Palestine Journal of Mathematics

Vol 15(2)(2026) , 187–202

EXPLORATION OF A SEMI-SYMMETRIC NON-METRIC CONNECTION IN THE CONTEXT OF RIEMANNIAN MANIFOLDS

R. Kumar, S. K. Chaubey, M. A. Khan and L. Chawngthu

Communicated by: Zafar Ahsan

MSC 2020 Classifications: Primary 53C05; Secondary 53C15, 53C25.

Keywords and phrases: Semi-symmetric non-metric connection, Levi-Civita connection, curvature tensors, group manifolds, Ricci soliton.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Corresponding Author: L. Chawngthu

Abstract. This article presents the introduction of a novel type of connection on Riemannian manifolds, referred to as the semi-symmetric non-metric connection, which serves as a generalization of the semi-symmetric recurrent-metric connection. The proposed connection is constructed by modifying the Levi-Civita connection through the integration of a specific class of tensor fields. A comprehensive literature review is undertaken to categorize various connections previously established within this theoretical framework by different researchers. The study derives several curvature tensors associated with the semi-symmetric non-metric connection on Riemannian manifolds and establishes the necessary conditions for the existence and validity of the proposed structure. Furthermore, the geometric implications of this connection are examined, with particular emphasis on its application to group manifolds. Special attention is given to group manifolds admitting Ricci soliton under the semi-symmetric non-metric connection. The article concludes with a detailed example that illustrates the practical relevance and significance of the proposed construction.

1 Introduction

A connection on a fibre bundle is a geometric tool that enables the comparison of fibres over nearby points in a smooth and consistent manner. At its core, it provides a systematic way to define parallel transport-the process of moving elements from one fibre to another along a curve in the base manifold-while preserving the structure of the bundle. This notion is essential for extending differential operations from the base manifold to the total space of the bundle.

In the context of differential geometry, connections allow for the definition of covariant derivatives, which generalize the concept of directional derivatives to curved spaces and arbitrary bundles. This makes it possible to study how tensor fields, vector fields, or more general sections of a bundle change from point to point in a geometrically meaningful way. A particularly important case is that of a linear connection on a vector bundle. Here, each fibre is a vector space, and the connection must respect the linear structure of these fibres. In other words, parallel transport along a path must act linearly on the fibres, ensuring that vector addition and scalar multiplication are preserved during the transport process. This linearity is crucial for maintaining the vector space structure across the bundle and underpins many geometric and physical theories.

Connections are also foundational in gauge theory, where fibre bundles (typically principal or vector bundles) represent fields or particles, and connections represent gauge fields. The curvature of the connection, in this setting, corresponds to the field strength, and parallel transport describes how internal degrees of freedom (like spin or charge) evolve as particles move through space-time. Moreover, connections on fibre bundles are central to the study of holonomy, curvature, and the topological and geometric properties of manifolds. They are key to understanding phenomena such as geodesics, conserved quantities in physics, and various types of curvature (Riemannian, affine, etc.) that influence both local and global structure.

The notion of a semi-symmetric connection on an n-dimensional differentiable manifold (M^n, g) was introduced by [1], formulated as a modification of the Levi-Civita connection Γ corresponding to a Riemannian metric g. The torsion tensor $\bar{T}$ of a linear connection $\bar{\Gamma}$ on (M^n, g) is defined as

$$\bar{T}(\varkappa_1, \varkappa_2) = \bar{\Gamma}_{\varkappa_1}\varkappa_2 - \bar{\Gamma}_{\varkappa_2}\varkappa_1 - [\varkappa_1, \varkappa_2], \tag{1.1}$$

for all vector fields $\varkappa_1, \varkappa_2$ on (M^n, g) and $[\,,]$ defines the Lie bracket. If the torsion tensor $\bar{T}$ satisfies $\bar{T}(\varkappa_1, \varkappa_2) = 0$, then the linear connection $\bar{\Gamma}$ referred as symmetric connection. Also, if there exists a 1-form η corresponding to a vector field ξ defined as

$$\eta(\varkappa_1) = g(\varkappa_1, \xi). \tag{1.2}$$

Then the linear connection $\bar{\Gamma}$ is termed as semi-symmetric [1] when the condition

$$\bar{T}(\varkappa_1, \varkappa_2) = \eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2. \tag{1.3}$$

is satisfied for the vector fields $\varkappa_1$ and $\varkappa_2$ on (M^n, g). A linear connection $\bar{\Gamma}$ is termed a metric connection [5] if it preserves the metric, that is, $\bar{\Gamma}g = 0$. If this condition is not satisfied, $\bar{\Gamma}$ is classified as a non-metric connection. Additionally, $\bar{\Gamma}$ is called a semi-symmetric non-metric connection if it possesses both semi-symmetric and non-metric properties [18].

