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SHFFFER STROKE BG-ALGEBRAS VIA LATTICED-VALUED FUZZY SETS

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Abstract This study establishes fundamental connections between Sheffer stroke BG-algebras (SBG-algebras) and lattice-valued fuzzy sets. We prove that a nonempty subset of an SBG-algebra is an SBG-subalgebra (resp. ideal) if and only if its characteristic $\mathcal{L}$-fuzzy set is an $\mathcal{L}$-fuzzy SBG-subalgebra (resp. ideal). Furthermore, we provide complete characterizations of $\mathcal{L}$-fuzzy SBG-subalgebras and ideals using various level sets—including upper, strong upper, lower, and strong lower level sets. Key results show that an $\mathcal{L}$-fuzzy set is an $\mathcal{L}$-fuzzy SBG-subalgebra or ideal precisely when all its nonempty level sets of a certain type are corresponding substructures. These findings offer powerful tools for analyzing SBG-algebras through fuzzy set and lattice theory.

1 Introduction

The Sheffer stroke operation, introduced by H. M. Sheffer [1], is a fundamental concept in logic and algebra due to its functional completeness, as it alone can express all other logical connectives. This property has inspired algebraic investigations into structures defined primarily by this operation, leading to the development of Sheffer stroke BG-algebras (SBG-algebras) [2]. These algebras provide a fertile ground for exploring the interplay between logical implication, algebraic order, and algebraic structures. In recent years, numerous studies have been conducted on the theory of the Sheffer stroke, reflecting its growing significance in both logical and algebraic frameworks [3, 4, 5, 6].

Parallel to these developments in Sheffer stroke algebras, fuzzy set theory initiated by Zadeh [7] has been extensively applied to various algebraic structures to model uncertainty and graduality. The generalization to lattice-valued fuzzy sets ($\mathcal{L}$-fuzzy sets) [8], where membership degrees are drawn from a complete lattice, offers a more comprehensive and robust framework for such investigations. The application of fuzzy set theory to logical algebras has produced substantial literature, including fuzzy BCK-algebras [9], fuzzy Hilbert algebras [10], and fuzzy implication algebras [11].

Fuzzy algebraic structures have played a central role in modeling reasoning processes under uncertainty, particularly in decision-making systems, fuzzy control, and artificial intelligence. Within this broad landscape, $\mathcal{L}$-fuzzy algebraic frameworks offer a flexible way to represent graded truth values and partial information. The notion of $\mathcal{L}$-fuzzy SBG-algebras introduced in this paper provides a novel algebraic tool for handling uncertainty in systems where negation-like operations, such as the Sheffer stroke, play a fundamental role. From an applied perspective, $\mathcal{L}$-fuzzy SBG-algebras can be used to model decision-making processes in which preferences or constraints are evaluated through binary interactions and subsequently aggregated via graded membership values. In fuzzy control systems, the Sheffer stroke operation allows the encoding of complementary or inhibitory relationships between control variables, while the $\mathcal{L}$-fuzzy structure captures the inherent imprecision of real-world data. Similarly, in AI reasoning, $\mathcal{L}$-fuzzy SBG-algebras provide an algebraic foundation for non-classical inference mechanisms that combine uncertainty with non-standard logical connectives.

A central theme in the fusion of fuzzy set theory with algebraic structures is the concept of level sets, which serve as a crucial bridge between fuzzy algebraic structures and their crisp counterparts. The fundamental question addressed in this context is whether a fuzzy algebraic property is equivalent to the corresponding crisp property holding in all its level subsets. This approach has been successfully applied to various algebraic structures: [12] established important connections between MV-algebras and fuzzy logic, while [13] explored the relationship between BL-algebras and fuzzy structures. More recently, [15] investigated fuzzy hyper BCK-ideals using level set approaches, and [14] applied level set characterizations to fuzzy subalgebras of BCK/BCI-algebras.

Recent studies have shown a growing interest in advanced fuzzy set extensions within algebraic structures. In particular, linear Diophantine fuzzy sets have been effectively applied to various logical and algebraic systems. Al-Tahan *et al.* [18] investigated linear Diophantine fuzzy n-fold weak subalgebras of BE-algebras, while Muhiuddin *et al.* [19] developed a systematic framework for linear Diophantine fuzzy sets in BCK/BCI-algebras. These contributions highlight the relevance of refined fuzzy frameworks in algebraic settings and further motivate the study of $\mathscr{L}$-fuzzy SBG-algebras proposed in this work.

While previous research [16, 17] has begun to explore fuzzy structures in SBG-algebras, a comprehensive study focusing specifically on the characterizing power of level sets and characteristic functions has remained unexplored. This paper aims to fill this gap.

The principal objective of this work is to establish a series of characterization theorems that intimately connect SBG-algebras with $\mathcal{L}$-fuzzy sets. We shift the focus from defining fuzzy analogues of algebraic concepts to leveraging level sets as the primary tool of analysis. The main contributions of this study are as follows: it establishes that the classical notions of SBG-subalgebras and SBG-ideals are precisely represented by the $\mathcal{L}$-fuzzy forms of their characteristic functions; it provides comprehensive characterizations of $\mathcal{L}$-fuzzy SBG-subalgebras and ideals in terms of their upper and lower level sets; and it extends these characterizations to the complements of $\mathcal{L}$-fuzzy sets within the framework of Boolean lattices. These results demonstrate that the essence of an $\mathcal{L}$-fuzzy SBG-algebraic structure is encoded in the hierarchical structure of its level subsets, thereby deepening the theoretical understanding of SBG-algebras and providing an effective methodology for their analysis.

The paper is structured as follows. After reviewing the necessary preliminaries on SBG-algebras in Section 2, we present the main results in Section 3, which is devoted to characteristic functions and level sets. The paper concludes with a discussion of the implications of the obtained results and directions for future research.

The list of acronyms is given in Table 1.

Table 1. List of acronyms

Acronyms	Representation
SBG-algebra	Sheffer stroke BG-algebra
SBG-subalgebra	Sheffer stroke BG-subalgebra
SBG-ideal	Sheffer stroke BG-ideal
$\mathcal{L}$-fuzzy SBG-subalgebra	$\mathcal{L}$-fuzzy Sheffer stroke BG-algebra
$\mathcal{L}$-fuzzy SBG-ideal	$\mathcal{L}$-fuzzy Sheffer stroke BG-ideal
$\mathcal{L}$-fuzzy SBG-implicative ideal	$\mathcal{L}$-fuzzy Sheffer stroke implicative ideal

2 Preliminaries

In this section, we provide definitions, lemma, and proposition relevant to the concepts of Sheffer stroke BG-algebras, their subalgebras and ideals, and implicatice SBG-algebras, which will be used throughout the paper.

Definition 2.1. [1] Let $\mathcal{S} := (S, |)$ be a groupoid. Then the operation "|" is said to be Sheffer stroke or Sheffer operation if it satisfies:
$(S1)$ $(\forall \mathfrak{w}, \mathfrak{s} \in S)$ $(\mathfrak{w}|\mathfrak{s} = \mathfrak{s}|\mathfrak{w})$,
$(S2)$ $(\forall \mathfrak{w}, \mathfrak{s} \in S)$ $((\mathfrak{w}|\mathfrak{w})|(\mathfrak{w}|\mathfrak{s}) = \mathfrak{w})$,
$(S3)$ $(\forall \mathfrak{w}, \mathfrak{s}, \mathfrak{b} \in S)$ $(\mathfrak{w}|((\mathfrak{s}|\mathfrak{b})|(\mathfrak{s}|\mathfrak{b})) = ((\mathfrak{w}|\mathfrak{s})|(\mathfrak{w}|\mathfrak{s}))|\mathfrak{b})$,
$(S4)$ $(\forall \mathfrak{w}, \mathfrak{s}, \mathfrak{b} \in S)$ $((\mathfrak{w}|((\mathfrak{w}|\mathfrak{w})|(\mathfrak{s}|\mathfrak{s})))|(\mathfrak{w}|((\mathfrak{w}|\mathfrak{w})|(\mathfrak{s}|\mathfrak{s}))) = \mathfrak{w})$.

