E)F = g(E, F)\zeta - \eta(F)E, \tag{2.8}$$

for any vector fields $E, F, G \in \mathfrak{X}(\mathcal{N})$.

3 $\mathcal{D}_\alpha$-Homothetic Deformation of $\mathcal{K}$-Contact Manifolds

Assume that the manifold $(\mathcal{N}^{2n+1}, g, \zeta, \varphi, \eta)$ is almost contact metric. The definition of a $\mathcal{D}_\alpha$-homothetic deformation is [52]

$$\varphi^\sharp = \varphi, \quad \zeta^\sharp = \frac{1}{\alpha}\zeta, \quad \eta^\sharp = \alpha\eta, \tag{3.1}$$

$$g^\sharp = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta, \tag{3.2}$$

where α is positive constant.

If $(\mathcal{N}, g, \zeta, \varphi, \eta)$ is a $\mathcal{K}$-contact manifold with Riemannian connection ∇, the connection $\nabla^\sharp$ of the $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $(\mathcal{N}^\sharp, g^\sharp, \zeta^\sharp, \varphi^\sharp, \eta^\sharp)$ can be computed from ∇ and $g^\sharp$. After adopting Koszul's formula with (2.5), (3.1) and (3.2), $\nabla^\sharp$ of $g^\sharp$ is given by

$$\nabla^\sharp_E F = \nabla_E F - \alpha(\alpha - 1)[\eta(F)\varphi E + \eta(E)\varphi F]. \tag{3.3}$$

In view of (3.3), we turn up

$$(\nabla^\sharp_E \varphi)F = (\nabla_E \varphi)F + (\alpha - 1)\eta(F)\varphi^2 E. \tag{3.4}$$

Now, $\mathcal{D}_\alpha$-homothetic deformed curvature tensor $\mathcal{R}^\sharp$ of $\mathcal{K}$-contact manifold is given by

$$\begin{aligned} \mathcal{R}^\sharp(E, F)G &= \mathcal{R}(E, F)G - (\alpha - 1)((g(\varphi F, G)\varphi E + g(\varphi G, E)\varphi F + 2g(\varphi F, E)\varphi G \\ &= +[g(E, G)\zeta - \eta(G)E]\eta(F) - [g(F, G)\zeta - \eta(G)F]\eta(F) \\ &= +\alpha[\eta(F)E - \eta(E)F]\eta(G)) \end{aligned} \tag{3.5}$$

4 r-Almost Newton-Ricci-Yamabe soliton on $\mathcal{D}_\alpha$–homothetic deformed $\mathcal{K}$-contact manifold

Let $\mathcal{M}^n$ is a connected and oriented hypersurface that is immersed in a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}^{(2n+1)}$. The Gauss formula for immersion is well known to be given by

$$\mathcal{R}^\sharp(E,F)G = (\mathcal{R}(E,F)G)^\top + \langle \mathcal{A}E, G\rangle \mathcal{A}F - \langle \mathcal{A}F, G\rangle \mathcal{A}E \tag{4.1}$$

for tangent vector fields $E, F, G \in \mathfrak{X}(\mathcal{M})$, where $(\ )^\top$ stands for a vector field's tangential component in $\mathfrak{X}(\mathcal{M})$ along $\mathcal{M}^n$.
In this case, $\mathcal{A} : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ denotes the second fundamental form (or shape operator) of $\mathcal{M}^n$ in $\mathcal{N}^{2n+1}$ with respect to a given orientation, and $R^\sharp$ and $\mathcal{R}$ stand for the curvature tensors of $\mathcal{N}^{2n+1}$ and $\mathcal{M}^n$, respectively.