The study of connections generalizing the classical Levi-Civita connection has been a fertile area of research in differential geometry. The notion of a semi-symmetric metric connection was first introduced by [13] in 1970, marking a foundational contribution to the field. This work initiated a sustained research program aimed at generalizing and refining the properties of such connections. Subsequent developments saw significant expansions of this concept. In 1992, [18] conducted a systematic investigation of semi-symmetric non-metric connections on Riemannian manifolds, elucidating their fundamental properties and establishing key theoretical results. This was followed in 1994 by the work of [30], who introduced and analyzed a distinct type of semi-symmetric non-metric connection, further broadening the scope of the theory.

The subject has since attracted considerable attention from numerous geometers. Prominent among them are [10] and [19], whose contributions have substantially advanced the structural and curvature properties of these connections. The exploration was extended to almost contact metric manifolds by [25], who examined the role of non-symmetric non-metric connections within this specialized geometric setting. A substantial body of research has been dedicated to the study of semi-symmetric non-metric connections on Riemannian manifolds, with notable contributions from [6, 12, 17, 23]. In addition, pivotal research efforts have explored various aspects, such as a special class of semi-symmetric non-metric η-connection on a Para-Sasakian manifolds by [29], the investigation of Lorentzian α-Sasakian manifolds admitting semi-symmetric non-metric connection by [7] and recently, the analysis of semi-symmetric recurrent-metric conjugate connection has been studied by [22]. Furthermore, the theory has been successfully applied to more complex geometric structures. For instance, [27] explored these connections in the context of Kenmotsu manifolds, while [28] investigated their properties on hyperbolic Kenmotsu manifolds, demonstrating the versatility and applicability of the framework.

For a more comprehensive overview of the existing literature and recent advancements in the study of generalized connections, readers are encouraged to consult the foundational and contemporary works of [4, 16, 20], among others. These contributions collectively provide a broad spectrum of perspectives, ranging from the theoretical foundations and classification of various types of connections to their applications in diverse geometric settings. The referenced studies delve into intricate properties of generalized connections, including their curvature, torsion, and compatibility conditions, as well as their roles in specialized manifolds such as Sasakian, para-Sasakian, and Lorentzian structures. Together, this rich body of research continues to deepen our understanding of the geometric and topological implications of generalized connections, fostering further exploration and inspiring novel directions in both pure and applied mathematics.

In this paper, we present several results concerning Bianchi's first and second identities [2] in the context of a semi-symmetric non-metric connection. These identities play a fundamental role in differential geometry and have numerous applications across various related fields.

The projective curvature tensor [15], conformal curvature tensor [9], concircular curvature tensor [11] and conharmonic curvature tensor [26], each associated with the Levi-Civita connection Γ on the Riemannian manifold (M^n, g) are defined as follow:

$$P(\varkappa_1,\varkappa_2)\varkappa_3 = R(\varkappa_1,\varkappa_2)\varkappa_3 - \frac{1}{n-1}\big[S(\varkappa_2,\varkappa_3)\varkappa_1 - S(\varkappa_1,\varkappa_3)\varkappa_2\big], \tag{1.4}$$

$$\begin{aligned} C(\varkappa_1,\varkappa_2)\varkappa_3 = {} & R(\varkappa_1,\varkappa_2)\varkappa_3 - \frac{1}{n-2}\big[S(\varkappa_2,\varkappa_3)\varkappa_1 - S(\varkappa_1,\varkappa_3)\varkappa_2 \\ & + g(\varkappa_2,\varkappa_3)Q\varkappa_1 - g(\varkappa_1,\varkappa_3)Q\varkappa_2\big] \\ & + \frac{r}{(n-1)(n-2)}\big[g(\varkappa_2,\varkappa_3)\varkappa_1 - g(\varkappa_1,\varkappa_3)\varkappa_2\big], \end{aligned} \tag{1.5}$$

$$W(\varkappa_1,\varkappa_2)\varkappa_3 = R(\varkappa_1,\varkappa_2)\varkappa_3 - \frac{r}{(n-1)(n-2)}\big[g(\varkappa_2,\varkappa_3)\varkappa_1 - g(\varkappa_1,\varkappa_3)\varkappa_2\big] \tag{1.6}$$