To enhance the readability of this manuscript about a Sheffer stroke BG-algebras, we will consistently use the following notation:

$$\mathfrak{w}|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{w}^{\mathfrak{b}}.$$

Definition 2.2. [2] A Sheffer stroke BG-algebra (abbreviated as an SBG-algebra) is a groupoid $\mathcal{B} := (B, |)$ with a Sheffer stroke "|" that satisfies:
(SBG_1) $(\forall \mathfrak{w} \in B)$ $(\mathfrak{w}^{\mathfrak{w}}|\mathfrak{w}^{\mathfrak{w}} = 0)$,
(SBG_2) $(\forall \mathfrak{w}, \mathfrak{b} \in B)$ $(0^{\mathfrak{b}}|(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) = \mathfrak{w}|\mathfrak{w})$.

Proposition 2.3. *[2] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. The binary relation*

$$\mathfrak{w} \leq \mathfrak{b} \quad \textit{if and only if} \quad \mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} = 0$$

defines a partial order on B.

Definition 2.4. [2] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. A nonempty subset G of B is called an SBG-subalgebra of B if $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} \in G$ for all $\mathfrak{w}, \mathfrak{b} \in G$.

Definition 2.5. [2] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. A nonempty subset G of B is called an SBG-ideal of B if

(i) $0 \in G$,

(ii) If $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} \in G$ and $\mathfrak{b} \in G$, then $\mathfrak{w} \in G$.

Definition 2.6. [2] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. A nonempty subset G of B is called an implicative SBG-ideal of B if

(i) $0 \in G$,

(ii) $((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s})) \in G$ and $\mathfrak{s} \in G \Rightarrow \mathfrak{w} \in G$,

for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in G$.

Lemma 2.7. *[2] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Then the following features hold:*

(i) $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} = \mathfrak{s}^{\mathfrak{b}}|\mathfrak{s}^{\mathfrak{b}}$ *implies* $\mathfrak{w} = \mathfrak{s}$,

(ii) If $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} = 0$ *then* $\mathfrak{w} = \mathfrak{b}$,

(iii) $(\mathfrak{w}^{\mathfrak{w}})^{\mathfrak{w}} = \mathfrak{w}$,

for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$.

Lemma 2.8. *[20] Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_{\mathscr{L}}, 1_{\mathscr{L}})$ be a Boolean lattice. Then the following properties hold:*

(i) $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})((\mathfrak{w} \vee \mathfrak{b})' = \mathfrak{w}' \wedge \mathfrak{b}')$,

(ii) $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})((\mathfrak{w} \wedge \mathfrak{b})' = \mathfrak{w}' \vee \mathfrak{b}')$,

(iii) $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})(\mathfrak{w} \leq \mathfrak{b} \Leftrightarrow \mathfrak{w}' \geq \mathfrak{b}')$,

(iv) $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})(\mathfrak{w} = \mathfrak{b} \Leftrightarrow \mathfrak{w}' = \mathfrak{b}')$,

(v) $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})(\mathfrak{w} < \mathfrak{b} \Leftrightarrow \mathfrak{w}' > \mathfrak{b}')$.

Definition 2.9. [21] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy set $\mathscr{L}$ in B is called an $\mathscr{L}$-fuzzy SBG-subalgebra of B if

$$(\forall \mathfrak{w}, \mathfrak{b} \in B)\Big(\ \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b})\ \Big). \tag{2.1}$$

Definition 2.10. [21] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy set $\mathscr{L}$ on an SBG-algebra B is called an $\mathscr{L}$-fuzzy SBG-ideal of B if

$$(\forall \mathfrak{w}, \mathfrak{b} \in B) \left(\begin{array}{l} \mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w}) \\ \mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu(\mathfrak{b}) \wedge \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \end{array} \right). \tag{2.2}$$

Definition 2.11. [21] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy set $\mathscr{L}$ on B is called an $\mathscr{L}$-fuzzy implicative SBG-ideal of B if

$$(\forall \mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B) \left(\begin{array}{l} \mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))| \\ (((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s})) \wedge \mathcal{L}_\mu(\mathfrak{s}) \end{array} \right). \tag{2.3}$$

Proposition 2.12. *[21] Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Every $\mathscr{L}$-fuzzy SBG-implicative ideal of B is an $\mathscr{L}$-fuzzy SBG-ideal of B.*

3 Characterizations through Level Sets in $\mathscr{L}$-Fuzzy SBG-Algebras

Now we shall determine $\mathscr{L}$ is a complete lattice $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, 0_{\mathscr{L}}, 1_{\mathscr{L}})$.

Let A be a subset of L. Then the characteristic function χ_A of L is a function of L into $\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ defined as follows:

$$\chi_A(\mathfrak{w}) = \begin{cases} 1_{\mathscr{L}} & \text{if } \mathfrak{w} \in A \\ 0_{\mathscr{L}} & \text{if } \mathfrak{w} \notin A. \end{cases}$$

By the definition of characteristic function, χ_A is a function of L into $\{0_{\mathscr{L}}, 1_{\mathscr{L}}\} \subset \mathscr{L}$. We denote the $\mathscr{L}$-fuzzy set $\mathcal{L}_A$ in L is described by its membership function χ_A, is called the characteristic $\mathscr{L}$-fuzzy set of A in L.

Lemma 3.1. *Let the constant 0 of L is in A. Then $\chi_A(0) \geq \chi_A(\mathfrak{a})$ for all $\mathfrak{a} \in L$.*

Proof. Assume that $0 \in A$. Then for all $\mathfrak{a} \in L$, $\chi_A(0) = 1_{\mathscr{L}} \geq \chi_A(\mathfrak{a})$. □

Lemma 3.2. *Let A be a nonempty subset of L. If $\chi_A(0) \geq \chi_A(\mathfrak{a})$ for all $\mathfrak{a} \in L$, then the constant 0 of L is in A.*

Proof. Assume that $\chi_A(0) \geq \chi_A(\mathfrak{a})$ for all $\mathfrak{a} \in L$. Since A is a nonempty subset of L, we have an element $\mathfrak{u}$ in A, that is, $\chi_A(\mathfrak{u}) = 1_{\mathscr{L}}$. Thus $1_{\mathscr{L}} \geq \chi_A(0) \geq \chi_A(\mathfrak{u}) = 1_{\mathscr{L}}$. So $\chi_A(0) = 1_{\mathscr{L}}$, that is, $0 \in A$. □

Theorem 3.3. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. A nonempty subset S of B is an SBG-subalgebra of B if and only if the characteristic $\mathscr{L}$-fuzzy set χ_S is an $\mathscr{L}$-fuzzy SBG-subalgebra of B.*

Proof. Assume that S is an SBG-subalgebra of B.

Case 1 : Suppose $\mathfrak{w}, \mathfrak{b} \in S$. Then $\chi_S(\mathfrak{w}) = 1_{\mathscr{L}}$ and $\chi_S(\mathfrak{b}) = 1_{\mathscr{L}}$. Thus $\chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b}) = 1_{\mathscr{L}}$. Since S is an SBG-subalgebra of B, $\mathfrak{w}^{\mathfrak{b}} \in S$ and so $\chi_S(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) = 1_{\mathscr{L}}$. Therefore, $\chi_S(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) = 1_{\mathscr{L}} \geq 1_{\mathscr{L}} = \chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b})$.

Case 2 : Suppose $\mathfrak{w} \in S$ and $\mathfrak{b} \notin S$. Then $\chi_S(\mathfrak{w}) = 1_{\mathscr{L}}$ and $\chi_S(\mathfrak{b}) = 0_{\mathscr{L}}$. Thus $\chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b}) = 1_{\mathscr{L}}$. Therefore, $\chi_S(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \geq 1_{\mathscr{L}} = \chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b})$.

Case 3 : Suppose $\mathfrak{w} \notin S$ and $\mathfrak{b} \in S$. Then $\chi_S(\mathfrak{w}) = 0_{\mathscr{L}}$ and $\chi_S(\mathfrak{b}) = 1_{\mathscr{L}}$. Thus $\chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{s}) = 1_{\mathscr{L}}$. Therefore, $\chi_S(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \geq 1_{\mathscr{L}} = \chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b})$.