In specifically, the hypersurface's scalar curvature $\mathcal{R}$ fulfills the requirements.

$$\mathcal{R}^\sharp = \sum_{1\le i,j\le n} \langle \mathcal{R}^\sharp(v_i, v_j)v_j, v_i\rangle + n^2\mathcal{H}^2 - |\mathcal{A}|^2, \tag{4.2}$$

wherein $\{v_1, \ldots, v_n\}$ is an orthonormal frame on $T\mathcal{M}$ and $|\cdot|$ indicates the Hilbert-Schmidt norm. In case, a space form $\mathcal{N}^{n+1}$ of constant sectional curvature c, we have the value

$$\mathcal{R}^\sharp = \alpha\mathcal{R} - n\alpha(\alpha-1) + n^2\mathcal{H}^2 - |\mathcal{A}|^2. \tag{4.3}$$

There are n algebraic invariants, which are the fundamental symmetric functions $\mathcal{R}_r$ of the hypersurface's primary curvatures $k_1, \ldots, k_n$, associated with the second fundamental form $\mathcal{A}$ of the hypersurface $\mathcal{M}^n$.

$$\mathcal{R}_0 = 1 \ \text{ and } \ \mathcal{R}_r = \sum_{i_1<\ldots<i_r} k_{i_1}\cdots k_{i_r}.$$

The following equation describes the r-th mean curvature of the immersion

$$\binom{n}{r}\mathcal{H}_r = \mathcal{R}_r.$$

If $r = 1$, we turn up $\mathcal{H}_1 = \frac{1}{n}tr(\mathcal{A}) = \mathcal{H}$ the mean curvature of $\mathcal{M}^n$.

The r-th *Newton transformation* is defined as $\mathcal{P}_r : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ for each $0 \le r \le n$. On the hypersurface $\mathcal{M}^n$ by using the identity operator ($\mathcal{P}_0 = I$) and the recurrence relation for $1 \le r \le n$

$$\mathcal{P}_r = \sum_{j=0}^{r}(-1)^{r-j}\binom{n}{j}\mathcal{H}_j\mathcal{A}^{(r)}, \tag{4.4}$$

where j times ($\mathcal{A}^{(0)} = I$) represent the composition of $\mathcal{A}$ with r. Observe that the second order linear differential operator $\mathcal{L}_r$ is connected to each Newton transformation $\mathcal{P}_r$, defined by $\mathcal{L}_r : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$

$$\mathcal{L}_r(\omega) = \text{tr}(\mathcal{P}_r \circ \hbar\text{ess}\ \omega).$$

We observe that $\mathcal{L}_0$ is just the Laplacian operator for $r = 0$. Additionally, it is apparent that

$$\begin{aligned}\text{div}_{\mathcal{M}}(\mathcal{P}_r\nabla\omega) &= \sum_{i=1}^{n}\langle(\nabla_{v_i}\mathcal{P}_r)(\nabla\omega), v_i\rangle + \sum_{i=1}^{n}\langle\mathcal{P}_r(\nabla_{v_i}\nabla\omega), v_i\rangle \\ &= \langle \text{div}_{\mathcal{M}}\mathcal{P}_r, \nabla\omega\rangle + \mathcal{L}_r(\omega),\end{aligned} \tag{4.5}$$

where the equation for the divergence of $\mathcal{P}_r$ on $\mathcal{M}^n$ is

$$\text{div}_{\mathcal{M}}\mathcal{P}_r = \text{tr}(\nabla\mathcal{P}_r) = \sum_{i=1}^{n}(\nabla_{v_i}\mathcal{P}_r)(v_i).$$

Because $\text{div}_{\mathcal{M}}\mathcal{P}_r = 0$, the equation (4.5) reduces to $\mathcal{L}_r(\omega) = \text{div}_{\mathcal{M}}(\mathcal{P}_r\nabla\omega)$ in particular when the ambient space has constant sectional curvature (see [48] for more information). The following lemma gives useful conclusions.

Lemma 4.1 ([48]). *If $\mathcal{M}$ without boundary is non-compact or compact and γ has compact support. Then we have*

(i) $\int_{\mathcal{M}} \mathcal{L}_r(\gamma) = 0$;

(ii) $\int_{\mathcal{M}} \gamma\mathcal{L}_r(\gamma) = -\int_{\mathcal{M}} \langle \mathcal{P}_r \nabla\gamma, \nabla\gamma\rangle$.