and

$$\begin{aligned} L(\varkappa_1,\varkappa_2)\varkappa_3 = {} & R(\varkappa_1,\varkappa_2)\varkappa_3 - \frac{1}{n-2}\big[S(\varkappa_2,\varkappa_3)\varkappa_1 - S(\varkappa_1,\varkappa_3)\varkappa_2 \\ & + g(\varkappa_2,\varkappa_3)Q\varkappa_1 - g(\varkappa_1,\varkappa_3)Q\varkappa_2\big], \end{aligned} \tag{1.7}$$

for all $\varkappa_1, \varkappa_2, \varkappa_3$ on (M^n, g), where R, S and Q are the Riemannian curvature tensor, the Ricci tensor and the Ricci-operator defined by $g(Q\varkappa_1, \varkappa_2) = S(\varkappa_1, \varkappa_2)$, respectively.

This research article is organized as follows: Following the introduction in Section 1, Section 2 is devoted to the formulation of a new class of semi-symmetric non-metric connections on a Riemannian manifold. In this section, we rigorously define the connection and establish its existence. In Section 3, we focus on the relationship between the curvature tensors corresponding to the Levi-Civita connection and the newly defined semi-symmetric non-metric connection. We derive transformation formulas linking the respective curvature tensors and investigate key properties of the curvature tensor associated with a semi-symmetric non-metric connection. This section also establishes necessary and sufficient conditions for a curvature tensor to be projectively invariant under the new connection. Moreover, we explore the interconnections among various curvature tensors-namely, the Riemannian curvature, conformal curvature, concircular curvature and conharmonic curvature tensors and identify the geometric conditions under which they coincide or differ in this new framework. Section 4 extends the study to group manifolds equipped with a semi-symmetric non-metric connection. We examine how the algebraic and geometric structure of group manifolds interacts with this type of connection. Particular emphasis is placed on group manifolds that admit Ricci solitons in the context of the semi-symmetric non-metric connection, shedding light on the role of such solitons in geometric evolution and curvature flows. Finally, in Section 5, we construct a nontrivial example of a 3-dimensional Riemannian manifold endowed with the proposed semi-symmetric non-metric connection. This example illustrates the applicability of the theory developed in earlier sections and is used to demonstrate and verify several theoretical results. Through explicit calculations, we highlight the distinctive features of the connection and its influence on the manifold's curvature properties.

2 A new class of a semi-symmetric non-metric connection

The present discussion focuses on introducing a new class of semi-symmetric non-metric connections, abbreviated as SSNMC, within the context of a Riemannian manifold (M^n, g). More specifically, a linear connection on (M^n, g) represented by $\bar{\Gamma}$ is introduced with the following definition

$$\bar{\Gamma}_{\varkappa_1}\varkappa_2 = \Gamma_{\varkappa_1}\varkappa_2 + \kappa\eta(\varkappa_2)\varkappa_1 + (\kappa-1)\eta(\varkappa_1)\varkappa_2, \tag{2.1}$$

where κ is a real number. Note that $\bar{\Gamma}$ on (M^n, g) is categorized as an SSNMC if it agrees with equations (1.2) and (1.3) and also satisfies the following condition that the Riemannian metric g holds

$$\begin{aligned} (\bar{\Gamma}_{\varkappa_1}g)(\varkappa_2,\varkappa_3) = {} & -2(\kappa-1)\eta(\varkappa_1)g(\varkappa_2,\varkappa_3) - \kappa\eta(\varkappa_2)g(\varkappa_1,\varkappa_3) - \kappa\eta(\varkappa_3)g(\varkappa_1,\varkappa_2) \\ & \neq 0, \end{aligned} \tag{2.2}$$

for all $\varkappa_1, \varkappa_2, \varkappa_3$ on (M^n, g). Consequently, the theorem can be stated as follows:

Theorem 2.1. *Let (M^n, g) be an n-dimensional Riemannian manifold endowed with the LCC Γ. Then there exists a distinct linear connection $\bar{\Gamma}$ on the manifold (M^n, g), called a semi-symmetric non-metric connection, defined by equation (2.1) and satisfies the conditions given in equations (1.3) and (2.2).*

Proof. It is known that a connection $\bar{\Gamma}$ can be written as the linear combination of the Levi-Civita connection Γ (abbreviated as LCC) and a (1,2) type tensor field H as

$$\bar{\Gamma}_{\varkappa_1}\varkappa_2 = \Gamma_{\varkappa_1}\varkappa_2 + H(\varkappa_1, \varkappa_2). \tag{2.3}$$