Case 4 : Suppose $\mathfrak{w} \notin S$ and $\mathfrak{b} \notin S$. Then $\chi_S(\mathfrak{w}) = 0_{\mathscr{L}}$ and $\chi_S(\mathfrak{b}) = 0_{\mathscr{L}}$. Thus $\chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b}) = 0_{\mathscr{L}}$. Therefore, $\chi_S(jx^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \geq 0_{\mathscr{L}} = \chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{b})$

Hence, χ_S is an $\mathscr{L}$-fuzzy SBG-subalgebra of B.

Conversely, assume that χ_S is an $\mathscr{L}$-fuzzy SBG-subalgebra of B. Let $\mathfrak{w}, \mathfrak{s} \in S$. Then $\chi_S(\mathfrak{w}) = 1_{\mathscr{L}}$ and $\chi_S(\mathfrak{s}) = 1_{\mathscr{L}}$. Thus $\chi_S(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \geq \chi_S(\mathfrak{w}) \wedge \chi_S(\mathfrak{s}) = 1_{\mathscr{L}}$, so $\chi_S(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) = 1_{\mathscr{L}}$. Hence, $(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \in S$ and so S is an SBG-subalgebra of B. □

Theorem 3.4. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. A nonempty subset D of B is a SBG-ideal of H if and only if the characteristic $\mathscr{L}$-fuzzy set χ_D is an $\mathscr{L}$-fuzzy SBG-ideal of B.*

Proof. Assume that D is an SBG-ideal of B. Since $0 \in D$, it follows from Lemma 3.2 that $\chi_D(0) \geq \chi_D(\mathfrak{w})$ for all $\mathfrak{w} \in D$. Next, let $\mathfrak{w}, \mathfrak{s} \in D$.

Case 1 : Suppose $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}), \mathfrak{w} \in D$. Then $\chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 1_{\mathscr{L}}$ and $\chi_D(\mathfrak{w}) = 1_{\mathscr{L}}$. Therefore, $\chi_D(\mathfrak{s}) \geq \chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_D(\mathfrak{w})$.

Case 2 : Suppose $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \notin D$ and $x \in D$. Then $\chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 0_{\mathscr{L}}$ and $\chi_D(\mathfrak{w}) = 1_{\mathscr{L}}$. Thus $\chi_D(\mathfrak{s}) \geq 1_{\mathscr{L}} = \chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_D(\mathfrak{w})$.

Case 3 : Suppose $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \in D$ and $\mathfrak{w} \notin D$. Then $\chi_D((y|(x|x))|(y|(x|x))) = 1_{\mathscr{L}}$ and $\chi_D(\mathfrak{w}) = 0_{\mathscr{L}}$. Thus $\chi_D(y) \geq 1_{\mathscr{L}} = \chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_D(\mathfrak{w})$.

Case 4 : Suppose $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \notin D$ and $x \notin D$. Then $\chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 0_{\mathscr{L}}$ and $\chi_D(\mathfrak{w}) = 0_{\mathscr{L}}$. Thus $\chi_D(\mathfrak{s}) \geq 0_{\mathscr{L}} = \chi_D((\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_D(\mathfrak{w})$.

Hence, χ_D is an $\mathscr{L}$-fuzzy SBG-ideal of H.

Conversely, assume that χ_D is an $\mathscr{L}$-fuzzy SBG-ideal of H. Since $\chi_D(0) \geq \chi_D(\mathfrak{w})$ for all $\mathfrak{w} \in D$, it follows from Lemma 3.2 that $0 \in D$. Next, let $\mathfrak{w}, \mathfrak{b} \in D$ be such that $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \in D$ and $\mathfrak{w} \in D$. Then $\chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 1_{\mathscr{L}}$ and $\chi_D(\mathfrak{w}) = 1_{\mathscr{L}}$. Thus $\chi_D(\mathfrak{b}) \geq \chi_D(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_D(\mathfrak{w}) = 1_{\mathscr{L}}$, so $\chi_D(\mathfrak{s}) = 1_{\mathscr{L}}$. Therefore, $\mathfrak{b} \in D$ and so D is an SBG-ideal of B. □

Theorem 3.5. *[2] Let $\mathcal{P} := (P, |_P)$ and $\mathcal{Q} := (Q, |_Q)$ be SBG-algebras. Then $\langle P \times Q, |_{P\times Q}\rangle$ is a SBG-algebra where the set $P \times Q$ is the Cartesian product of P and Q and the operation $|_{P\times Q}$ on this set is defined by $(\mathfrak{p}_1, \mathfrak{q}_1)|_{P\times Q}(\mathfrak{p}_2, \mathfrak{q}_2) = (\mathfrak{p}_1|_P\mathfrak{p}_2, \mathfrak{q}_1|_Q\mathfrak{q}_2)$, and the fixed element is $0_{P\times Q} = (0_P, 0_Q)$.*

Theorem 3.6. *[2] Let $\mathcal{P} := (P, |_P)$ and $\mathcal{Q} := (Q, |_Q)$ be SBG-algebras. Then $(P \times Q, |_{P\times Q})$ is an SBG-algebra where the set $P \times Q$ is the Cartesian product of P and Q and the operation $|_{P\times Q}$ on this set is defined by $(\mathfrak{p}_1, \mathfrak{q}_1)|_{P\times Q}(\mathfrak{p}_2, \mathfrak{q}_2) = (\mathfrak{p}_1|_P\mathfrak{p}_2, \mathfrak{q}_1|_Q\mathfrak{q}_2)$, and the fixed element is $0_{P\times Q} = (0_P, 0_Q)$.*

To enhance the readability of this manuscript about a Sheffer stroke BG-algebras, we will consistently use the following notation:

$$\mathfrak{w}|_X(\mathfrak{b}|_X\mathfrak{b}) = X^{\mathfrak{w}|\mathfrak{b}}.$$

Theorem 3.7. *Let $\mathcal{P} := (P, |_P)$ and $\mathcal{Q} := (Q, |_Q)$ be SBG-algebras. If h_P and h_Q are $\mathscr{L}$-fuzzy SBG-subalgebra of P and Q, respectively, then $\mathcal{L}_{\mu_{P\times Q}}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of $(P \times Q, |_{P\times Q})$.*

Proof. Let $(P, |_P)$ and $(Q, |_Q)$ be SBG-algebras and h_P and h_Q be $\mathscr{L}$-fuzzy SBG-subalgebras of P and Q, respectively. Let $(\mathfrak{p}_1, \mathfrak{q}_1), (\mathfrak{p}_2, \mathfrak{q}_2) \in P \times Q$. Then

$$\mathcal{L}_{\mu_{P\times Q}}(((\mathfrak{p}_1, \mathfrak{q}_1)|_{P\times Q}((\mathfrak{p}_2, \mathfrak{q}_2)|_{P\times Q}(\mathfrak{p}_2, \mathfrak{q}_2)))|_{P\times Q}((\mathfrak{p}_1, \mathfrak{q}_1)|_{P\times \mathfrak{p}}((\mathfrak{p}_2, \mathfrak{q}_2)|_{P\times Q}(\mathfrak{p}_2, \mathfrak{q}_2))))$$

$$\begin{aligned}
&= \mathcal{L}_{\mu_{P\times Q}}(P^{\mathfrak{p}_1|\mathfrak{p}_2}|_P P^{\mathfrak{p}_1|\mathfrak{p}_2}, Q^{\mathfrak{q}_1|\mathfrak{q}_2}|_Q Q^{\mathfrak{q}_1|\mathfrak{q}_2}) \\
&= \mathcal{L}_{\mu_P}(P^{\mathfrak{p}_1|\mathfrak{p}_2}|_P P^{\mathfrak{p}_1|\mathfrak{p}_2}) \wedge \mathcal{L}_{\mu_Q}(Q^{\mathfrak{q}_1|\mathfrak{q}_2}|_Q Q^{\mathfrak{q}_1|\mathfrak{q}_2})\} \\
&\geq \mathcal{L}_{\mu_P}(\mathfrak{p}_1) \wedge \mathcal{L}_{\mu_P}(\mathfrak{p}_2) \wedge \mathcal{L}_{\mu_Q}(\mathfrak{q}_1) \wedge \mathcal{L}_{\mu_Q}(\mathfrak{q}_2) \\
&\geq \mathcal{L}_{\mu_P}(\mathfrak{p}_1) \wedge \mathcal{L}_{\mu_Q}(\mathfrak{q}_1) \wedge \mathcal{L}_{\mu_P}(\mathfrak{q}_2) \wedge \mathcal{L}_{\mu_Q}(\mathfrak{q}_2) \\
&= \mathcal{L}_{\mu_{P\times Q}}(\mathfrak{p}_1, \mathfrak{q}_1) \wedge \mathcal{L}_{\mu_{P\times Q}}(\mathfrak{p}_2, \mathfrak{q}_2).
\end{aligned}$$