The so-called traceless second fundamental form of the hypersurface, denoted by $\Phi = \mathcal{A} - \mathcal{H}I$, will likewise work for our purposes. Take into account that tr $(\Phi) = 0$ and $|\Phi|^2 = \text{tr}(\Phi^2) = |\mathcal{A}|^2 - n\mathcal{H}^2 \geq 0$ are equivalent if and only if $\mathcal{M}^n$ is totally umbilical.
Let us study the Yau's lemma, which is Theorem 3 of [55], to conclude this topic.

Lemma 4.2. *Let ω be a subharmonic and nonnegative function on a complete Riemannian manifold $\mathcal{M}^n$. If $\omega \in \mathcal{L}^p(\mathcal{M})$, for some $p > 1$, then ω is a constant.*

Here we adopt the symbol $\mathcal{L}^p(\mathcal{M}) = \{\omega : \mathcal{M}^n \to \mathbb{R} : \int_{\mathcal{M}} |\omega|^p <\}$, for each $p \geq 1$.
We end these considerations by providing an example.

Example 4.3. Let the standard immersion of $\mathbb{S}^n$ into $\mathbb{S}^{n+1}$, which we know that is totally geodesic. In particular, $P_r \equiv 0$, for all $1 \leq r \leq n$, and choosing $\Lambda = \frac{2\delta-(n-1)\varepsilon}{2(n-1)}$, we obtain the equation (1.5) is fulfilled by the immersion.

Moreover, if the scalar curvature of $\mathcal{M}^n$ is constant, the equation (1.5) becomes valid.

$$\delta\mathcal{R}ic + \mathcal{P}_{\mathrm{r}} \circ \hbar\text{ess}(\gamma) = \mu g, \tag{4.6}$$

where $\mu = \Lambda - \frac{\varepsilon}{2}\mathcal{R}$. So, we can recall Example 2 of [10] as another example of gradient r-Almost-Newton Ricci-Yamabe soliton.

5 Results of Triviality

With the gradient r-Newton-Ricci-Yamabe soliton (gradient r-NRYS) closed and Λ constant, we spend this part to presenting our key findings. The Riemannian manifold with constant sectional curvature c is denoted by the symbol $\mathcal{N}_c^{n+1}$ throughout the text. More specifically:

Theorem 5.1. *If $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed gradient r-NRYS immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded above or bounded below (referring to quadratic form) and any one of the following propositions is valid, then we have*

i) $\delta > \frac{-n\varepsilon}{2}$ *and* $\mathcal{R}^\sharp \geq 0$ *and* $\Lambda \geq 0$, *or* $\mathcal{R}^\sharp \leq 0$ *and* $\Lambda \leq 0$,

ii) $\delta < \frac{-n\varepsilon}{2}$ *and* $\mathcal{R}^\sharp \geq 0$ *and* $\Lambda \leq 0$, *or* $\mathcal{R}^\sharp \leq 0$ *and* $\Lambda \geq 0$,

iii) $\delta \neq \frac{-n\varepsilon}{2}$ *and either* $\mathcal{R}^\sharp \geq \frac{2n\Lambda}{2\delta+n\varepsilon}$ *or* $\mathcal{R}^\sharp \leq \frac{2n\Lambda}{2\delta+n\varepsilon}$,

the scalar curvature $\mathcal{R}^\sharp$ is constant of $\mathcal{M}^n$ and $\mathcal{M}^n$ is trivial.