Using equation (2.3) in (1.1), we get

$$\bar{T}(\varkappa_1, \varkappa_2) = H(\varkappa_1, \varkappa_2) - H(\varkappa_2, \varkappa_1). \tag{2.4}$$

Taking the inner product of (2.4) with respect to $\varkappa_3$ becomes

$$g(\bar{T}(\varkappa_1, \varkappa_2), \varkappa_3) = g(H(\varkappa_1, \varkappa_2), \varkappa_3) - g(H(\varkappa_2, \varkappa_1), \varkappa_3). \tag{2.5}$$

From (1.3) and (2.5), we have

$$g(H(\varkappa_1, \varkappa_2), \varkappa_3) - g(H(\varkappa_2, \varkappa_1), \varkappa_3) = \eta(\varkappa_2)g(\varkappa_1, \varkappa_3) - \eta(\varkappa_1)g(\varkappa_2, \varkappa_3). \tag{2.6}$$

Equation (2.1) reduces to

$$\begin{aligned}(\bar{\Gamma}_{\varkappa_1}g)(\varkappa_2, \varkappa_3) &= -g(\bar{\Gamma}_{\varkappa_1}\varkappa_2 - \Gamma_{\varkappa_1}\varkappa_2, \varkappa_3) - g(\varkappa_2, \bar{\Gamma}_{\varkappa_1}\varkappa_3 - \Gamma_{\varkappa_1}\varkappa_3) \\ &= -H'(\varkappa_1, \varkappa_2, \varkappa_3),\end{aligned} \tag{2.7}$$

where

$$H'(\varkappa_1, \varkappa_2, \varkappa_3) = g(H(\varkappa_1, \varkappa_2), \varkappa_3) + g(H(\varkappa_1, \varkappa_3), \varkappa_2).$$

By utilizing equations (2.4), (2.5) and (2.7), we arrive at

$$\begin{aligned}&g(\bar{T}(\varkappa_1, \varkappa_2), \varkappa_3) + g(\bar{T}(\varkappa_3, \varkappa_1), \varkappa_2) + g(\bar{T}(\varkappa_3, \varkappa_2), \varkappa_1) \\ &= 2g(H(\varkappa_1, \varkappa_2), \varkappa_3) - H'(\varkappa_1, \varkappa_2, \varkappa_3) + H'(\varkappa_3, \varkappa_1, \varkappa_2) - H'(\varkappa_2, \varkappa_1, \varkappa_3).\end{aligned}$$

Combining equations (2.2) and (2.7), we obtain

$$2g(H(\varkappa_1, \varkappa_2), \varkappa_3) = 2\kappa\eta(\varkappa_2)g(\varkappa_1, \varkappa_3) + 2(\kappa - 1)\eta(\varkappa_1)g(\varkappa_2, \varkappa_3). \tag{2.8}$$

From (2.8), we get

$$H(\varkappa_1, \varkappa_2) = \kappa\eta(\varkappa_2)\varkappa_1 + (\kappa - 1)\eta(\varkappa_1)\varkappa_2. \tag{2.9}$$

Using (2.9) in (2.3), it gives

$$\bar{\Gamma}_{\varkappa_1}\varkappa_2 = \Gamma_{\varkappa_1}\varkappa_2 + \kappa\eta(\varkappa_2)\varkappa_1 + (\kappa - 1)\eta(\varkappa_1)\varkappa_2.$$

Conversely, we can show that if $\bar{\Gamma}$ satisfies equation (2.1), then equations (1.3) and (2.2) are also satisfied. Hence, the proof is complete. □

We deduce equation (1.2) as follows

$$(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2) = (\Gamma_{\varkappa_1}\eta)(\varkappa_2) - (2\kappa - 1)\eta(\varkappa_1)\eta(\varkappa_2). \tag{2.10}$$

From (2.10), we have

$$(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2) - (\bar{\Gamma}_{\varkappa_2}\eta)(\varkappa_1) = (\Gamma_{\varkappa_1}\eta)(\varkappa_2) - (\Gamma_{\varkappa_2}\eta)(\varkappa_1). \tag{2.11}$$

The extensive study of different types of connections by various researchers provides the motivation for introducing a new class of semi-symmetric non-metric connection (SSNMC) on Riemannian manifolds. The connection defined in equation (2.1) admits the following special cases:

1. If $\kappa = 0$, then (2.1) reduces to $\bar{\Gamma}_{\varkappa_1}\varkappa_2 = \Gamma_{\varkappa_1}\varkappa_2 - \eta(\varkappa_1)\varkappa_2$, which is known as a semi-symmetric recurrent metric connection (in short, SSRMC) defined by Andonie and Smaranda [21] and Liang [30],
2. If $\kappa = 1$, then (2.1) reduces to $\bar{\Gamma}_{\varkappa_1}\varkappa_2 = \Gamma_{\varkappa_1}\varkappa_2 + \eta(\varkappa_2)\varkappa_1$, said to be a SSNMC defined by Agashe and Chafle [18],
3. If $\kappa = \frac{1}{2}$, then (2.1) reduces to $\bar{\Gamma}_{\varkappa_1}\varkappa_2 = \Gamma_{\varkappa_1}\varkappa_2 + \frac{1}{2}\big[\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2\big]$, known as a new type of SSNMC defined by Chaubey and Yildiz [14],
4. De et al. [8] defined a special type of SSNMC on a Riemannian manifold as follows:

$$\bar{\nabla}_{\varkappa_1}\varkappa_2 = \nabla_{\varkappa_1}\varkappa_2 + a\omega(\varkappa_1)\varkappa_2 + b\omega(\varkappa_2)\varkappa_1,$$

where a and b are non-zero real numbers and $\omega(\varkappa_1) = g(\varkappa_1, \rho)$. If we set $a = \kappa \neq 0$ and $b = \kappa - 1 \neq 0$, then we recover a special type of SSNMC. Since no restriction is imposed on κ, the connection $\bar{\Gamma}$ defined in (2.1) thus generalizes this special class of SSNMC.

Remark 2.2. Remark that in [8], authors have considered that κ is non-zero scalar and defined the notion of special type of SSNMC. In this manuscript, we are not imposing any restriction on κ. Thus, our connection defined in (2.1) is generalization of semi-symmetric recurrent metric connection (Andonie and Smaranda [21] and Liang [30]), SSNMC (Agashe and Chafle [18]), a new type of SSNMC (Chaubey and Yildiz [14]) and a special type of SSNMC (De et al. [8]).

Proposition 2.3. *On a Riemannian manifold (M^n, g), the 1-form η linked to the connection $\bar{\Gamma}$ is closed precisely when it is closed with respect to the LCC Γ associated with the metric g. This equivalence provides a necessary and sufficient condition for η to be closed under $\bar{\Gamma}$.*

Definition 2.4. [8] The vector field ξ is irrotational if $g(\varkappa_1, \Gamma_{\varkappa_2}\xi) = g(\varkappa_2, \Gamma_{\varkappa_1}\xi)$ and the integral curves of the vector field ξ are geodesic if $\Gamma_\xi \xi = 0$.

Putting $\varkappa_2 = \xi$ in (2.1), we get

$$\bar{\Gamma}_{\varkappa_1}\xi = \Gamma_{\varkappa_1}\xi + \kappa\eta(\xi)\varkappa_1 + (\kappa - 1)\eta(\varkappa_1)\xi.$$

Taking the inner product with $\varkappa_2$ in the latest equation and gives

$$g(\varkappa_2, \bar{\Gamma}_{\varkappa_1}\xi) - g(\varkappa_1, \bar{\Gamma}_{\varkappa_2}\xi) = g(\varkappa_2, \Gamma_{\varkappa_1}\xi) - g(\varkappa_1, \Gamma_{\varkappa_2}\xi),$$

which shows that the vector field ξ is irrotational relative to $\bar{\Gamma}$ if and only if it is irrotational relative to Γ.

Again, plugging $\varkappa_1 = \varkappa_2 = \xi$ in equation (2.1), we get

$$\bar{\Gamma}_\xi \xi = \Gamma_\xi \xi + (2\kappa - 1)\eta(\xi)\xi. \tag{2.12}$$

Given that $\Gamma_\xi \xi = -(2\kappa - 1)\eta(\xi)\xi$ and $\kappa \neq \frac{1}{2}$, it follows from the above equation that $\bar{\Gamma}_\xi \xi = 0$. If $\kappa = \frac{1}{2}$ in (2.12), then we have

$$\bar{\Gamma}_\xi \xi = \Gamma_\xi \xi.$$

We therefore establish the following theorem:

Theorem 2.5. *If a Riemannian manifold (M^n, g) admitting an SSNMC $\bar{\Gamma}$, then*

(i) the unit vector field ξ is irrotational under SSNMC $\bar{\Gamma}$ precisely when it is irrotational under the LCC Γ,

(ii) the integral curves of the unit vector field ξ are geodesic with respect to SSNMC $\bar{\Gamma}$ if $\Gamma_\xi \xi = -(2\kappa - 1)\eta(\xi)\xi$, provided that $\kappa \neq \frac{1}{2}$,

(iii) for the real number $\kappa = \frac{1}{2}$, the integral curves of the unit vector ξ are geodesic with SSNMC $\bar{\Gamma}$ if and only if the integral curves of the unit vector field ξ are also geodesic with respect to the LCC Γ.

Theorem 2.6. *On a Riemannian manifold (M^n, g), an SSNMC $\bar{\Gamma}$ defined in (2.1) has the following properties:*

(i) $'\bar{T}(\varkappa_1, \varkappa_2, \varkappa_3) + {}'\bar{T}(\varkappa_2, \varkappa_1, \varkappa_3) = 0$,

(ii) $'\bar{T}(\varkappa_1, \varkappa_2, \varkappa_3) + {}'\bar{T}(\varkappa_2, \varkappa_3, \varkappa_1) + {}'\bar{T}(\varkappa_3, \varkappa_1, \varkappa_2) = 0$.

Proof. We define $'\bar{T}(\varkappa_1, \varkappa_2, \varkappa_3) = g(\bar{T}(\varkappa_1, \varkappa_2), \varkappa_3)$ on (M^n, g). Therefore, (1.3) yields

$$'\bar{T}(\varkappa_1, \varkappa_2, \varkappa_3) = \eta(\varkappa_2)g(\varkappa_1, \varkappa_3) - \eta(\varkappa_1)g(\varkappa_2, \varkappa_3). \tag{2.13}$$

The result is obtained from equation (2.13), which completes the proof of the theorem. □

Theorem 2.7. *If (M^n, g) is an n-dimensional Riemannian manifold equipped with an SSNMC $\bar{\Gamma}$, then $\bar{T}$ is cyclic parallel if and only if the corresponding 1-form η is closed.*

Proof. The covariant derivative of equation (1.3) yields

$$\begin{aligned}(\bar{\Gamma}_{\varkappa_1}\bar{T})(\varkappa_2, \varkappa_3) = (\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_3)\varkappa_2 - (\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2)\varkappa_3 - (3\kappa - 2)\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 \\ + (3\kappa - 2)\eta(\varkappa_1)\eta(\varkappa_2)\varkappa_3 + \kappa\eta(\xi)\{g(\varkappa_1, \varkappa_2)\varkappa_3 - g(\varkappa_1, \varkappa_3)\varkappa_2\}.\end{aligned} \tag{2.14}$$

From (2.14), the cyclic sum for all $\varkappa_1, \varkappa_2, \varkappa_3$ on (M^n, g) gives

$$\begin{aligned}&(\bar{\Gamma}_{\varkappa_1}\bar{T})(\varkappa_2, \varkappa_3) + (\bar{\Gamma}_{\varkappa_2}\bar{T})(\varkappa_3, \varkappa_1) + (\bar{\Gamma}_{\varkappa_3}\bar{T})(\varkappa_1, \varkappa_2) \\ &= \{(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_3) - (\bar{\Gamma}_{\varkappa_3}\eta)(\varkappa_1)\}\varkappa_2 + \{(\bar{\Gamma}_{\varkappa_3}\eta)(\varkappa_2) \\ &- (\bar{\Gamma}_{\varkappa_2}\eta)(\varkappa_3)\}\varkappa_1 + \{(\bar{\Gamma}_{\varkappa_2}\eta)(\varkappa_1) - (\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2)\}\varkappa_3.\end{aligned} \tag{2.15}$$

Also, the equation

$$(\bar{\Gamma}_{\varkappa_1}\bar{T})(\varkappa_2, \varkappa_3) + (\bar{\Gamma}_{\varkappa_2}\bar{T})(\varkappa_3, \varkappa_1) + (\bar{\Gamma}_{\varkappa_3}\bar{T})(\varkappa_1, \varkappa_2) = 0$$

can be deduced from Proposition 2.3 and equation (2.15), noted that the 1-form η associated with $\bar{\Gamma}$ is closed. Hence, verified. □

