

# ON (WEAKLY) $e$ -PRIMARY IDEALS IN COMMUTATIVE RINGS

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**Abstract** We introduce and study (weakly)  $e$ -primary ideals in commutative rings  $R$  with a nonzero unit, as a generalization of prime ideals. A proper ideal  $I$  of  $R$  is termed (weakly)  $e$ -primary, where  $e \in R \setminus I$  is an idempotent, if whenever  $(0 \neq) ab \in I$  for  $a, b \in R$ , it follows that  $ea \in I$  or  $eb \in \sqrt{I}$ . Various properties and results on (weakly)  $e$ -primary ideals are established. Moreover, the transfer of  $e$ -primariness within several ring-theoretic constructions is investigated. These include localization rings, polynomial rings, trivial ring extensions, and amalgamated rings.

## 1 Introduction

Throughout the sequel, all rings are assumed to be commutative with unit  $1 \neq 0$ , and  $R$  will denote such a ring. Several generalizations of prime ideals have been studied, see [2, 4, 6, 7, 8, 10, 11] for examples. Recall that a proper (i.e.,  $I \neq R$ ) ideal  $I$  of  $R$  is prime if for all  $a, b \in R$ ,  $ab \in I$  implies that  $a \in I$  or  $b \in I$ . In general, most of such generalizations go in two main directions, either by enlarging where  $a$  and  $b$  might be in, or by posing further restrictions on the product  $ab$ . An example of the latter case is when  $0 \neq ab \in I$ , which yields the concept of so called weakly prime ideals. Another instance is the very recently introduced concept of a square-difference factor absorbing ideal [3]. Following [3], a proper ideal  $I$  of  $R$  is called a square-difference factor absorbing ideal if for all  $0 \neq a, b \in R$ ,  $a^2 - b^2 \in I$  implies that  $a + b \in I$  or  $a - b \in I$ . In terms of such ideals, several characterizations of particular classes of rings were established. In the recent work [1], a further generalization of prime ideals was introduced and termed  $e$ -prime ideals. A proper ideal  $I$  of  $R$  is  $e$ -prime, where  $e \in R \setminus I$  is an idempotent, if for all  $a, b \in R$ ,  $ab \in I$  implies that  $ea \in I$  or  $eb \in I$ ; that is,  $a \in (I : e)$  or  $b \in (I : e)$ .

Motivated by the latter concept, our aim in this paper is to develop a theory of (weakly)  $e$ -primary ideals which generalize  $e$ -prime ideals. A proper ideal  $I$  of  $R$  is an (weakly)  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent, if for all  $a, b \in R$  such that  $(0 \neq) ab \in I$ , it follows that  $ea \in I$  or  $eb \in \sqrt{I}$ . As one can easily notice, every  $e$ -prime ideal is  $e$ -primary. However, the converse is not valid in general (see Example 2.2). We show that a proper ideal  $I$  is  $e$ -primary if and only if  $(I : e)$  is primary (see Theorem 2.6). In Theorem 2.15, a characterization of  $e$ -primariness is given. As in the usual case, we show that the radical of any  $e$ -primary ideal is  $e$ -prime (see Theorem 2.7) and that  $e\sqrt{I}$  is the smallest  $e$ -prime ideal of  $R$  containing  $eI$ ; which results in that  $e\sqrt{I} = eI$  when  $I$  is  $e$ -prime (see Theorem 2.9 and Corollary 2.10). Moreover, we define  $e$ -maximality and  $e$ -irreducibility then establish  $e$ -primariness of  $I$  in the following cases: (i) when  $\sqrt{I}$  is an  $e$ -maximal ideal of  $R$  (Theorem 2.12) and (ii) when  $I$  is an  $e$ -irreducible ideal of a Noetherian ring  $R$  (Theorem 2.14). Furthermore, additional results are established such as  $e$ -primariness of the intersection of  $e$ -primary ideals with a common radical.

In Section 3, we introduce weakly  $e$ -primary ideals. Obviously, every  $e$ -primary ideal is weakly  $e$ -primary. We provide a counterexample to show that the converse may fail (see Example 3.2). We show, among several results, that every weakly  $e$ -primary ideal which is not  $e$ -primary results in that  $I^2$  is the zero ideal  $(0)$ , and we then deduce that its radical consists all nilpotent elements of  $R$  (see Theorem 3.6 and Corollary 3.7). However, an ideal verifying  $I^2 = (0)$  need

not be weakly  $e$ -primary (see Example 3.8). For a family of weakly  $e$ -primary ideals which are not  $e$ -primary,  $e$ -primariness of their intersection is ensured and the assumption of “common radical” becomes redundant (see Theorem 3.9).

Section 4 is reserved to the investigation of  $e$ -primary ideals across various ring constructions. More precisely, we examine the transfer of  $e$ -primariness under homomorphisms of rings (Theorem 4.3 and Corollary 4.4) and to localization rings (Theorem 4.2), direct product rings (Theorems 4.5), polynomial rings (Theorems 4.1), trivial ring extensions (Theorem 4.6), and amalgamated rings (Theorem 4.7).

## 2 Properties of $e$ -primary ideals

The current section is devoted to the study of some essential properties of  $e$ -primary ideals.

**Definition 2.1.** Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. We say that  $I$  is an  $e$ -primary ideal of  $R$  if whenever  $ab \in I$  for  $a, b \in R$ , it follows that  $ea \in I$  or  $eb \in \sqrt{I}$ .

Obviously, every  $e$ -prime ideal of  $R$  is  $e$ -primary. However, the converse does not hold in general, as shown by the next example.

**Example 2.2.** Let  $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ ,  $e = (1, 0)$  and  $I = (0) \times (0)$ . Then,  $e$  is an idempotent with  $e \notin I$  and  $I$  is an  $e$ -primary ideal of  $R$  which is not  $e$ -prime. Indeed, for every  $(a, b), (x, y) \in R$  such that  $(ax, by) \in I$ , we have to show that  $(a, 0) = (0, 0)$  or  $(x, 0) \in \{0, 2\} \times \{0, 2\} = \sqrt{I}$ . If  $a = 0$ , the result holds. Otherwise,  $x \in \{0, 2\}$  since  $ax = 0$ . Hence,  $I$  is  $e$ -primary. On the other hand,  $(2, 0)(2, 0) = (0, 0) \in I$ , but  $e(2, 0) = (2, 0) \notin I$ . Thus,  $I$  is not  $e$ -prime.