Proof. In light of Lemma 4.1 and structural equation, we obtain

$$0 = \int_{\mathcal{M}} \mathcal{L}_r(\gamma) = \int_M \left(n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp\right).$$

Therefore, if (i) and (ii) are true, we derive $\mathcal{R}^\sharp = \Lambda = 0$ and $\mathcal{L}_r(\gamma) = 0$ from the structural equation. There is a positive constant $C > 0$ such that because the quadratic form of $\mathcal{P}_r$ is bounded above or below

$$0 = \mathcal{L}_r(\gamma) \leq C\Delta\gamma \text{ or } 0 = \mathcal{L}_r(\gamma) \geq -C\Delta\gamma,$$

respectively. γ is a subharmonic function as a result. Hopf's theorem tells us that γ is a constant function since $\mathcal{M}$ is compact. Therefore, the soliton is trivial. Finally (iii) follows identically to (i) and (ii). ◻

Remark 5.2. *The items (i) and (ii) in the above theorem entails that $\mathcal{M}$ is steady and $\mathcal{R}^\sharp = 0$. Since $\mathcal{M}^n$ trivial, we get $\mathcal{R}ic \equiv 0$. Finally, (iii) implies $\mathcal{R}^\sharp = \frac{-\Lambda n}{\rho n-1}$. Since $\mathcal{M}$ is trivial, we obtain*

$$\delta\mathcal{R}ic = \left(\Lambda - \frac{n\varepsilon\Lambda}{(2\delta+\varepsilon)}\right) g = \frac{2\delta\Lambda}{(2\delta+n\varepsilon)} g = \delta\frac{\mathcal{R}^\sharp}{n} g,$$

i.e., $\mathcal{M}^n$ is Einstein.

Theorem 5.3. *If $(\mathcal{M}^n, \gamma, \Lambda, \delta, \varepsilon)$ be a closed gradient r-NRYS immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded above or bounded below (referring to quadratic form) and $\delta \neq \frac{-n\varepsilon}{2}$. Then the scalar curvature of $\mathcal{M}^n$ is constant, $\mathcal{M}^n$ is Einstein and trivial.*

Proof. In viwe of structural equation Lemma 4.1 we have

$$\int_{\mathcal{M}} |n\Lambda - (2\delta+n\varepsilon)\mathcal{R}^\sharp|^2 = \int_{\mathcal{M}} \left(n\Lambda - (2\delta+n\varepsilon)\mathcal{R}^\sharp\right) \mathcal{L}_r(\gamma) = \left(n\Lambda - (2\delta+n\varepsilon)\mathcal{R}^\sharp\right) \int_{\mathcal{M}} \mathcal{L}_r(\gamma) = 0.$$

Hence, we obtain $\mathcal{R}^\sharp = \frac{2n\Lambda}{2\delta+n\varepsilon}$ and $\mathcal{L}_r(\gamma) = 0$. Adopting that $\mathcal{P}_r$ is bounded above or bounded below (referring to quadratic form) to demonstrate that $\mathcal{M}$ is trivial, we can adopt the same steps as in the proof of Theorem 5.1. Last but not least, since M^n is trivial, we can move on to Remark. 5.2 to conclude that $\mathcal{M}^n$ is Einstein. □

We established a Schur-type inequality in the following Theorem.

Theorem 5.4. *If $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed gradient r-NRYS immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\delta > \frac{-n\varepsilon}{2}$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{nC}{(n-2)\left(\delta + \frac{n\varepsilon}{2}\right)} \|\overset{\circ}{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\|\nabla^2\gamma - \frac{\Delta\gamma}{n} g\right\|_{\mathcal{L}^2}. \tag{5.1}$$

Proof. The contracted second Bianchi identity states

$$\mathrm{div}(\mathcal{R}ic) - \frac{1}{2}\nabla\mathcal{R}^\sharp = 0,$$

and hence that

$$\mathrm{div}(\overset{\circ}{\mathcal{R}ic}) = \frac{n-2}{2n}\nabla\mathcal{R}^\sharp,$$

where $\overset{\circ}{\mathcal{R}ic}$ is the traceless Ricci tensor. Since $\mathcal{M}$ is compact we get using our assumption on $\mathcal{P}_r$ that