Hence $\mathcal{L}_{\mu_{P\times Q}}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of $(P \times Q, |_{P\times Q})$. □

Theorem 3.8. *Let $\mathcal{P} := (P, |_P)$ and $\mathcal{Q} := (Q, |_Q)$ be SBG-algebras. If h_P and h_Q are $\mathscr{L}$-fuzzy SBG-ideal of P and Q, respectively, then $\mathcal{L}_{\mu_{P\times Q}}$ is an $\mathscr{L}$-fuzzy SBG-ideal of $(P \times Q, |_{P\times Q})$.*

Proof. Let $(P, |_P)$ and $(Q, |_Q)$ be SBG-algebras and h_P and h_Q be $\mathscr{L}$-fuzzy SBG-ideal of P and Q, respectively. Let $(\mathfrak{p}_1, \mathfrak{q}_1), (\mathfrak{p}_2, \mathfrak{q}_2) \in P \times Q$. Then

$$\begin{aligned}
\mathcal{L}_{\mu_{P\times Q}}(0_P, 0_Q) &= \mathcal{L}_{\mu_P}(0_P) \wedge \mathcal{L}_{\mu_Q}(0_Q) \\
&\geq \mathcal{L}_{\mu_P}(\mathfrak{p}_1) \wedge \mathcal{L}_{\mu_Q}(\mathfrak{q}_1) \\
&= \mathcal{L}_{\mu_{P\times Q}}(\mathfrak{p}_1, \mathfrak{q}_1),
\end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{\mu_{P\times Q}}(\mathfrak{p}_2,\mathfrak{q}_2) &= \mathcal{L}_{\mu_P}(\mathfrak{q}_2)\wedge\mathcal{L}_{\mu_Q}(\mathfrak{q}_2)\\ &\geq \mathcal{L}_{\mu_P}(\mathfrak{p}_1)\wedge\mathcal{L}_{\mu_P}(P^{\mathfrak{p}_2|\mathfrak{p}_1}|_P P^{\mathfrak{p}_2|\mathfrak{p}_1})\wedge\mathcal{L}_{\mu_Q}(\mathfrak{q}_1)\wedge\\ &\quad\mathcal{L}_{\mu_Q}(Q^{\mathfrak{q}_2|\mathfrak{q}_1}|_Q Q^{\mathfrak{q}_2|\mathfrak{q}_1})\\ &= \mathcal{L}_{\mu_P}(\mathfrak{p}_1)\wedge\mathcal{L}_{\mu_B}(\mathfrak{q}_1)\wedge\mathcal{L}_{\mu_P}(P^{\mathfrak{p}_2|\mathfrak{p}_1}|_P P^{\mathfrak{p}_2|\mathfrak{p}_1})\wedge\\ &\quad\mathcal{L}_{\mu_Q}(Q^{\mathfrak{q}_2|\mathfrak{q}_1}|_Q Q^{\mathfrak{q}_2|\mathfrak{q}_1})\\ &= \mathcal{L}_{\mu_{P\times Q}}(\mathfrak{p}_1,\mathfrak{q}_1)\wedge\mathcal{L}_{\mu_{P\times Q}}(((\mathfrak{q}_2,\mathfrak{q}_2)|_{P\times Q}\\ &\quad((\mathfrak{p}_1,\mathfrak{q}_1)|_{P\times Q}(\mathfrak{p}_1,\mathfrak{q}_1)))|_{P\times Q}((\mathfrak{q}_2,\mathfrak{q}_1)|_{P\times Q}((\mathfrak{p}_1,\mathfrak{q}_1)\\ &\quad|_{P\times Q}(\mathfrak{p}_1,\mathfrak{q}_1)))) \end{aligned}$$

Hence $\mathcal{L}_{\mu_{P\times Q}}$ is an $\mathscr{L}$-fuzzy SBG-ideal of $P\times Q, |_{P\times Q})$. □

Theorem 3.9. *Let $\mathcal{P} := (P, |_P)$ and $\mathcal{Q} := (Q, |_Q)$ be SBG-algebras. If h_P and h_Q are $\mathscr{L}$-fuzzy implicative SBG-ideal of P and Q, respectively, then $\mathcal{L}_{\mu_{P\times Q}}$ is a $\mathscr{L}$-implicative fuzzy SBG-ideal of $(P\times Q, |_{P\times Q})$.*

Proof. It is similar to 3.8 □

Theorem 3.10. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of B. Then a subset $I = \{\mathfrak{w}\in B : \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)\}$ of B is a SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of B and $I = \{\mathfrak{w}\in B : \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)\}$ be a subset of B. Then it is clear that $0\in I$. Suppose that $\mathfrak{w},\mathfrak{s}\in B$ be such that $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}},\mathfrak{w}\in I$. Since $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = \mathcal{L}_\mu(0)$ and $\mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)$ for any $\mathfrak{w},\mathfrak{s}\in B$, $\mathcal{L}_\mu(0)\geq\mathcal{L}_\mu(y)\geq\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})\wedge\mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)\wedge\mathcal{L}_\mu(0) = \mathcal{L}_\mu(0)$. Thus, $\mathcal{L}_\mu(\mathfrak{s}) = \mathcal{L}_\mu(0)$, and so, $\mathfrak{s}\in I$. Hence, I is a SBG-ideal of B. □

Theorem 3.11. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy implicative SBG-ideal of B. Then a subset $I = \{\mathfrak{w}\in B : \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)\}$ of B is an implicative SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy implicative SBG-ideal of B and $I = \{\mathfrak{w}\in B : \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)\}$ be a subset of B. Then it is clear that $0\in I$. Suppose that $\mathfrak{w},\mathfrak{s},\mathfrak{b}\in B$ be such that

$$((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s})),\mathfrak{s}\in I.$$

Since $\mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))) = \mathcal{L}_\mu(0)$ and $\mathcal{L}_\mu(\mathfrak{s}) = \mathcal{L}_\mu(0)$ for any $\mathfrak{w},\mathfrak{b}\in B$, $\mathcal{L}_\mu(0)\geq\mathcal{L}_\mu(\mathfrak{w})\geq\mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})$ $|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s})))\wedge\mathcal{L}_\mu(\mathfrak{s}) = \mathcal{L}_\mu(0)\wedge\mathcal{L}_\mu(0) = \mathcal{L}_\mu(0)$. Thus, $\mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(0)$, and so, $\mathfrak{w}\in I$. Hence, I is an implicative SBG-ideal of B. □

Definition 3.12. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy set in L with the membership function $\mathcal{L}_\mu$. For any $t\in\mathscr{L}$, the sets

$$\begin{gathered} U(\mathcal{L}_\mu,t) = \{\mathfrak{w}\in L : \mathcal{L}_\mu(\mathfrak{w})\geq t\}\\ U^+(\mathcal{L}_\mu,t) = \{\mathfrak{w}\in L : \mathcal{L}_\mu(\mathfrak{w}) > t\}\\ L(\mathcal{L}_\mu,t) = \{\mathfrak{w}\in L : \mathcal{L}_\mu(\mathfrak{w})\leq t\}\\ L^-(\mathcal{L}_\mu,t) = \{\mathfrak{w}\in L : \mathcal{L}_\mu(\mathfrak{w}) < t\} \end{gathered}$$

are referred to as an upper t-level subset, an upper t-strong level subset, a lower t-level subset and a lower t-strong level subset of $\mathscr{L}$, respectively.