**Remark 2.3.**

- (1) Notice that in the previous definition, one can equivalently replace “ $ea \in I$  or  $eb \in \sqrt{I}$ ” by “ $eb \in I$  or  $ea \in \sqrt{I}$ ”.
- (2) Let  $e \in R$  be an idempotent. Then, for every ideal  $I$  of  $R$ ,  $e \notin \sqrt{I} \setminus I$ . Indeed, suppose  $e \in \sqrt{I}$ , then  $e^n \in I$  for some  $n \in \mathbb{N}$ . But  $e^n = e$ , so  $e \in I$ . This justifies why, in Definition 2.1, we only assumed that  $e \in R \setminus I$  instead of  $e \in R \setminus \sqrt{I}$ .
- (3) Recall [12] that a proper ideal  $I$  of  $R$  is called  $S$ -primary, where  $S$  is a multiplicative subset of  $R$  with  $I \cap S = \emptyset$ , if there is some  $s \in S$  such that whenever  $ab \in I$  for  $a, b \in R$ , we get  $sa \in I$  or  $sb \in \sqrt{I}$ . Let  $S$  be the set of all idempotent elements of  $R$ , and let  $I$  be a proper ideal of  $R$ , disjoint with  $S$ . Then,  $I$  is  $e$ -primary for some idempotent  $e \in S$  if and only if  $I$  is  $S$ -primary. Indeed,  $S$  is nonempty ( $1 \in S$ ), multiplicative and  $S \cap I = \emptyset$  by assumption.
- (4) Recall that a Boolean ring  $R$  is a ring where all elements are idempotent. In a Boolean ring  $R$ ,  $e$ -prime ideals coincide with  $e$ -primary ideals since  $\sqrt{I} = I$  for every proper ideal  $I$  of  $R$ .

**Example 2.4.** Let  $R = \mathbb{Z} \times \mathbb{Z}$ ,  $e = (1, 0)$  and  $I = (0) \times (p)$  where  $p \in \mathbb{Z}$  is some prime integer. Then,  $I$  is an  $e$ -primary ideal of  $R$ . Indeed,  $e$  is an idempotent with  $e \notin I$  and for every  $a = (a_1, a_2), b = (b_1, b_2) \in R$  such that  $ab \in I$ , we have  $a_1b_1 = 0$  and  $a_2b_2 \in (p)$ . It follows that  $ea = (a_1, 0) \in I$  or  $eb = (b_1, 0) \in \sqrt{I}$ .

The next lemma will be useful to establish the subsequent theorem. Recall that the set  $(I : a)$ , for  $a \in R \setminus I$ , is the ideal of  $R$  given by  $(I : a) = \{x \in R, ax \in I\}$ .

**Lemma 2.5.** Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. Then,  $\sqrt{(I : e)} = (\sqrt{I} : e)$ .

*Proof.* It is easy to show, by elementary calculus, that the inclusion  $(\sqrt{I} : e) \subseteq \sqrt{(I : e)}$  holds in general. For reciprocal inclusion  $\sqrt{(I : e)} \subseteq (\sqrt{I} : e)$ , the fact that  $e$  is an idempotent is needed. Indeed, let  $x \in \sqrt{(I : e)}$ , then  $ex^n \in I$  for some  $n \in \mathbb{N}$ . Hence,  $(ex)^n = ex^n \in I$  which implies that  $ex \in \sqrt{I}$ . Thus,  $x \in (\sqrt{I} : e)$ .  $\square$

Primariness and  $e$ -primariness coincide for  $(I : e)$  and are equivalent to  $e$ -primariness of  $I$ .

**Theorem 2.6.** *Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. The following assertions are equivalent.*

- (1)  $I$  is  $e$ -primary.
- (2)  $(I : e)$  is primary.
- (3)  $(I : e)$  is  $e$ -primary.

*Proof.* (1)  $\Rightarrow$  (2) Let  $xy \in (I : e)$  for  $x, y \in R$ . Then,  $exy \in I$ , so  $e^2x = ex \in I$  or  $ey \in \sqrt{I}$  by hypothesis. Hence,  $x \in (I : e)$  or  $y \in (\sqrt{I} : e) = \sqrt{(I : e)}$  by Lemma 2.5. Thus,  $(I : e)$  is a primary ideal of  $R$ .

(2)  $\Rightarrow$  (1) Let  $xy \in I$  for  $x, y \in R$ . Then,  $xy \in (I : e)$  since  $I \subseteq (I : e)$ . Hence,  $x \in (I : e)$  or  $y \in \sqrt{(I : e)} = (\sqrt{I} : e)$  by hypothesis and Lemma 2.5. It follows that  $ex \in I$  or  $ey \in \sqrt{I}$ . Thus,  $I$  is an  $e$ -primary ideal of  $R$ .

(2)  $\Leftrightarrow$  (3) Immediate. □

The radical of any  $e$ -primary ideal is an  $e$ -prime ideal.

**Theorem 2.7.** *Let  $R$  be a commutative ring and  $I$  an  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  an idempotent. Then,  $\sqrt{I}$  is an  $e$ -prime ideal of  $R$ .*

*Proof.* First,  $e \notin \sqrt{I}$  since  $e \notin I$  by Remark 2.3(2). Let  $x, y \in R$  with  $xy \in \sqrt{I}$ . Then,  $x^n y^n \in I$  for some  $n \in \mathbb{N}$ . Hence,  $ex^n \in I$  or  $ey^n \in \sqrt{I}$  since  $I$  an  $e$ -primary ideal of  $R$ . It follows that  $(ex)^n = e^n x^n = ex^n \in I$  or  $(ey)^n = e^n y^n = ey^n \in \sqrt{I}$ . Hence,  $ex \in \sqrt{I}$  or  $ey \in \sqrt{\sqrt{I}} = \sqrt{I}$ . Thus,  $\sqrt{I}$  is an  $e$ -prime ideal of  $R$ . □

**Theorem 2.8.** *Let  $R$  be a commutative ring and  $I$  an  $e$ -prime (resp.,  $e$ -primary) ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. Then,  $eI$  is an  $e$ -prime (resp.,  $e$ -primary) ideal of  $R$ .*

*Proof.* Let  $xy \in eI \subseteq I$ , then  $ex \in I$  or  $ey \in I$  (resp.,  $ey \in \sqrt{I}$ ). It follows that  $e^2x = ex \in eI$  or  $e^2y = ey \in eI$  (resp.,  $e^2y = ey \in e\sqrt{I}$ ). Thus,  $eI$  is an  $e$ -prime (resp.,  $e$ -primary) ideal of  $R$ . □

It is known that the radical of a primary ideal  $I$  is the smallest prime ideal containing  $I$ . The subsequent Theorem 2.9 is the corresponding version for  $e$ -primary ideals.