Proposition 2.8. *If a Riemannian manifold (M^n, g) admits an SSNMC $\bar{\Gamma}$, then the Lie derivatives of a Riemannian metric g holds the following relation*

$$(\bar{\pounds}_\xi g)(\varkappa_1, \varkappa_2) = (\pounds_\xi g)(\varkappa_1, \varkappa_2) + 2\eta(\xi)g(\varkappa_1, \varkappa_2) - 2\eta(\varkappa_1)\eta(\varkappa_2), \tag{2.16}$$

where $\bar{\pounds}_\xi$ and $\pounds_\xi$ denote the Lie derivatives along the vector field ξ corresponding to an SSNMC $\bar{\Gamma}$ and the LCC Γ, respectively.

Proof. We know that

$$(\pounds_\xi g)(\varkappa_1, \varkappa_2) = g(\Gamma_{\varkappa_1}\xi, \varkappa_2) + g(\varkappa_1, \Gamma_{\varkappa_2}\xi). \tag{2.17}$$

From equations (2.1), (2.17) and with the help of Lie derivative's definition, we obtain

$$(\bar{\pounds}_\xi g)(\varkappa_1, \varkappa_2) = (\pounds_\xi g)(\varkappa_1, \varkappa_2) + 2\eta(\xi)g(\varkappa_1, \varkappa_2) - 2\eta(\varkappa_1)\eta(\varkappa_2).$$

Thus, the proposition has been demonstrated. □

If the vector field ξ is a Killing vector field on the Riemannian manifold (M^n, g), this means that the metric tensor g is preserved along the flow generated by ξ. As a result, the Lie derivative of g with respect to ξ, i.e., $\pounds_\xi g$ vanishes. According to Proposition 2.8, we have the following result:

Corollary 2.9. *If ξ is a Killing vector field on a Riemannian manifold (M^n, g) that admits an SSNMC $\bar{\Gamma}$, then*

$$(\bar{\pounds}_\xi g)(\varkappa_1, \varkappa_2) = 2[\eta(\xi)g(\varkappa_1, \varkappa_2) - \eta(\varkappa_1)\eta(\varkappa_2)]. \tag{2.18}$$

3 Curvature tensor associated with a semi-symmetric non-metric connection

Let (M^n, g) be an n-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection $\bar{\Gamma}$. Then the curvature tensor $\bar{R}$ of an SSNMC $\bar{\Gamma}$ on (M^n, g) is defined by the equation

$$\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 = \bar{\Gamma}_{\varkappa_1}\bar{\Gamma}_{\varkappa_2}\varkappa_3 - \bar{\Gamma}_{\varkappa_2}\bar{\Gamma}_{\varkappa_1}\varkappa_3 - \bar{\Gamma}_{[\varkappa_1,\varkappa_2]}\varkappa_3.$$

Using (2.1) in the above equation, we get

$$\begin{aligned}\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 = R(\varkappa_1, \varkappa_2)\varkappa_3 + \kappa\{\alpha(\varkappa_1, \varkappa_3)\varkappa_2 - \alpha(\varkappa_2, \varkappa_3)\varkappa_1\} \\ + (\kappa - 1)\{\alpha(\varkappa_1, \varkappa_2) - \alpha(\varkappa_2, \varkappa_1)\}\varkappa_3,\end{aligned} \tag{3.1}$$

where $R(\varkappa_1, \varkappa_2)\varkappa_3 = \Gamma_{\varkappa_1}\Gamma_{\varkappa_2}\varkappa_3 - \Gamma_{\varkappa_2}\Gamma_{\varkappa_1}\varkappa_3 - \Gamma_{[\varkappa_1,\varkappa_2]}\varkappa_3$ is a Riemannian curvature tensor with respect to the LCC Γ and α is a (0,2) type tensor field defined by

$$\alpha(\varkappa_1, \varkappa_2) = g(A\varkappa_1, \varkappa_2) = (\Gamma_{\varkappa_1}\eta)(\varkappa_2) - \kappa\eta(\varkappa_1)\eta(\varkappa_2) \tag{3.2}$$

and

$$A\varkappa_1 = \Gamma_{\varkappa_1}\xi - \kappa\eta(\varkappa_1)\xi. \tag{3.3}$$