$$\begin{aligned}
\int_{\mathcal{M}} |n\Lambda - (2\delta+n\varepsilon)\mathcal{R}^\sharp|^2 &= \int_{\mathcal{M}} \left(n\Lambda - (2\delta+n\varepsilon)\mathcal{R}^\sharp\right) \mathcal{L}_r(\gamma) = \int_{\mathcal{M}} \left(n\Lambda - (2\delta+n\varepsilon)\mathcal{R}^\sharp\right) \mathrm{div}(\mathcal{P}_r\nabla\gamma) \\
&= (2\delta+n\varepsilon)\int_{\mathcal{M}} \langle\nabla\mathcal{R}^\sharp, \mathcal{P}_r\nabla\Psi\rangle \leq C(2\delta+n\varepsilon)\int_{\mathcal{M}} \langle\nabla\mathcal{R}^\sharp, \nabla\gamma\rangle \\
&= \frac{2nC(2\delta+n\varepsilon)}{n-2}\int_{\mathcal{M}} \langle\mathrm{div}(\overset{\circ}{\mathcal{R}ic}), \nabla\gamma\rangle \\
&= \frac{2nC(2\delta+n\varepsilon)}{n-2}\int_{\mathcal{M}} \langle\overset{\circ}{\mathcal{R}ic}, \nabla^2\gamma\rangle \\
&= \frac{2nC(2\delta+n\varepsilon)}{n-2}\int_{\mathcal{M}} \langle\overset{\circ}{\mathcal{R}ic}, \nabla^2\gamma - \frac{\Delta\Psi}{n} g\rangle \\
&\leq \frac{2nC(2\delta+n\varepsilon)}{n-2} \|\overset{\circ}{\mathcal{R}ic}\|_{\mathcal{L}^2} \left\|\nabla^2\Psi - \frac{\Delta\gamma}{n} g\right\|_{\mathcal{L}^2},
\end{aligned}$$

wherein we employed that $\langle\overset{\circ}{\mathcal{R}ic}, g\rangle = 0$. Provided that $\mathcal{M}$ is compact, we get

$$n\Lambda = (2\delta+n\varepsilon)\overline{\mathcal{R}},$$

where $\overline{\mathcal{R}}$ indicate for the average of $\mathcal{R}^\sharp$. Therefore,

$$(2\delta+n\varepsilon)^2\int_M|\mathcal{R}^\sharp-\overline{\mathcal{R}}|^2=\int_{\mathcal{M}}|n\Lambda-(2\delta+n\varepsilon)\mathcal{R}^\sharp|^2,$$

i.e.,

$$(2\delta+n\varepsilon)^2\int_{\mathcal{M}}|\mathcal{R}^\sharp-\overline{\mathcal{R}}|^2\leq\frac{2nC(2\delta+n\varepsilon)}{n-2}\|\overset{\circ}{\mathcal{R}ic}\|_{\mathcal{L}^2}\left\|\nabla^2\gamma-\frac{\Delta\gamma}{n}g\right\|_{\mathcal{L}^2},$$

i.e.,

$$\int_{\mathcal{M}}|\mathcal{R}^\sharp-\overline{\mathcal{R}}|^2\leq\frac{2nC}{(n-2)(2\delta+n\varepsilon)}\|(\overset{\circ}{\mathcal{R}ic})\|_{\mathcal{L}^2}\left\|\nabla^2\gamma-\frac{\Delta\gamma}{n}g\right\|_{\mathcal{L}^2}. \tag{5.2}$$

This completes the proof. □

Remark 5.5. *Due to the fact that both sides of the expression (5.1) diminish in the foregoing theorem if M^n" is Einstein, the equality is maintained. To demonstrate the rigidity would be a fascinating problem.*

6 complete non-compact r-Newton-Ricci-Yamabe solitons

This section begins with the following finding.