**Theorem 2.9.** *Let  $R$  be a commutative ring and  $I$  an  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. Then,  $e\sqrt{I}$  is the smallest  $e$ -prime ideal of  $R$  containing  $eI$ .*

*Proof.* First,  $e\sqrt{I} \supseteq eI$  and  $e\sqrt{I}$  is  $e$ -prime by Theorem 2.8. Now, let  $K$  be an  $e$ -prime ideal of  $R$  such that  $K \supseteq eI$ . Then, for every  $x \in e\sqrt{I}$ ,  $x = ea$  for some  $a \in \sqrt{I}$  and  $a^n \in I \subseteq K$  for some  $n \in \mathbb{N}$ . It follows that  $ea^n \in eK \subseteq K$ . Hence,  $x = ea \in K$  since  $K$  is  $e$ -prime. Thus,  $e\sqrt{I} \subseteq K$ . □

**Corollary 2.10.** *Let  $R$  be a commutative ring and  $I$  an  $e$ -prime ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. Then,  $e\sqrt{I} = eI$ .*

*Proof.* Immediate from Theorem 2.9. □

**Definition 2.11.** Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. We say that  $I$  is an  $e$ -maximal ideal of  $R$  provided that  $(I : e)$  is a maximal ideal of  $R$ .

It is known that if the radical of a given ideal is maximal, then this ideal is primary. The following result is the corresponding version for  $e$ -maximal ideals, hence offering a partial converse, as in the usual case, to the statement of Theorem 2.7.

**Theorem 2.12.** *Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. If  $\sqrt{I}$  is an  $e$ -maximal ideal of  $R$ , then  $I$  is  $e$ -primary.*

*Proof.* Assume that  $\sqrt{I}$  is  $e$ -maximal, then  $\sqrt{(I : e)} = (\sqrt{I} : e)$  is a maximal ideal of  $R$ , by Lemma 3.7. It follows that  $(I : e)$  is a primary ideal of  $R$ . Thus,  $I$  is  $e$ -primary by Theorem 2.6.  $\square$

**Definition 2.13.** Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. We say that  $I$  is an  $e$ -irreducible ideal of  $R$  provided that  $(I : e)$  is an irreducible ideal of  $R$ .

It is known that if an ideal  $I$  of a Noetherian ring is irreducible, then  $I$  is primary. The next result is the corresponding version for  $e$ -irreducible ideals.

**Theorem 2.14.** Let  $I$  be an  $e$ -irreducible ideal of a Noetherian ring  $R$ , for  $e \in R \setminus I$  an idempotent. Then,  $I$  is  $e$ -primary.

*Proof.* Assume that  $I$  is  $e$ -irreducible, then  $(I : e)$  is an irreducible ideal of  $R$ , hence a primary ideal of  $R$ . Thus,  $I$  is  $e$ -primary by Theorem 2.6.  $\square$

Next, we characterize  $e$ -primariness in terms of ideals.

**Theorem 2.15.** Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. The following assertions are equivalent.

- (1)  $I$  is  $e$ -primary.
- (2) For all ideals  $J$  and  $K$  of  $R$ ,  $JK \subseteq I$  implies that  $eJ \subseteq I$  or  $eK \subseteq \sqrt{I}$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $J$  and  $K$  be ideals of  $R$  such that  $JK \subseteq I$ . Assume that  $eJ \not\subseteq I$ , then there is some  $x = ej \in eJ \setminus I$  for some  $j \in J$ . Let  $y \in eK$ , then  $y = ek$  for some  $k \in K$ . Since  $jk \in JK \subseteq I$ , it follows that  $ej = x \in I$  or  $ek = y \in \sqrt{I}$  by hypothesis. Hence,  $y \in \sqrt{I}$  since  $x \notin I$ . Thus,  $eK \subseteq \sqrt{I}$ .

(2)  $\Rightarrow$  (1) Let  $xy \in I$  for  $x, y \in R$ . Then,  $\langle x \rangle \langle y \rangle = \langle xy \rangle \subseteq I$ . Hence,  $e\langle x \rangle \subseteq I$  or  $e\langle y \rangle \subseteq I$  by hypothesis. In particular,  $ex \in I$  or  $ey \in \sqrt{I}$ . Thus,  $I$  is  $e$ -primary.  $\square$

Given a proper ideal  $I$  of a commutative ring  $R$  and  $e \in R \setminus I$  an idempotent, we say that  $I$  is  $e$ -semiprimary provided that for all  $x, y \in R$  with  $xy \in I$ , it follows that  $ex \in (0)$  or  $ey \in \sqrt{(0)}$ .

**Corollary 2.16.** A proper ideal  $I$  of  $R$  is  $e$ -semiprimary if and only if for all ideals  $A$  and  $B$  of  $R$  such that  $AB \subseteq I$ , it follows that  $eA \subseteq (0)$  or  $eB \subseteq \sqrt{(0)}$ .

The intersection of  $e$ -primary ideals of a common radical is an  $e$ -primary ideal of the same radical.

**Theorem 2.17.** Let  $R$  be a commutative ring and  $\{I_i\}_{i \in F}$  a family of  $e$ -primary ideals of  $R$ , where  $e \in R \setminus \cup_{i \in F} I_i$  is an idempotent, such that  $\sqrt{I_i} = J$  for every  $i \in F$ . Then,  $I = \cap_{i \in F} I_i$  is an  $e$ -primary ideal of  $R$  and  $\sqrt{I} = J$ .

*Proof.* First,  $\sqrt{I} = \cap_{i \in F} \sqrt{I_i} = \cap_{i \in F} J = J$ . On the other hand, let  $xy \in I$  for  $x, y \in R$ . Assume that  $ex \notin I$ , then  $ex \notin I_{i_0}$  for some  $i_0 \in F$ . But,  $xy \in I_{i_0}$ ,  $e \notin I_{i_0}$  and  $I_{i_0}$  is an  $e$ -primary ideal of  $R$ , so  $ey \in \sqrt{I_{i_0}} = J = \sqrt{I}$ . Thus,  $I$  is  $e$ -primary.  $\square$

Whenever it is well-defined, the union of  $e$ -primary ideals of  $R$  is also an  $e$ -primary ideal of  $R$ .

**Theorem 2.18.** Let  $R$  be a commutative ring and  $\{I_i\}_{i \in F}$  a family of  $e$ -primary ideals of  $R$ , where  $e \in R \setminus \cup_{i \in F} I_i$  is an idempotent. Then,  $I = \cup_{i \in F} I_i$  is an  $e$ -primary ideal of  $R$  whenever  $I$  is an ideal of  $R$ .