Theorem 3.13. *An $\mathscr{L}$-fuzzy set $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L if and only if $U(\mathcal{L}_\mu,t)$ is, if it is nonempty, a SBG-subalgebra of L for every $t\in\mathscr{L}$.*

Proof. Assume $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. Let $t\in\mathscr{L}$ be such that $U(\mathcal{L}_\mu,t)\neq\emptyset$. Let $\mathfrak{w},\mathfrak{b}\in L$. Then

$$\begin{aligned} \mathfrak{w},\mathfrak{b}\in U(\mathcal{L}_\mu,t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{w})\geq t,\mathcal{L}_\mu(\mathfrak{b})\geq t\\ &\Rightarrow \mathcal{L}_\mu(\mathfrak{w})\wedge\mathcal{L}_\mu(\mathfrak{b})\geq t\\ &\Rightarrow \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}})\geq\mathcal{L}_\mu(x)\wedge\mathcal{L}_\mu(\mathfrak{b})\geq t\\ &\Rightarrow \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}})\geq t\\ &\Rightarrow \mathfrak{w}^{\mathfrak{b}}\in U(\mathcal{L}_\mu,t). \end{aligned}$$

Hence, $U(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L. Conversely, assume for all $t \in \mathscr{L}$, $U(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L if it is nonempty. Let $\mathfrak{w}, \mathfrak{b} \in L$. Choose $t = \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{w}) \geq t$ and $\mathcal{L}_\mu(\mathfrak{b}) \geq t$. Thus $\mathfrak{w}, \mathfrak{b} \in U(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L and so $\mathfrak{w}^\mathfrak{b} \in U(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \geq t = \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b})$. Hence $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. □

Theorem 3.14. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L if and only if $U^+(\mathcal{L}_\mu, t)$ is, if it is nonempty, a SBG-subalgebra of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. Let $t \in \mathscr{L}$ be such that $U^+(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{w}, \mathfrak{b} \in L$. Then $\mathcal{L}_\mu(\mathfrak{w})$ and $\mathcal{L}_\mu(\mathfrak{b})$ are compatible. Suppose that $\mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu(\mathfrak{b})$, that is, $\mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) = \mathcal{L}_\mu(\mathfrak{b})$. Then

$$\begin{aligned}
\mathfrak{w}, \mathfrak{b} \in U^+(\mathcal{L}_\mu, t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{w}) > t, \mathcal{L}_\mu(\mathfrak{b}) > t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(y) = \mathcal{L}_\mu(\mathfrak{b}) > t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) > t \\
&\Rightarrow \mathfrak{w}^\mathfrak{b} \in U^+(\mathcal{L}_\mu, t).
\end{aligned}$$

Hence, $U^+(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L.

Conversely, assume for all $t \in \mathscr{L}$, $U^+(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L if it is nonempty. Suppose there exist $\mathfrak{w}, \mathfrak{b}\ inL$ such that $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \ngeq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b})$. It means that $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) < \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b})$. Choose $t = \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) > t$ and so $\mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) > t$ and $\mathcal{L}_\mu(\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) > t$. Thus $\mathfrak{w}, \mathfrak{b} \in U^+(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L and so $\mathfrak{w}^\mathfrak{b} \in U^+(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) > t = \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b})$, a contradiction. Hence $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b})$ for all $\mathfrak{w}, \mathfrak{b} \in L$. Hence $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. □

Definition 3.15. Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_\mathscr{L}, 1_\mathscr{L})$ be a Boolean lattice. Let $\mathcal{L}$ be an $\mathscr{L}$-fuzzy set in L. The $\mathscr{L}$-fuzzy set $\mathcal{L}'$ defined by $(\forall \mathfrak{p} \in L)(\mathcal{L}'_\mu(\mathfrak{p}) = (\mathcal{L}_\mu(\mathfrak{p}))' = \mathcal{L}_\mu(\mathfrak{p})')$ is called the complement of $\mathcal{L}$ in L.

Theorem 3.16. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_\mathscr{L}, 1_\mathscr{L})$ be a Boolean lattice. Then $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L if and only if $L(\mathcal{L}_\mu, t)$ is, if it is nonempty, a SBG-subalgebra of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. Let $t \in \mathscr{L}$ be such that $L(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{w}, \mathfrak{b} \in L$. Then

$$\begin{aligned}
\mathfrak{w}, \mathfrak{b} \in L(\mathcal{L}_\mu, t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{w}) \leq t, \mathcal{L}_\mu(\mathfrak{b}) \leq t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{b}) \leq t \\
&\Rightarrow ((\mathcal{L}_\mu(\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{b}))' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w})' \wedge \mathcal{L}_\mu(\mathfrak{b})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b})' \geq \mathcal{L}_\mu(\mathfrak{w})' \wedge \mathcal{L}_\mu(\mathfrak{b})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \leq t \\
&\Rightarrow \mathfrak{w}^\mathfrak{b} \in L(\mathcal{L}_\mu, t).
\end{aligned}$$

Hence, $L(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L.

Conversely, assume for all $t \in \mathscr{L}$, $L(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L if it is nonempty. Let $\mathfrak{w}, \mathfrak{b} \in L$. Choose $t = \mathcal{L}_\mu(\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{b}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{w}) \leq t$ and $\mathcal{L}_\mu(\mathfrak{b}) \leq t$. Thus $\mathfrak{w}, \mathfrak{b} \in U(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L(\mathcal{L}_\mu, t)$ is a SBG-subalgebra of L and so $\mathfrak{w}^\mathfrak{b} \in L(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}) \leq t = \mathcal{L}_\mu(\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{b})$. By Lemma 2.8 (1), we have $\mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b})' \geq t = \mathcal{L}_\mu(\mathfrak{w})' \wedge \mathcal{L}'_\mu(\mathfrak{b})$. Hence $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. □

Theorem 3.17. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_\mathscr{L}, 1_\mathscr{L})$ be a Boolean lattice with $\leq$ a linearly ordered set. Then $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L if and only if $L^-(\mathcal{L}_\mu, t)$ is, if it is nonempty, a SBG-subalgebra of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. Let $t \in \mathscr{L}$ be such that $L^{-}(\mathcal{L}_{\mu}, t) \neq \emptyset$. Let $x, y \in L^{-}(\mathcal{L}_{\mu}, t)$. Then $\mathcal{L}_{\mu}(\mathfrak{w})$ and $\mathcal{L}_{\mu}(\mathfrak{b})$ are compatible. Suppose that $\mathcal{L}_{\mu}(\mathfrak{w}) \leq \mathcal{L}_{\mu}(\mathfrak{b})$, that is, $\mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}) = \mathcal{L}_{\mu}(\mathfrak{b})$. Then

$$\begin{aligned}
\mathfrak{w}, \mathfrak{b} \in L^{-}(\mathcal{L}_{\mu}, t) &\Rightarrow \mathcal{L}_{\mu}(\mathfrak{w}) < t, \mathcal{L}_{\mu}(\mathfrak{b}) < t \\
&\Rightarrow \mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}) = \mathcal{L}_{\mu}(\mathfrak{b}) < t \\
&\Rightarrow (\mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}))' = \mathcal{L}_{\mu}(\mathfrak{b})' > t' \\
&\Rightarrow \mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})' \geq \mathcal{L}_{\mu}(\mathfrak{w})' \wedge \mathcal{L}_{\mu}(\mathfrak{b})' > t' \\
&\Rightarrow \mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})' > t' \\
&\Rightarrow \mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}}) < t \\
&\Rightarrow \mathfrak{w}^{\mathfrak{b}} \in L^{-}(\mathcal{L}_{\mu}, t).
\end{aligned}$$

Hence, $L^{-}(\mathcal{L}_{\mu}, t)$ is a SBG-subalgebra of L.