Theorem 6.1. *Let $(\mathcal{M}^n,g,\gamma,\lambda,\delta,\varepsilon)$ be a complete r-NRYS immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$, such that the potential function Ψ is non-negative and belongs $\mathcal{L}^p(\mathcal{M})$, for some $p>1$. If any one of the following assertions is valid, then*

(i) $\delta>\frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded above (referring to quadratic form) and $\mathcal{R}^\sharp\geq\frac{-\Lambda n}{\rho n-1}$,

(ii) $\delta>\frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\mathcal{R}^\sharp\leq\frac{2n\Lambda}{(2\delta+n\varepsilon)}$,

iii.) $\delta<\frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\mathcal{R}^\sharp\geq\frac{2n\Lambda}{(2\delta+n\varepsilon)}$,

iv.) $\delta<\frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded above (referring to quadratic form) and $\mathcal{R}^\sharp\leq\frac{2n\Lambda}{(2\delta+n\varepsilon)}$,

then $\mathcal{R}^\sharp=\frac{2n\Lambda}{(2\delta+n\varepsilon)}$, $\mathcal{M}^n$ is Einstein and trivial.

Proof. Let $\mathcal{P}_r$ is bounded above (referring to quadratic form) there exists a positive constant $\sigma>0$ such that

$$0\leq n\Lambda-(2\delta+n\varepsilon)\mathcal{R}=\mathcal{L}_r(\gamma)\leq\sigma\Delta\gamma.$$

Thus, γ is a subharmonic function, so in light of Lemma 4.2 we turn up γ is a constant. Therefore, $\mathcal{R}=\frac{2n\Lambda}{(2\delta+n\varepsilon)}$ and $\mathcal{M}$ is trivial. Since $\mathcal{M}$ is trivial we have from structural equation that

$$\delta\mathcal{R}ic=\left(\Lambda-\frac{n\varepsilon\Lambda}{(2\delta+\varepsilon)}\right)g=\frac{2\delta\Lambda}{(2\delta+n\varepsilon)}g=\delta\frac{\mathcal{R}^\sharp}{n}g,$$

i.e., $\mathcal{M}^n$ is Einstein.

Eventually, if holds (ii), there exists a positive constant $\sigma>0$ such that

$$\mathcal{L}_r(\gamma)\geq-\sigma\Delta\gamma.$$

Hence that following the same steps as is adequate (i). The cases (iii) and (iv) are analogous. □

Theorem 6.2. *Let $(\mathcal{M}^n,g,\gamma,\Lambda,\delta,\varepsilon)$ be a complete gradient r-ARYS of non-negative scalar curvature immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$. Assume that the potential function γ fulfills the condition*

$$\int_{\mathcal{M}_{\mathcal{B}(b,r)}}\frac{\gamma}{d(x,b)^2}<\infty, \tag{6.1}$$

where $d(x,b)$ is the distance function from b in $\mathcal{M}$ and $\mathcal{B}(b,r)$ is a ball with radius $r>0$ and center at b. If any one of the following assertions is valid *(i)* *$\mathcal{M}$ is non-expanding, $\mathcal{P}_r$ is bounded above (referring to quadratic form) and $\delta > \frac{-n\varepsilon}{2}$,*

(ii) *$\mathcal{M}$ is expanding, $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\delta < \frac{-n\varepsilon}{2}$,*

then $\mathcal{R}^\sharp = 0$.

Proof. Let at look the item (i), the item (ii) is analogous. Taking the trace in the structural equation, we get

$$n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp = \mathcal{L}_r(\gamma). \tag{6.2}$$

Consider a cut-off function that was proposed in [16], $\Psi_r \in C_0^\infty(\mathcal{B}(b, 2r))$ for $r > 0$ such that

$$\begin{cases} 0 \leq \Psi_r \leq 1 & \text{in } \mathcal{B}(b, 2r) \\ \Psi_r = 1 & \text{in } \mathcal{B}(b, r) \\ |\nabla\Psi_r|^2 \leq \frac{C}{r^2} & \text{in } \mathcal{B}(b, 2r) \\ \Delta\Psi_r \leq \frac{C}{r^2} & \text{in } \mathcal{B}(b, 2r), \end{cases} \tag{6.3}$$