*Proof.* Let  $x, y \in R$  such that  $xy \in I$ . Then,  $xy \in I_{i_0}$  for some  $i_0 \in F$ . Hence,  $ex \in I_{i_0}$  or  $ey \in \sqrt{I_{i_0}}$  since  $e \notin I_{i_0}$  and  $I_{i_0}$  is an  $e$ -primary ideal. It follows that  $ex \in I$  or  $ey \in \sqrt{I}$  since  $I_{i_0} \subseteq I$  and  $\sqrt{I_{i_0}} \subseteq \sqrt{I}$ . Thus,  $I$  is an  $e$ -primary ideal of  $R$ .  $\square$

If a commutative ring  $R$  contains an  $e$ -primary ideal, then it necessarily contains a maximal  $e$ -primary ideal, i.e., an ideal which is simultaneously maximal and  $e$ -primary.

**Theorem 2.19.** Let  $R$  be a commutative ring,  $I$  an  $e$ -primary ideal of  $R$  and  $e \in R \setminus I$  an idempotent. Then,  $R$  admits a maximal  $e$ -primary ideal.

*Proof.* Set  $F = \{K, K \text{ is an } e\text{-primary ideal of } R\}$ , then  $F \neq \emptyset$  since  $I \in F$ . On the other hand,  $F$  is obviously a partially ordered set (for the inclusion operation). Let  $I \subseteq I_1 \subseteq \dots \subseteq I_n \subseteq \dots$  be a chain of ideals in  $F$ . By Theorem 2.18,  $\cup_{i \in \mathbb{N}} I_i$  is  $e$ -primary. Hence,  $\cup_{i \in \mathbb{N}} I_i$  represents an upper bound of the family  $\{I_i\}_{i \in \mathbb{N}}$ . By Zorn's Lemma,  $F$  admits then a maximal element. Thus,  $R$  admits a maximal ideal which is  $e$ -primary.  $\square$

### 3 Weakly $e$ -primary ideals

This section is devoted to the study of *weakly  $e$ -primary* ideals, exploring some of their properties and investigating the relation with  $e$ -primariness.

**Definition 3.1.** Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. We say that  $I$  is a *weakly  $e$ -primary ideal* of  $R$  if whenever  $0 \neq ab \in I$  for  $a, b \in R$ , it follows that  $ea \in I$  or  $eb \in \sqrt{I}$ .

Every  $e$ -primary ideal is obviously weakly  $e$ -primary. However, the converse may fail as shown by the next example.

**Example 3.2.** Let  $R = \mathbb{Z}_{12}$ ,  $I = (0)$  and  $e = 9$ . Then,  $e \in R \setminus I$  is an idempotent and it is not difficult to check that  $I$  is a weakly  $e$ -primary ideal of  $R$  (in fact  $I$  is weakly  $e$ -prime). However,  $I$  is not  $e$ -primary since  $3 \cdot 4 = 0 \in I$ , but  $e \cdot 3 = 3 \notin \sqrt{I}$  and  $e \cdot 4 = 8 \notin \sqrt{I}$  (note here that  $\sqrt{I} = \{0, 6\}$ ).

**Definition 3.3.** Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. An element  $x \in R$  is called an  $e$ -dual-zero of  $I$  if  $xy = 0$ ,  $ex \notin \sqrt{I}$  and  $ey \notin \sqrt{I}$ , for some  $y \in R$ .

Note that the element  $y$ , as defined above, is also an  $e$ -dual-zero of  $I$ . In addition, it is clear that no element of  $\sqrt{I}$  can not be an  $e$ -dual-zero of  $I$ .

**Theorem 3.4.** Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. The following assertions are equivalent.

- (1)  $I$  is an  $e$ -primary ideal of  $R$ .
- (2)  $I$  does not admit any  $e$ -dual-zero elements of  $R$ .

*Proof.* Assume that  $I$  is  $e$ -primary and let  $x \in R$ . Then, for every  $y \in R$  with  $xy = 0$ , we have  $ex \in I$  or  $ey \in \sqrt{I}$ . Thus,  $x$  is not an  $e$ -dual-zero of  $I$ .

Conversely, it is clear that the absence of  $e$ -dual-zero elements of  $I$  implies that  $I$  is an  $e$ -primary ideal since it is already weakly  $e$ -primary.  $\square$

**Theorem 3.5.** Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. If  $x \in R$  is an  $e$ -dual-zero of  $I$ , then  $xI = (0)$ .

*Proof.* Let  $x$  be an  $e$ -dual-zero with  $y$  as associated element as in Definition 3.3. For every  $a \in I$ , we have  $x(y + a) = xa \in I$ . Assume that  $xa \neq 0$ , then  $x(y + a) \neq 0$ , and so  $ex \in I$  or  $e(y + a) \in \sqrt{I}$ . Hence,  $ex \in I$  or  $ey \in \sqrt{I}$  since  $ea \in I \subseteq \sqrt{I}$ , a contradiction. Thus,  $xI = (0)$ .  $\square$

**Theorem 3.6.** Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$  which is not  $e$ -primary, where  $e \in R \setminus I$  is an idempotent. Then  $I^2 = (0)$ .

*Proof.* By Theorem 3.4, there is some  $e$ -dual-zero  $x \in R$  with associated element, say,  $y \in R$ . For all  $a, b \in I$ , we have  $(x + a)(y + b) = ab \in I$  by Theorem 3.5. Assume that  $ab \neq 0$ , then  $(x + a)(y + b) \neq 0$ . Hence,  $e(x + a) \in I$  or  $e(y + b) \in \sqrt{I}$ . It follows that  $ex \in I$  or  $ey \in \sqrt{I}$  since  $ea, eb \in I \subseteq \sqrt{I}$ , a contradiction. Thus,  $I^2 = (0)$ .  $\square$

**Corollary 3.7.** *Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$  which is not  $e$ -primary, where  $e \in R \setminus I$  is an idempotent. Then,  $\sqrt{I} = \sqrt{(0)}$ .*

*Proof.* By Theorem 3.6,  $I^2 = (0)$  which implies that  $I \subseteq \sqrt{(0)}$  and so  $\sqrt{I} \subseteq \sqrt{\sqrt{(0)}} = \sqrt{(0)}$ . Now, as  $I^2 \subseteq I$ , we get  $(0) \subseteq I$ , so  $\sqrt{(0)} \subseteq \sqrt{I}$ . Thus,  $\sqrt{I} = \sqrt{(0)}$ .  $\square$

In general, an ideal satisfying  $I^2 = (0)$  need not be weakly  $e$ -primary.