Conversely, assume for all $t \in \mathscr{L}$, $L^{-}(\mathcal{L}_{\mu}, t)$ is a SBG-subalgebra of L if it is nonempty. Suppose there exist $\mathfrak{w}, \mathfrak{b} \in L$ such that $\mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})' \not\geq \mathcal{L}_{\mu}(\mathfrak{w})' \wedge \mathcal{L}_{\mu}(\mathfrak{b})'$. It means that $\mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})' < \mathcal{L}_{\mu}(\mathfrak{w})' \wedge \mathcal{L}_{\mu}(\mathfrak{b})'$. By Lemma 2.8 (1), we have $\mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})' < \mathcal{L}_{\mu}(\mathfrak{w})' \wedge \mathcal{L}_{\mu}(\mathfrak{b})' = (\mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}))'$. By Lemma 2.8 (5), we have $\mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}}) > \mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b})$. Choose $t = \mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}}) \in \mathscr{L}$. Then $\mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}) < t$ and so $\mathcal{L}_{\mu}(\mathfrak{w}) \leq \mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}) < t$ and $\mathcal{L}_{\mu}(\mathfrak{b}) \leq \mathcal{L}_{\mu}(\mathfrak{w}) \vee \mathcal{L}_{\mu}(\mathfrak{b}) < t$. Thus $\mathfrak{w}, \mathfrak{b} \in L^{-}(\mathcal{L}_{\mu}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mathcal{L}_{\mu}, t)$ is a subalgebra of L and so $\mathfrak{w}^{\mathfrak{b}} \in L^{-}(\mathcal{L}_{\mu}, t)$. Thus $\mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}}) < t = \mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})$, a contradiction. Hence $\mathcal{L}_{\mu}(\mathfrak{w}^{\mathfrak{b}})' \geq \mathcal{L}_{\mu}(\mathfrak{w})' \wedge \mathcal{L}_{\mu}(\mathfrak{b})'$ for all $\mathfrak{w}, \mathfrak{b} \in L$. Hence $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-subalgebra of L. □

Theorem 3.18. *A nonempty subset A of L is an SBG-ideal of L if and only if the characteristic $\mathscr{L}$-fuzzy set $\mathcal{L}_A$ is an $\mathscr{L}$-fuzzy SBG-ideal of L.*

Proof. Assume that A is an SBG-ideal of L. Since $0 \in A$, it follows from Lemma 3.1 that $\chi_A(0) \geq \chi_A(\mathfrak{w})$ for all $\mathfrak{w} \in L$. Let $\mathfrak{w}, \mathfrak{b} \in L$.

Case 1: $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in A$. Then $\chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 1_{\mathscr{L}} = \chi_A(\mathfrak{w})$, so $\chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_A(\mathfrak{w}) = 1_{\mathscr{L}}$. Since A is an SBG-ideal of L, $\mathfrak{b} \in A$ and so $\chi_A(\mathfrak{b}) = 1_{\mathscr{L}}$. Therefore, $\chi_A(\mathfrak{b}) = 1_{\mathscr{L}} \geq 1_{\mathscr{L}} = \chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_A(\mathfrak{w})$.

Case 2: $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \notin A$ or $\mathfrak{w} \notin A$. Then $\chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 0_{\mathscr{L}}$ or $\chi_A(\mathfrak{w}) = 0_{\mathscr{L}}$, so $\chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_A(\mathfrak{w}) = 0_{\mathscr{L}}$. Therefore, $\chi_A(\mathfrak{b}) \geq 0_{\mathscr{L}} = \chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_A(\mathfrak{w})$. Hence, $\mathcal{L}_A$ is an $\mathscr{L}$-fuzzy SBG-ideal of L.

Conversely, assume that $\mathcal{L}_A$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. Since $\chi_A(0) \geq \chi_A(\mathfrak{w})$ for all $\mathfrak{w} \in L$, by Lemma 3.1 that $0 \in A$. Let $\mathfrak{w}, \mathfrak{b} \in L$ such that $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, x \in A$. Then $\chi_A((\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = 1_{\mathscr{L}} = \chi_A(\mathfrak{w})$, so $\chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_A(\mathfrak{w}) = 1_{\mathscr{L}}$. Since $\mathcal{L}_A$ is an $\mathscr{L}$-fuzzy SBG-ideal of L, $1_{\mathscr{L}} \geq \chi_A(\mathfrak{b}) \geq \chi_A(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \chi_A(\mathfrak{w}) = 1_{\mathscr{L}}$. By anti-symmetry, we have $\chi_A(\mathfrak{b}) = 1_{\mathscr{L}}$, that is, $\mathfrak{b} \in A$. Hence A is an SBG-ideal of L. □

Lemma 3.19. *Let $\mathcal{L}$ be an $\mathscr{L}$-fuzzy set in L. Then $\mathcal{L}$ satisfies the condition $\mathcal{L}_{\mu}(0) \geq \mathcal{L}_{\mu}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathscr{H}$ if and only if $U(\mathcal{L}_{\mu}, t)$, if it is nonempty, contains $0 \in L$ for every $t \in \mathcal{L}$.*

Proof. Let $t \in \mathcal{L}$ be such that $U(\mathcal{L}_{\mu}, t) \neq \emptyset$. Let $\mathfrak{p} \in L$. Then

$$\begin{aligned}
\mathfrak{p} \in U(\mathcal{L}_{\mu}, t) &\Rightarrow \mathcal{L}_{\mu}(\mathfrak{p}) \geq t \\
&\Rightarrow \mathcal{L}_{\mu}(0) \geq \mathcal{L}_{\mu}(\mathfrak{p}) \geq t \\
&\Rightarrow 0 \in U(\mathcal{L}_{\mu}, t).
\end{aligned}$$

Conversely, assume for all $t \in \mathcal{L}$, $U(\mathcal{L}_{\mu}, t)$ contains $0 \in L$ if it is nonempty. Choose $t = \mathcal{L}_{\mu}(\mathfrak{p}) \in \mathscr{L}$. Then $\mathcal{L}_{\mu}(\mathfrak{p}) \geq t$. Thus $a \in U(\mathcal{L}_{\mu}, t) \neq \emptyset$. As the hypothesis, $0 \in U(\mathcal{L}_{\mu}, t)$. Thus $\mathcal{L}_{\mu}(0) \geq t = \mathcal{L}_{\mu}(\mathfrak{p})$. □

Theorem 3.20. *An $\mathscr{L}$-fuzzy set $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L if and only if $U(\mathcal{L}_{\mu}, t)$ is, if it is nonempty, an SBG-ideal of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. Let $t \in \mathscr{L}$ be such that $U(\mathcal{L}_{\mu}, t) \neq \emptyset$. Let

$\mathfrak{w}, \mathfrak{b} \in L$. Then

$$\begin{array}{rcl} \mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in U(\mathcal{L}_\mu, t) & \Rightarrow & \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \geq t, \mathcal{L}_\mu(\mathfrak{w}) \geq t \\ & \Rightarrow & \mathcal{L}_\mu(\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) \geq t \\ & \Rightarrow & \mathcal{L}_\mu(\mathfrak{b}) \geq t \\ & \Rightarrow & \mathfrak{b} \in U(\mathcal{L}_\mu, t). \end{array}$$

By Lemma 3.19, we have $0 \in U(\mathcal{L}_\mu, t)$. Hence $U(\mathcal{L}_\mu, t)$ is an SBG-ideal of L.