wherein $C > 0$ is a constant. Now adopting (6.2), integration by parts and that $\mathcal{P}_r$ is bounded from above (referring to quadratic form), we get

$$\begin{aligned} 0 \leq \int_{\mathcal{B}(b,2r)} \Psi_r\mathcal{R}^\sharp &= \int_{\mathcal{B}(b,2r)} \Psi_r \left(\frac{1}{-(2\delta + n\varepsilon)}\mathcal{L}_r(\gamma) + \frac{n\Lambda}{(2\delta + n\varepsilon)}\right) \\ &\leq \frac{1}{-(2\delta + n\varepsilon)} \int_{\mathcal{B}(b,2r)} \Psi_r\mathcal{L}_r(\gamma) \leq \frac{C_1}{-(2\delta + n\varepsilon)} \int_{\mathcal{B}(b,2r)} \Psi_r\Delta\gamma \\ &\leq \frac{C_1}{-(2\delta + n\varepsilon)} \int_{\mathcal{B}(b,2r)-\mathcal{B}(b,r)} \gamma\Delta\Psi_r \\ &\leq \frac{C_1}{-(2\delta + n\varepsilon)} \int_{B(q,2r)-B(q,r)} \frac{C_2}{r^2}\gamma \to 0, \end{aligned}$$

as $r \to \infty$. Since $\psi_r = 1$ in $B(q, r)$, from above inequality we have $\mathcal{R}^\sharp = 0$. This conclude the proof. □

Remark 6.3. *The theorem above still guarantees that a gradient r-ANRYS is in fact a gradient r-ANRS in [10]. Therefore, any gradient r-ANRYS satisfying the conditions of Theorem 6.2 is a gradient r-ANRS with scalar curvature $\mathcal{R} = 0$.*

Theorem 6.4. *Let $(\mathcal{M}^n, g, \gamma, -\frac{n\varepsilon}{2})$ be a non-expanding gradient traceless r-NRYS immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded above (referring to quadratic form) with a non-negative potential function γ. If $\gamma \in \mathcal{L}^p(\mathcal{M})$, for some $p > 1$, then $\mathcal{M}$ is steady, Einstein and trivial.*

Proof. Given that for non-expanding solitons $\Lambda \geq 0$, it follows from the structural equation that

$$\mathcal{L}_r(\gamma) = n\Lambda \geq 0.$$

From hypothesis on P_r, there exists a positive constant $C > 0$ such that

$$0 \leq n\Lambda = L_r(\Psi) \leq C\Delta\Psi,$$

i.e., γ is a subharmonic non-negative function. Hence, from Lemma 4.2 we turn up γ constant, and $0 = \Delta\gamma \geq n\Lambda \geq 0$. Therefore $\Lambda = 0$ and $\mathcal{M}$ is trivial. Finally, since $\mathcal{M}$ is trivial, we obtain $\mathcal{R}ic = \frac{\mathcal{R}}{n}g$, so M is Einstein.

□

7 Non existence results

The following lemma from Caminha et al. [9] will be used in this section

Lemma 7.1. *Let E be a smooth vector field on the n-dimensional, non compact, complete, oriented Riemannian manifold $\mathcal{M}^n$, such that $div_{\mathcal{M}}E$ does not change the sign on $\mathcal{M}$. If $|E| \in \mathcal{L}^1(\mathcal{M})$, then $div_{\mathcal{M}}E = 0$.*

We illustrate the following by adopting Cunha et al. theory [10]

Theorem 7.2. *If $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a complete r-ANRYS immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold $\mathcal{N}_c^{n+1}$, with bounded second fundamental form and potential function $\gamma : \mathcal{M}^n \to \mathbb{R}$ such that $|\nabla\gamma| \in \mathcal{L}^1(\mathcal{M})$, then we have*

(i) if $\alpha \leq 0$, $\Lambda > 0$ and $\delta < \frac{-n\varepsilon}{2}$, then $\mathcal{M}^n$ can not be minimal,

(ii) if $\alpha < 0$, $\Lambda \geq 0$ and $\delta < \frac{-n\varepsilon}{2}$, then $\mathcal{M}^n$ can not be minimal,

iii) if $\alpha = 0$, $\Lambda \geq 0$, $\delta < \frac{-n\varepsilon}{2}$, and $\mathcal{M}^n$ is minimal, then $\mathcal{M}^n$ is steady and isometric to the Euclidean space.