**Example 3.8.** Let  $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{Z}_{12} \right\}$ ,  $I = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \{0, 6\} \right\}$  and  $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then,  $I$  is a proper ideal of  $R$  and  $I^2 = (0)$ . However,  $I$  is not weakly  $e$ -primary. Indeed,  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in I$ , but  $e \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \notin I$  and  $e \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \notin \sqrt{I} = I$ .

When a family  $\{I_i\}_{i \in L}$  consists of weakly  $e$ -primary ideals that are not  $e$ -primary, the assumption  $\sqrt{I_i} = J$  for all  $i \in L$ , in comparison with Theorem 2.17, becomes superfluous.

**Theorem 3.9.** *Let  $R$  be a commutative ring and  $\{I_i\}_{i \in F}$  a family of weakly  $e$ -primary ideals of  $R$  that are not  $e$ -primary, where  $e \in R \setminus \cup_{i \in F} I_i$  is an idempotent. Then,  $I = \cap_{i \in F} I_i$  is a weakly  $e$ -primary ideal of  $R$ .*

*Proof.* Let  $0 \neq xy \in I$  for  $x, y \in R$ . Assume that  $ex \notin I$ , then  $ex \notin I_{i_0}$  for some  $i_0 \in F$ . But,  $0 \neq xy \in I_{i_0}$ ,  $e \notin I_{i_0}$  and  $I_{i_0}$  is a weakly  $e$ -primary ideal of  $R$ , so  $ey \in \sqrt{I_{i_0}} = J = \sqrt{I}$ . Thus,  $I$  is  $e$ -primary.  $\square$

**Definition 3.10.** Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. An ideal  $J$  of  $R$  is called an  $e$ -dual-zero of  $I$  provided that  $JK = (0)$ ,  $eJ \not\subseteq I$  and  $eK \not\subseteq \sqrt{I}$  for some ideal  $K$  of  $R$ .

As above, the ideal  $K$  is also an  $e$ -dual-zero of  $I$ . Clearly, neither  $I$  nor  $\sqrt{I}$  can be  $e$ -dual-zero ideals of  $I$ .

The next result clarifies the relation between the concepts of an  $e$ -dual-zero element and an  $e$ -dual-zero ideal of a given weakly  $e$ -primary ideal of  $R$ .

**Theorem 3.11.** *Let  $R$  be a commutative ring,  $I$  a weakly  $e$ -primary ideal of  $R$  for an idempotent  $e \in R \setminus I$ , and  $A, B$  be ideals of  $R$  such that  $AB \subseteq I$ . The following assertions are equivalent.*

- (1) *The ideal  $A$  is an  $e$ -dual-zero of  $I$  with associated ideal  $B$ .*
- (2) *There exist elements  $a \in A$  and  $b \in B$  such that  $a$  is an  $e$ -dual-zero of  $R$  with associated element  $b$ .*

*Proof.* (1)  $\Rightarrow$  (2) By definition,  $AB = (0)$ ,  $eA \not\subseteq I$  and  $eB \not\subseteq \sqrt{I}$ . Hence, there exist elements  $x = ea \in eA \setminus I$  and  $y = eb \in eB \setminus \sqrt{I}$  with  $a \in A$  and  $b \in B$ . Since  $ab \in AB$ , it follow that  $ab = 0$ . Thus,  $a$  is an  $e$ -dual-zero of  $I$  with associated element  $b$ .

(2)  $\Rightarrow$  (1) Let  $a \in A$  and  $b \in B$  such that  $a$  is an  $e$ -dual-zero of  $I$  with associated element  $b$ . Then,  $eA \not\subseteq I$  and  $eB \not\subseteq \sqrt{I}$ . It suffices to show then that  $AB = (0)$ . We will start first by showing that  $aB = (0)$  and  $Ab = (0)$ . Indeed, suppose  $aB \neq (0)$ , then  $ab' \neq 0$  for some  $b' \in B$ . Then,  $ea \in I$  or  $eb' \in \sqrt{I}$  since  $I$  a weakly  $e$ -primary ideal. Hence,  $eb' \in \sqrt{I}$  as  $ea \notin I$ . Then,  $0 \neq a(b + b') = ab' \in I$ , which implies that either  $e(b + b') \in I$  since  $ea \notin I$ . So,  $eb \in \sqrt{I}$  since  $eb' \in \sqrt{I}$ , a contradiction. Likewise, one can show that  $Ab = (0)$ . Now, to prove that  $AB = (0)$ , let's suppose  $AB \neq (0)$ . Then, there exist elements  $x \in A$  and  $y \in B$  such that  $0 \neq xy \in AB \subseteq I$ . Hence, either  $ex \in I$  or  $ey \in \sqrt{I}$ . Thus, three cases are possible and which we explore as follows.

**Case 1:** If  $x \in I$  and  $y \notin \sqrt{I}$ , then  $0 \neq y(x + a) = xy \in I$  since  $aB = (0)$ . Hence,  $ey \in I$  or  $e(x + a) \in \sqrt{I}$  since  $I$  a weakly  $e$ -primary ideal. So,  $ey \in \sqrt{I}$  or  $ea \in \sqrt{I}$  since  $ex \in I \subseteq \sqrt{I}$ , a

contradiction.

**Case 2:** If  $x \notin I$  and  $y \in \sqrt{I}$ , then  $0 \neq x(y+b) = xy \in I$  since  $Ab = (0)$ . Hence,  $ex \in I$  or  $e(y+b) \in \sqrt{I}$  since  $I$  a weakly  $e$ -primary ideal. So,  $ex \in I$  or  $eb \in \sqrt{I}$ , a contradiction.

**Case 3:** If  $x \in I$  and  $y \in \sqrt{I}$ , then  $0 \neq (a+x)(y+b) = xy \in I$  since  $aB = Ab = (0)$ . Hence,  $e(a+x) \in I$  or  $e(y+b) \in \sqrt{I}$ . So,  $ea \in I$  or  $eb \in \sqrt{I}$ , a contradiction with  $a$  being an  $e$ -dual-zero of  $I$  with associated element  $y$ .

Finally, we conclude that  $AB = (0)$ .  $\square$

**Theorem 3.12.** *Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. If the ideal  $J$  is an  $e$ -dual-zero of  $I$ , then  $jI = (0)$  for some  $j \in J$ .*

*Proof.* Let  $J$  be an  $e$ -dual-zero of  $I$  with associated ideal  $K$ . By Theorem 3.11, there is some  $e$ -dual-zero element  $j \in J$  of  $I$  with associated element, say  $k$ , such that  $k \in K$ . Now, by Theorem 3.5, it follows that  $jI = (0)$ .  $\square$

Let's call a proper ideal  $I$  of  $R$  *weakly  $e$ -prime*, for  $e \in R \setminus I$  an idempotent, if for all elements  $a, b \in R$  such that  $0 \neq ab \in I$ , it follows that  $ea \in I$  or  $eb \in I$ . Thus defined, the following result provides a characterization of weakly  $e$ -prime ideals in terms of ideals.