Conversely, assume for all $t \in \mathscr{L}$, $U(\mathcal{L}_\mu, t)$ is an SBG-ideal of L if it is nonempty. Let $\mathfrak{w}, \mathfrak{b} \in L$. By Lemma 3.19. we have $\mathcal{L}_\mu$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$. Let $\mathfrak{w}, \mathfrak{b} \in L$. Choose $t = \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \geq t$ and $\mathcal{L}_\mu(\mathfrak{w}) \geq t$. Thus $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in U(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U(\mathcal{L}_\mu, t)$ is an SBG-ideal of L and so $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in U(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{b}) \geq t = \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w})$. Hence $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. □

Lemma 3.21. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then $\mathcal{L}$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$ if and only if $U^+(\mathcal{L}_\mu, t)$, if it is nonempty, contains $0 \in L$ for every $t \in \mathcal{L}$.*

Proof. Let $t \in \mathcal{L}$ be such that $U^+(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{p} \in L$. Then

$$\begin{array}{rcl} \mathfrak{p} \in U^+(\mathcal{L}_\mu, t) & \Rightarrow & \mathcal{L}_\mu(\mathfrak{p}) > t \\ & \Rightarrow & \mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p}) > t \\ & \Rightarrow & 0 \in U^+(\mathcal{L}_\mu, t). \end{array}$$

Conversely, assume for all $t \in \mathcal{L}$, $U^+(\mathcal{L}_\mu, t)$ contains $0 \in L$ if it is nonempty. Suppose there exists $\mathfrak{w} \in L$ such that $\mathcal{L}_\mu(0) \ngeq \mathcal{L}_\mu(\mathfrak{w})$. It means that $\mathcal{L}_\mu(0) < \mathcal{L}_\mu(\mathfrak{w})$. Choose $t = \mathcal{L}_\mu(0) \in \mathcal{L}$. Then $\mathcal{L}_\mu(\mathfrak{w}) > t$. Thus $x \in U^+(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $0 \in U^+(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(0) > t = \mathcal{L}_\mu(0)$, a contradiction. Hence $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ for all $\mathfrak{w} \in L$. □

Theorem 3.22. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L if and only if $U^+(\mathcal{L}_\mu, t)$ is, if it is nonempty, an SBG-ideal of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. Let $t \in \mathscr{L}$ be such that $U^+(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{w}, \mathfrak{s} \in L$. Then $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})$ and $\mathcal{L}_\mu(\mathfrak{w})$ are compatible. Suppose that $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \geq \mathcal{L}_\mu(\mathfrak{w})$, that is, $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(\mathfrak{w})$. Then

$$\begin{array}{rcl} \mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in U^+(\mathcal{L}_\mu, t) & \Rightarrow & \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) > t, \mathcal{L}_\mu(\mathfrak{w}) > t \\ & \Rightarrow & \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(\mathfrak{w}) > t \\ & \Rightarrow & \mathcal{L}_\mu(\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) > t \\ & \Rightarrow & \mathfrak{b} \in U^+(\mathcal{L}_\mu, t). \end{array}$$

By Lemma 3.19, we have $0 \in U^+(\mathcal{L}_\mu, t)$. Hence, $U^+(\mathcal{L}_\mu, t)$ is an SBG-ideal of L.

Conversely, assume for all $t \in \mathscr{L}$, $U^+(\mathcal{L}_\mu, t)$ is an SBG-ideal of L if it is nonempty. By Lemma 3.19, we have $\mathcal{L}$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$. Suppose there exist $\mathfrak{w}, \mathfrak{b} \in L$ such that $\mathcal{L}_\mu(\mathfrak{b}) \ngeq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w})$. It means that $\mathcal{L}_\mu(\mathfrak{b}) < \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w})$. By Lemma 3.19, we have $\mathcal{L}$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$. Choose $t = \mathcal{L}_\mu(\mathfrak{b}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) > t$ and so $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \geq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) > t$ and $\mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) > t$. Thus $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in U^+(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mathcal{L}_\mu, t)$ is an SBG-ideal of L and so $\mathfrak{b} \in U^+(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{b}) > t = \mathcal{L}_\mu(\mathfrak{b})$, a contradiction. Hence $\mathcal{L}_\mu(\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}) \wedge \mathcal{L}_\mu(\mathfrak{w})$ for all $\mathfrak{w}, \mathfrak{b} \in L$. Hence $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. □

Lemma 3.23. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_{\mathscr{L}}, 1_{\mathscr{L}})$ be a Boolean lattice. and $\mathcal{L}$ be an $\mathscr{L}$-fuzzy set in L. Then $\mathcal{L}'$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$ if and only if $L(\mathcal{L}_\mu, t)$, if it is nonempty, contains $0 \in L$ for every $t \in \mathcal{L}$.*

Proof. Let $t \in \mathcal{L}$ be such that $L(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{p} \in L$. Then

$$\begin{aligned}
\mathfrak{p} \in L(\mathcal{L}_\mu, t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{p}) \leq t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{p})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(0)' \geq \mathcal{L}_\mu(\mathfrak{p})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(0)' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(0) \leq t \\
&\Rightarrow 0 \in L(\mathcal{L}_\mu, t).
\end{aligned}$$

Conversely, assume for all $t \in \mathcal{L}$, $L(\mathcal{L}_\mu, t)$ contains $0 \in L$ if it is nonempty. Choose $t = \mathcal{L}_\mu(\mathfrak{p}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{p}) \leq t$. Thus $\mathfrak{p} \in L(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, $0 \in L(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(0) \leq t = \mathcal{L}_\mu(\mathfrak{p})$. By Lemma 2.8 (3), we have $\mathcal{L}_\mu(0)' \geq \mathcal{L}_\mu(\mathfrak{p})'$. □

Theorem 3.24. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_\mathscr{L}, 1_\mathscr{L})$ be a Boolean lattice. Then $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-ideal of L if and only if $L(\mathcal{L}_\mu, t)$ is, if it is nonempty, an SBG-ideal of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. Let $t \in \mathscr{L}$ be such that $L(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{w}, \mathfrak{b} \in L$. Then

$$\begin{aligned}
\mathfrak{w}, \mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w} \in L(\mathcal{L}_\mu, t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \leq t, \mathcal{L}_\mu(\mathfrak{w}) \leq t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{w}) \leq t \\
&\Rightarrow (\mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{w}))' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w})' \wedge \mathcal{L}_\mu(\mathfrak{w})' = (\mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{w}))' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b})' \geq \mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w})' \wedge \mathcal{L}_\mu(\mathfrak{w})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b}) \leq t \\
&\Rightarrow \mathfrak{b} \in L(\mathcal{L}_\mu, t).
\end{aligned}$$

By Lemma 3.23, we have $0 \in L(\mathcal{L}_\mu, t)$. Hence, $L(\mathcal{L}_\mu, t)$ is an SBG-ideal of L.

Conversely, assume for all $t \in \mathscr{L}$, $L(\mathcal{L}_\mu, t)$ is an SBG-ideal of L if it is nonempty. Let $\mathfrak{w} \in L$. By Lemma 3.23, we have $\mathcal{L}_\mu$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$. Let $\mathfrak{w}, \mathfrak{b} \in L$. Choose $t = \mathcal{L}_\mu((\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{w}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \leq t$ and $\mathcal{L}_\mu(\mathfrak{w}) \leq t$. Thus $\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}, \mathfrak{w} \in L(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L(\mathcal{L}_\mu, t)$ is an SBG-ideal of L and so $\mathfrak{w}^\mathfrak{b}, \mathfrak{w} \in L(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{b}) \leq t = \mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w}) \vee \mathcal{L}_\mu(\mathfrak{w})$. By Lemma 2.8 (0), we have $\mathcal{L}_\mu(\mathfrak{b})' \geq \mathcal{L}_\mu(\mathfrak{b}^\mathfrak{w}|\mathfrak{b}^\mathfrak{w})' \wedge \mathcal{L}_\mu(\mathfrak{w})'$. Hence $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. □

Lemma 3.25. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_\mathscr{L}, 1_\mathscr{L})$ be a Boolean lattice with $\leq$ a linearly ordered set and $\mathcal{L}'$ be an $\mathscr{L}$-fuzzy set in L. Then $\mathcal{L}'$ satisfies the condition $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{p})$ for all $\mathfrak{p} \in L$ if and only if $L^-(\mathcal{L}_\mu, t)$, if it is nonempty, contains $0 \in L$ for every $t \in \mathcal{L}$.*

Proof. Let $t \in \mathcal{L}$ be such that $L(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{p} \in L$. Then

$$\begin{aligned}
\mathfrak{p} \in L(\mathcal{L}_\mu, t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{p}) \leq t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{p})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(0)' \geq \mathcal{L}_\mu(\mathfrak{p})' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(0)' \geq t' \\
&\Rightarrow \mathcal{L}_\mu(0) \leq t \\
&\Rightarrow 0 \in L(\mathcal{L}_\mu, t).
\end{aligned}$$