Proof. By using the equation (4.5) we can determine that the operator $\mathcal{L}_r$ is a divergent-type operator if the ambient space has a constant sectional curvature. On the other hand, since the Newton transformation $\mathcal{P}_r$ has bounded norm, it follows from (4.4) that $\mathcal{M}^n$ has bounded second basic form. More specifically

$$|\mathcal{P}_r\nabla\gamma| \leq |\mathcal{P}_r|\,|\nabla\gamma| \in \mathcal{L}^1(\mathcal{M}). \tag{7.1}$$

Let us suppose that $\mathcal{M}^n$ is minimal using (i) and (ii). The scalar curvature of $\mathcal{M}^n$ thus fulfills $\mathcal{R}^\sharp \leq 0 (\mathcal{R}^\sharp < 0)$ according to equation (4.3) and the assumption that $\alpha \leq 0$ ($\alpha < 0$). Thus, contracting (1.5), we find that

$$L_r(\Psi) = n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp > 0.$$

In both situations, the fact that follows contradicts Lemma 7.1. The first two claims are now validated by this. Given that the ambient space has $\alpha = 0$ and that $\mathcal{M}^n$ is minimal for the (iii) claim, the equation (4.3) becomes applicable.

$$\mathcal{R}^\sharp = -|\mathcal{A}|^2 \leq 0. \tag{7.2}$$

So, since $\Lambda \geq 0$ and $\rho < \frac{1}{n}$ we possess that

$$L_r(\Psi) = n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp \geq 0.$$

Now, using the fact that $\mathcal{L}_r$ is a divergent-type operator and $|\mathcal{P}_r\nabla\gamma| \in \mathcal{L}^1(\mathcal{M})$, again from Lemma 7.1 we have $\mathcal{L}_r\gamma = 0$ on $\mathcal{M}^n$. So, we get the conclusion that $0 \geq \mathcal{R}^\sharp = \frac{2n\Lambda}{(2\delta+n\varepsilon)} \geq 0$, that is, $\mathcal{R}^\sharp = \Lambda = 0$. This means that $|\mathcal{A}|^2 = 0$. Hence, the gradient r-ANRYS is steady, totally geodesic and flat. □

8 Conclusion remarks

A concept that extends the idea of Ricci and Yamabe solitons to immersions to constant sectional curvature space is the r-almost-Newton-Ricci-Yamabe solitons immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold. Through characterizations and nonexistence results, these novel objects were approached. In addition, we obtained constant scalar curvature under certain conditions and established several triviality results for the compact case.

The context of this work was r-almost Newton Ricci-Yamabe solitons immersed in a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold. For compact gradient r-Almost Newton-Ricci-Yamabe

solitons, we obtained the triviality requirements.
The hypersurface of an immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold with a bounded second fundamental form was the focus of our computation, and it satisfied the requirements for a r-Almost Newton-Ricci-Yamabe soliton on the hypersurface to be totally umbilical. Furthermore, it was demonstrated that a complete r-almost Newton-Ricci-Yamabe soliton on the hypersurface of immersed into a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold is admitted by the stable r-Almost Newton-Ricci-Yamabe soliton.

Additionally, we derived a Schur-type inequality and Hopf's strong maximum principle in terms of the submerged r-Almost Newton-Ricci-Yamabe soliton in a compact, totally geodesic $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifolds.
Our results further develop knowledge of the isometry of the Euclidean sphere $\mathcal{S}^{4m}$ with respect to the immersed r-Almost Newton-Ricci-Yamabe soliton in a $\mathcal{D}_\alpha$-homothetic deformed $\mathcal{K}$-contact manifold.

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