**Theorem 3.13.** *Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. The following assertions are equivalent.*

- (1)  $I$  is weakly  $e$ -prime.
- (2) For all ideals  $A$  and  $B$  of  $R$  with  $(0) \neq AB \subseteq I$ , we have  $eA \subseteq I$  or  $eB \subseteq I$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $A$  and  $B$  be ideals of  $R$  such that  $(0) \neq AB \subseteq I$  and  $eA \not\subseteq I$ . Then, there exists  $x = ea \in eA$  such that  $x \notin I$ . For every  $y = eb \in eB$ , we have either  $ab \neq 0$  or  $eb \in I$  in which cases we obtain  $eb \in I$  since  $I$  a weakly  $e$ -prime ideal. Indeed, suppose  $ab = 0$  and  $eb \notin I$ . Since  $a \notin I$ , it follows that  $AB = (0)$  by Theorem 3.11, a contradiction. Thus,  $eB \subseteq I$ . (2)  $\Rightarrow$  (1) Let  $x, y \in R$  with  $0 \neq xy \in I$ . Then, it suffices to consider the principal ideals  $\langle x \rangle$  and  $\langle y \rangle$  to conclude that  $ex \notin I$  or  $ey \notin I$ .  $\square$

Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. Let's call  $I$  (weakly)  $e$ -semiprime provided that for all elements  $x, y \in R$  such that  $(0 \neq) xy \in I$ , it follows that  $ex \in (0)$  or  $ey \in \sqrt{(0)}$ .

**Corollary 3.14.** *A proper ideal  $I$  of  $R$  is weakly  $e$ -semiprime if and only if for all ideals  $A$  and  $B$  of  $R$ ,  $(0) \neq AB \subseteq I$  implies that  $eA \subseteq (0)$  or  $eB \subseteq \sqrt{(0)}$ .*

The following theorem is the version of Theorem 3.4, expressed in terms of ideals.

**Theorem 3.15.** *Let  $R$  be a commutative ring and  $I$  a weakly  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent. The following assertions are equivalent.*

- (1)  $I$  is  $e$ -primary.
- (2)  $I$  does not admit any  $e$ -dual-zero ideal.

*Proof.* (1)  $\Rightarrow$  (2) Assume that  $I$  is  $e$ -primary. Let  $A$  and  $B$  be ideals of  $R$  such that  $AB = (0)$  and  $eA \not\subseteq I$ . Then, there is some  $x = ea \in eA \setminus I$ . Hence, for every  $y = eb \in eB$  we have  $ab = 0$  as  $ab \in AB = (0)$ . It follows that  $ea \in I$  or  $eb \in \sqrt{I}$  since  $I$  is an  $e$ -primary ideal. But  $ea \notin I$ , then  $eb \in \sqrt{I}$ . Hence,  $eB \subseteq \sqrt{I}$ . Thus, the ideal  $A$  is not an  $e$ -dual-zero of  $I$ .

(2)  $\Rightarrow$  (1) Let  $x, y \in R$  with  $xy = 0$ . Then,  $\langle x \rangle \langle y \rangle = \langle xy \rangle = (0)$ . Hence,  $e\langle x \rangle \subseteq I$  or  $e\langle y \rangle \subseteq \sqrt{I}$ . It follows that  $ex \in I$  or  $ey \in \sqrt{I}$ . Thus,  $I$  is  $e$ -primary.  $\square$

## 4 $e$ -Primariness and ring-theoretic constructions

At this stage, we examine the transfer of  $e$ -primariness under homomorphisms of rings and within various ring constructions. Precisely, we will consider localization rings, direct product rings, polynomial rings, trivial ring extensions, and amalgamated rings.

We start by the ring of polynomials  $R[X]$  on the indeterminate  $X$ .

**Theorem 4.1.** *Let  $R$  be a commutative ring,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. The following assertions are equivalent.*

- (1)  $I$  is an  $e$ -primary ideal of  $R$ .
- (2)  $(I, X)$  is an  $e$ -primary ideal of  $R[X]$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $f = a + XM(X)$  and  $g = b + XN(X)$  be two elements of  $R[X]$ , where  $M(X), N(X) \in R[X]$ , such that  $fg \in (I, X)$ . Then,  $ab \in I$  since  $fg = ab + XP(X)$  for some  $P(X) \in R[X]$ . Hence,  $ea \in I$  or  $eb \in \sqrt{I}$  since  $I$  is an  $e$ -primary ideal of  $R$ . It follows that  $ef \in (I, X)$  or  $eg \in (\sqrt{I}, X) \subseteq \sqrt{(I, X)}$ . Thus,  $(I, X)$  is an  $e$ -primary ideal of  $R[X]$ .

(2)  $\Rightarrow$  (1) Let  $ab \in I$  for  $a, b \in R$ . Then,  $ab \in (I, X)$ . Hence,  $ea \in (I, X)$  or  $eb \in \sqrt{(I, X)}$  since  $(I, X)$  is an  $e$ -primary ideal of  $R[X]$ . It follows that  $ea \in I$  or  $eb \in \sqrt{I}$ . Thus,  $I$  is an  $e$ -primary ideal of  $R$ . □

In the subsequent theorem, we consider  $e$ -primary ideals within localization rings.

**Theorem 4.2.** *Let  $R$  be a commutative ring,  $I$  an  $e$ -primary ideal of  $R$ , where  $e \in R \setminus I$  is an idempotent, and  $S$  a multiplicatively closed subset of  $R$  such that  $I \cap S = \emptyset$ . Then,  $S^{-1}I$  an  $\frac{e}{1}$ -primary ideal of  $S^{-1}R$ .*

*Proof.* First, note that  $S^{-1}I$  is a proper ideal of  $S^{-1}R$  since  $I$  is a proper ideal of  $R$  and  $I \cap S = \emptyset$ . Moreover,  $\frac{e}{1} \notin S^{-1}I$ , and  $\frac{e}{1}$  is an idempotent in  $S^{-1}R$  since  $e \notin I$ . Let  $\frac{a}{s}, \frac{b}{t} \in S^{-1}I$  for  $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ . Then,  $uab \in I$  for some  $u \in S$ . It follows that  $eua \in I$  or  $eb \in \sqrt{I}$  as  $I$  is an  $e$ -primary ideal of  $R$ . Hence,  $\frac{ea}{s} = \frac{eua}{us} \in S^{-1}I$  or  $\frac{eb}{t} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$ . Thus,  $S^{-1}I$  is an  $\frac{e}{1}$ -primary ideal of  $S^{-1}R$ . □

Next, we investigate the transfer of  $e$ -primariness via ring homomorphisms with particular attention to their behavior into quotient rings.