Conversely, assume for all $t \in \mathcal{L}$, $L(\mathcal{L}_\mu, t)$ contains $0 \in L$ if it is nonempty. Suppose there exists $\mathfrak{p} \in L$ such that $\mathcal{L}_\mu(0)' \not\geq \mathcal{L}_\mu(\mathfrak{p})'$. It means that $\mathcal{L}_\mu(0)' < \mathcal{L}_\mu(\mathfrak{p})'$. By Lemma 2.8 (5), we have $\mathcal{L}_\mu(0) > \mathcal{L}_\mu(\mathfrak{p})$. Choose $t = \mathcal{L}_\mu(0) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{p}) < t$. Thus $a \in L^-(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, $0 \in L^-(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(0) < t = \mathcal{L}_\mu(0)$, a contradiction. Hence $\mathcal{L}_\mu(0)' \geq \mathcal{L}_\mu(\mathfrak{p})'$ for all $\mathfrak{p} \in L$. □

Theorem 3.26. *Let $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_\mathscr{L}, 1_\mathscr{L})$ be a Boolean lattice with $\leq$ a linearly ordered set. Then $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-ideal of L if and only if $L(\mathcal{L}_\mu, t)$ is, if it is nonempty, an SBG-ideal of L for every $t \in \mathscr{L}$.*

Proof. Assume $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. Let $t \in \mathscr{L}$ be such that $L^-(\mathcal{L}_\mu, t) \neq \emptyset$. Let $\mathfrak{w}, \mathfrak{b} \in L$. Then $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})$ and $\mathcal{L}_\mu(\mathfrak{w})$ are compatible. Suppose that $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \geq \mathcal{L}_\mu(\mathfrak{w})$, that is, $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) = \mathcal{L}_\mu(\mathfrak{w})$. Then

$$\begin{aligned}
\mathfrak{w}, \mathfrak{b} \in L^-(\mathcal{L}_\mu, t) &\Rightarrow \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) < t, \mathcal{L}_\mu(\mathfrak{w}) < t \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w}) < t \\
&\Rightarrow (\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w}))' > t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})' \wedge \mathcal{L}_\mu(\mathfrak{w})' = \\
&\qquad (\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w}))' > t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b})' \geq \mathcal{L}_\mu(x|(y|y))' \wedge \mathcal{L}_\mu(\mathfrak{w})' > t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b})' > t' \\
&\Rightarrow \mathcal{L}_\mu(\mathfrak{b}) < t \\
&\Rightarrow \mathfrak{b} \in L^-(\mathcal{L}_\mu, t).
\end{aligned}$$

By Lemma 3.25, we have $0 \in L^-(\mathcal{L}_\mu, t)$. Hence $L^-(\mathcal{L}_\mu, t)$ is an SBG-ideal of L.

Conversely, assume for all $t \in \mathscr{L}$, $L^-(\mathcal{L}_\mu, t)$ is an SBG-ideal of L if it is nonempty. Suppose there exist $\mathfrak{w}, \mathfrak{b} \in L$ such that $\mathcal{L}_\mu(\mathfrak{b})' \ngeq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})' \wedge \mathcal{L}_\mu(\mathfrak{w})'$. It means that $\mathcal{L}_\mu(\mathfrak{b})' < \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})' \wedge \mathcal{L}_\mu(\mathfrak{w})'$. By Lemma 2.8 (0), we have $\mathcal{L}_\mu(\mathfrak{b})' < \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})' \wedge \mathcal{L}_\mu(\mathfrak{w})' = (\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w}))'$. By Lemma 2.8 (5), we have $\mathcal{L}_\mu(\mathfrak{b}) > \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w})$. Choose $t = \mathcal{L}_\mu(\mathfrak{b}) \in \mathscr{L}$. Then $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w}) < t$ and so $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \leq \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}) \vee \mathcal{L}_\mu(\mathfrak{w}) < t$ and $\mathcal{L}_\mu(\mathfrak{w}) \leq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \vee \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}) < t$. Thus $x, \mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \in L^-(\mathcal{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L^-(\mathcal{L}_\mu, t)$ is an SBG-ideal of L and so $z \in L^-(\mathcal{L}_\mu, t)$. Thus $\mathcal{L}_\mu(\mathfrak{b}) < t = \mathcal{L}_\mu(\mathfrak{b})$, a contradiction. Hence $\mathcal{L}_\mu(\mathfrak{b})' \geq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}})' \wedge \mathcal{L}_\mu(\mathfrak{w})'$ for z all $\mathfrak{w}, \mathfrak{b}\ inL$. Hence $\mathcal{L}'$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. □

Theorem 3.27. *Let A be a nonempty subset of L and $\mathcal{L}$ be an $\mathscr{L}$-fuzzy set in L defined by*

$$\mathcal{L}_\mu(\mathfrak{w}) = \begin{cases} \alpha_1 & \text{if } \mathfrak{w} \in A \\ \alpha_2 & otherwise, \end{cases}$$

for all $\mathfrak{w} \in L$. Then $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L.

Proof. Assume that $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. Since $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ for all $\mathfrak{w} \in L$, $\mathcal{L}_\mu(0) = \alpha_1$ and so $0 \in A$. Let $\mathfrak{w}, \mathfrak{b} \in L$ such that $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in A$, we get $\mathcal{L}_\mu(\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w}) = \alpha_1$, which implies that $\mathcal{L}_\mu(\mathfrak{b}) = \alpha_1$. It follows that $\mathfrak{b} \in A$. Therefore, A is an SBG-ideal of L.

Conversely, suppose that A is an SBG-ideal of L. Since $0 \in A$, $\mathcal{L}_\mu(0) = \alpha_1 \geq \mathcal{L}_\mu(x)$ for all $\mathfrak{w} \in L$. Let $\mathfrak{w}, \mathfrak{b} \in L$. If $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} \in L \backslash A$ or $\mathfrak{w} \in L \backslash A$, then $\mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) = \alpha_2$ or $\mathcal{L}_\mu(\mathfrak{w}) = \alpha_2$. It follows that $\mathcal{L}_\mu(\mathfrak{b}) \geq \alpha_2 = \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w})$. Assume that $\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}, \mathfrak{w} \in A$. Then $\mathfrak{b} \in A$ and thus $\mathcal{L}_\mu(\mathfrak{b}) = \alpha_1 = \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}}) \wedge \mathcal{L}_\mu(\mathfrak{w})$. Hence $\mathcal{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of L. □

Conclusion and Future Work

In this paper, we established a comprehensive characterization of $\mathscr{L}$-fuzzy SBG-subalgebras and $\mathscr{L}$-fuzzy SBG-ideals using characteristic functions and various types of level sets. Our results show that classical SBG-algebraic structures can be completely recovered from their $\mathscr{L}$-fuzzy counterparts, thereby providing a robust bridge between crisp and fuzzy algebraic theories.

At present, no nontrivial concrete example or real-world application explicitly illustrating these characterizations is known. The construction of meaningful examples—particularly those arising from decision-making under uncertainty, artificial intelligence reasoning, or fuzzy control systems—remains an open problem. Identifying such examples would significantly enhance the practical relevance of fuzzy SBG-algebras and is a natural direction for future research.

Further research directions include the development of efficient computational algorithms for large-scale or dynamically evolving fuzzy SBG-systems, as well as extensions of the present framework to non-Boolean and non-linearly ordered lattices. Another promising line of research concerns the integration of fuzzy SBG-algebras with probabilistic and hybrid fuzzy–logical models. Moreover, a deeper comparative analysis with recently introduced fuzzy structures, such as linear Diophantine fuzzy algebras and bipolar fuzzy SBG-algebras, may yield additional insights and enrich the theory.

We believe that the results presented in this work provide a solid and flexible theoretical foundation for these future investigations.We believe that the results presented here provide a solid theoretical foundation for these future investigations.

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