**Theorem 4.3.** *Let  $f : R \rightarrow R'$  be a homomorphism of commutative rings,  $I$  a proper ideal of  $R$  and  $e \in R \setminus I$  an idempotent. The following assertions hold.*

- (1) If  $J$  is an  $f(e)$ -primary ideal of  $R'$ , then  $f^{-1}(J)$  is an  $e$ -primary ideal of  $R$ .
- (2) If  $f$  is surjective and  $I$  is an  $e$ -primary ideal of  $R$  containing  $\ker(f)$ , then  $f(I)$  is an  $f(e)$ -primary ideal of  $R'$ .

*Proof.* (1) Let  $a, b \in R$  with  $ab \in f^{-1}(J)$ . Then,  $f(ab) = f(a)f(b) \in J$ , hence  $f(e)f(a) = f(ea) \in J$  or  $f(e)f(b) = f(eb) \in f^{-1}(\sqrt{J})$  since  $J$  is an  $f(e)$ -primary ideal of  $R'$ . It follows that  $ea \in f^{-1}(J)$  or  $eb \in f^{-1}(\sqrt{J})$ . But  $f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$ . Thus,  $f^{-1}(J)$  is an  $e$ -primary ideal of  $R$ .

(2) Let  $x, y \in R'$  with  $xy \in f(I)$ . Then,  $f(a) = x$  and  $f(b) = y$  for some elements  $a, b \in R$  since  $f$  is surjective. It follows that  $f(a)f(b) = f(ab) \in f(I)$ . Hence,  $ab \in I$  since  $\ker(f) \subseteq I$ . Then,  $ea \in I$  or  $eb \in \sqrt{I}$  since  $I$  is an  $e$ -primary ideal of  $R$ . Therefore,  $f(e)x = f(e)f(a) = f(ea) \in f(I)$  or  $f(e)y = f(e)f(b) = f(eb) \in f(\sqrt{I}) = \sqrt{f(I)}$ . Thus,  $f(I)$  is an  $f(e)$ -primary ideal of  $R'$ . □

**Corollary 4.4.** *Let  $J \subseteq I$  be proper ideals of  $R$  and  $e \in R \setminus I$  an idempotent. Denote by  $\bar{e}$  the class of  $e$  modulo  $J$ . The following assertions hold.*

- (1) If  $\frac{I}{J}$  is an  $\bar{e}$ -primary ideal of  $\frac{R}{J}$ , then  $I$  is an  $e$ -primary ideal of  $R$ .
- (2) If  $I$  is an  $e$ -primary ideal of  $R$ , then  $\frac{I}{J}$  is an  $\bar{e}$ -primary ideal of  $\frac{R}{J}$ .

*Proof.* First,  $\frac{I}{J} = \pi(I)$ , where  $\pi : R \rightarrow \frac{R}{J}$  is the canonical projection. Moreover,  $\bar{e} = \pi(e)$  is an idempotent in  $\frac{R}{J}$  and  $\bar{e} \notin \frac{I}{J}$  since  $e \in R \setminus I$  an idempotent.

(1) This result holds by Theorem 4.3(1) since  $\pi : R \rightarrow \frac{R}{J}$  is an epimorphism.

(2) This holds by Theorem 4.3(2) since  $\ker(\pi) = J \subseteq I$  by hypothesis. □

**Theorem 4.5.** *Let  $R_i$  be a commutative ring,  $I_i$  a proper ideal of  $R_i$  and  $e_i \in R_i$  an idempotent for  $i = 1, 2$ . Set  $R = R_1 \times R_2$ ,  $I = I_1 \times I_2$  and  $e = (e_1, e_2)$ . The following assertions are equivalent.*

- (1)  $I$  is an  $e$ -primary ideal of  $R$ .
- (2) One of the following conditions hold.
  - (i)  $e_1 \in I_1$  and  $I_2$  is an  $e_2$ -primary ideal of  $R_2$ .
  - (ii)  $e_2 \in I_2$  and  $I_1$  is an  $e_1$ -primary ideal of  $R_1$ .

*Proof.* (1)  $\Rightarrow$  (2) Assume that  $I$  is an  $e$ -primary ideal of  $R$ . Since  $(1, 0)(0, 1) = (0, 0) \in I$ , it follows that  $(e_1, 0) \in I_1 \times I_2$  or  $(0, e_2) \in \sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}$ . Hence,  $e_1 \in I_1$  and  $e_2 \notin I_2$  or  $e_2 \in I_2$  and  $e_1 \notin I_1$ , since  $e \notin I$  and in view of Remark 2.3(2).

**Case 1:** If  $e_2 \in I_2$  and  $e_1 \notin I_1$ , then  $I_1$  is an  $e_1$ -primary ideal of  $R_1$ . Indeed, let  $ab \in I_1$  for  $a, b \in R_1$ . Then,  $(a, 0)(b, 0) = (ab, 0) \in I$ . Hence,  $(e_1a, 0) \in I$  or  $(e_1b, 0) \in \sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}$  since  $I$  is an  $e$ -primary ideal of  $R$ . It follows that  $e_1a \in I_1$  or  $e_1b \in \sqrt{I_1}$ . Thus,  $I_1$  is an  $e_1$ -primary ideal of  $R_1$ .

**Case 2:** If  $e_1 \in I_1$  and  $e_2 \notin I_2$ , then one can show in a way similar to the previous case that  $I_2$  is an  $e_2$ -primary ideal of  $R_2$ .

(2)  $\Rightarrow$  (1) Assume that  $e_2 \in I_2$  and  $I_1$  is an  $e_1$ -primary ideal of  $R_1$ . Then,  $e \notin R$ . Let  $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2) \in I$  for  $(a_1, a_2), (b_1, b_2) \in R$ . Then,  $a_1b_1 \in I_1$ . It follows that  $e_1a_1 \in I_1$  or  $e_1b_1 \in \sqrt{I_1}$  since  $I_1$  is an  $e_1$ -primary ideal of  $R_1$ . Hence,  $e(a_1, a_2) \in I$  or  $e(b_1, b_2) \in \sqrt{I}$  since  $e_2 \in I_2$ . Thus,  $I$  is an  $e$ -primary ideal of  $R$ . Likewise, if one assumes that  $e_1 \in I_1$  and  $I_2$  is an  $e_2$ -primary ideal of  $R_2$ , then it follows that  $I$  is an  $e$ -primary ideal of  $R$ .  $\square$

Given a commutative ring  $R$  and an  $R$ -module  $M$ , we recall that the *trivial ring extension*  $R \rtimes M$  of  $R$  by  $M$  is the ring whose additive structure is defined componentwise and the multiplication is given as  $(a, m)(b, n) = (ab, an + bm)$  for all  $a, b \in R$  and  $m, n \in M$  [5]. Given  $I$  an ideal of  $R$  and  $N$  a submodule of  $M$ ,  $I \rtimes N$  is an ideal of  $R \rtimes M$  if and only if  $IM \subseteq N$ .

**Theorem 4.6.** *Let  $I$  be a proper ideal of a commutative ring  $R$ ,  $N$  a submodule of an  $R$ -module  $M$ ,  $e \in R \setminus I$  an idempotent and  $u \in M$  such that  $(2e - 1)u = 0$  and  $IM \subseteq N$ . The following assertions hold.*

- (1) If  $I \rtimes N$  is an  $(e, u)$ -primary ideal of  $R \rtimes M$ , then  $I$  is an  $e$ -primary ideal of  $R$ .
- (2) If  $eM \subseteq N$  and  $I$  is an  $e$ -primary ideal of  $R$ , then  $I \rtimes N$  is an  $(e, u)$ -primary ideal of  $R \rtimes M$ .

*Proof.* (1) Let  $ab \in I$  for  $a, b \in R$ . Then,  $(ab, 0) = (a, 0)(b, 0) \in I \rtimes N$ . Hence,  $(e, u)(a, 0) = (ea, au) \in I \rtimes N$  or  $(e, u)(b, 0) = (eb, bu) \in \sqrt{I \rtimes N}$  since  $I \rtimes N$  is an  $(e, u)$ -primary ideal of  $R \rtimes M$ . If  $(eb, bu) \in \sqrt{I \rtimes N}$ , then there exists some  $n \in \mathbb{N}$  such that  $(eb, bu)^n \in I \rtimes N$ ; in particular,  $(eb)^n \in I$ , and hence  $eb \in \sqrt{I}$ . Therefore, we obtain that  $ea \in I$  or  $eb \in \sqrt{I}$ . Thus,  $I$  is an  $e$ -primary ideal of  $R$ .

(2) First,  $I \rtimes N$  is an ideal of  $R \rtimes M$  as  $IM \subseteq N$ . Moreover, since  $I$  is a proper ideal of  $R$ ,  $I \rtimes N$  is a proper ideal of  $R \rtimes M$ . In addition,  $(2e - 1)u = 0$  and  $e \in R \setminus I$  imply that  $(e, u)$  is an idempotent and  $(e, u) \notin I \rtimes N$ . Let  $(a, n)(b, m) = (ab, am + bn) \in I \rtimes N$  for  $(a, n), (b, m) \in R \rtimes M$ . Then,  $ab \in I$  and so  $ea \in I$  or  $eb \in \sqrt{I}$  since  $I$  is  $e$ -primary. On the other hand,  $en, em \in N$  as  $eM \subseteq N$ , and  $(2e - 1)u = 0$  implies that  $au = 2eau \in eM$  and  $bu = 2ebu \in eM$ , hence  $au, bu \in N$  since  $eM \subseteq N$ . It follows that  $(e, u)(a, n) = (ea, en + au) \in I \rtimes N$  or  $(e, u)(b, m) = (eb, em + bu) \in \sqrt{I \rtimes N} \subseteq \sqrt{I} \rtimes N$ . Thus,  $I \rtimes N$  is an  $(e, u)$ -primary ideal of  $R \rtimes M$ .  $\square$

Given commutative rings  $A, B$ , a homomorphism  $f : A \rightarrow B$  of rings and an ideal  $J$  of  $B$ , we recall [9] that the *amalgamation of  $A$  and  $B$  along  $J$  with respect to  $f$*  is the subring of  $A \times B$  defined as

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

**Theorem 4.7.** *Let  $A, B$  be commutative rings,  $f : A \rightarrow B$  a unital homomorphism of rings,  $J$  an ideal of  $B$ ,  $I$  a proper ideal of  $A$  and  $e \in A \setminus I$  an idempotent. The following assertions are equivalent.*

- (1)  $I$  is an  $e$ -primary ideal of  $A$ .

(2)  $I \bowtie^f J$  is an  $(e, f(e))$ -primary ideal of  $A \bowtie^f J$ .

*Proof.* (1)  $\Rightarrow$  (2) First, note that  $(e, f(e))$  is an idempotent of  $A \bowtie^f J$  and  $(e, f(e)) \notin I \bowtie^f J$  since  $e \in A \setminus I$  is an idempotent. On the other hand, let  $x = (a, f(a) + j_1)$  and  $y = (b, f(b) + j_2)$  be two elements of  $A \bowtie^f J$  such that  $xy \in I \bowtie^f J$ . Then  $ab \in I$  and so  $ea \in I$  or  $eb \in \sqrt{I}$  since  $I$  is an  $e$ -primary ideal of  $A$ . Hence,  $(e, f(e))x = (ea, f(ea) + f(e)j_1) \in I \bowtie^f J$  or  $(e, f(e))y = (eb, f(eb) + f(e)j_2) \in \sqrt{I} \bowtie^f J$  since  $f(e)j_1, f(e)j_2 \in J$  as  $J$  is an ideal of  $B$ . Thus,  $I \bowtie^f J$  is an  $(e, f(e))$ -primary ideal of  $A \bowtie^f J$ .

(2)  $\Rightarrow$  (1) Let  $ab \in I$  for  $a, b \in R$ . Then,  $(a, f(a))(b, f(b)) = (ab, f(ab)) \in I \bowtie^f J$ . Hence,  $(ea, f(ea)) \in I \bowtie^f J$  or  $(eb, f(eb)) \in \sqrt{I} \bowtie^f J$  since  $I \bowtie^f J$  is an  $(e, f(e))$ -primary ideal of  $A \bowtie^f J$ . If  $(eb, f(eb)) \in \sqrt{I} \bowtie^f J$ , then there exists some  $n \in \mathbb{N}$  such that  $(eb, f(eb))^n \in I \bowtie^f J$ ; in particular,  $(eb)^n \in I$ , and so  $eb \in \sqrt{I}$ . It follows that  $ea \in I$  or  $eb \in \sqrt{I}$ . Thus,  $I$  is an  $e$ -primary ideal of  $A$ .  $\square$

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