The symmetry of the tensor field α is equivalent to the closure of the 1-form η that can be observed from equation (3.2). Taking the inner product with $\varkappa_4$ in (3.1), and setting $\varkappa_1 = \varkappa_4 = e_i$ for $1 \le i \le n$, we obtain the following expression

$$\bar{S}(\varkappa_2, \varkappa_3) = S(\varkappa_2, \varkappa_3) - (\kappa n - 1)\alpha(\varkappa_2, \varkappa_3) + (\kappa - 1)\alpha(\varkappa_3, \varkappa_2), \tag{3.4}$$

where $\bar{S}$ and S are the Ricci tensors of $\bar{\Gamma}$ and Γ, respectively. Let $\bar{Q}$ and Q be the Ricci operators of $\bar{\Gamma}$ and Γ, respectively. Then we have $\bar{S}(\varkappa_2, \varkappa_3) = g(\bar{Q}\varkappa_2, \varkappa_3)$ and $S(\varkappa_2, \varkappa_3) = g(Q\varkappa_2, \varkappa_3)$.

Let us assume an orthonormal frame field on (M^n, g) and contracting (3.4) with respect to $\varkappa_1$ and $\varkappa_2$, then

$$\bar{r} = r - \kappa(n - 1)a, \tag{3.5}$$

where $a = traceA$, $\bar{r}$ and r are the scalar curvatures of (M^n, g) with respect to $\bar{\Gamma}$ and Γ, respectively, and

$$a = \sum_{i=1}^{n} \alpha(e_i, e_j), \quad 1 \le i \le n.$$

As a consequence of equation (3.5), we arrive at the following proposition:

Proposition 3.1. *Let (M^n, g) be a Riemannian manifold admitting an SSNMC $\bar{\Gamma}$, then the scalar curvatures $\bar{r}$ and r of (M^n, g) corresponding to $\bar{\Gamma}$ and Γ coincide if and only if either $a = 0$ or $\bar{\Gamma}$ reduces to an SSRMC.*

Interchanging $\varkappa_2$ and $\varkappa_3$ in (3.4), we obtain

$$\bar{S}(\varkappa_3, \varkappa_2) = S(\varkappa_3, \varkappa_2) - (\kappa n - 1)\alpha(\varkappa_3, \varkappa_2) + (\kappa - 1)\alpha(\varkappa_2, \varkappa_3). \tag{3.6}$$

After using equations (3.2), (3.4) and (3.6), it forms

$$\begin{aligned}\bar{S}(\varkappa_2, \varkappa_3) - \bar{S}(\varkappa_3, \varkappa_2) &= [\kappa(n + 1) - 2]\{\alpha(\varkappa_3, \varkappa_2) - \alpha(\varkappa_2, \varkappa_3)\} \\ &= [\kappa(n + 1) - 2]\mathsf{d}\eta(\varkappa_3, \varkappa_2),\end{aligned} \tag{3.7}$$

where the operator d is the exterior derivative of g. Based on (3.7) and Proposition 2.3, the following proposition is established:

Proposition 3.2. *Let (M^n, g) be an n-dimensional Riemannian manifold endowed with an SSNMC $\bar{\Gamma}$. Then the Ricci tensor $\bar{S}$ of $\bar{\Gamma}$ is symmetric if and only if either $\kappa = \frac{2}{n+1}$ or its exterior derivative d vanishes.*

Theorem 3.3. *Let (M^n, g) be an n-dimensional Riemannian manifold equipped with an SSNMC $\bar{\Gamma}$, then the curvature tensor $\bar{R}$ of an SSNMC $\bar{\Gamma}$ satisfies the following identities for all $\varkappa_1, \varkappa_2, \varkappa_3$ and $\varkappa_4$ on (M^n, g):*

(i) $\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 + \bar{R}(\varkappa_2, \varkappa_1)\varkappa_3 = 0,$

(ii) $\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 + \bar{R}(\varkappa_2, \varkappa_3)\varkappa_1 + \bar{R}(\varkappa_3, \varkappa_1)\varkappa_2 = 0,$ *provided that* η *is closed,*

(iii) $(\bar{\Gamma}_{\varkappa_1}\bar{R})(\varkappa_2, \varkappa_3)\varkappa_4 + (\bar{\Gamma}_{\varkappa_2}\bar{R})(\varkappa_3, \varkappa_1)\varkappa_4 + (\bar{\Gamma}_{\varkappa_3}\bar{R})(\varkappa_1, \varkappa_2)\varkappa_4$
$= 2\eta(\varkappa_1)\bar{R}(\varkappa_2, \varkappa_3)\varkappa_4 + 2\eta(\varkappa_2)\bar{R}(\varkappa_3, \varkappa_1)\varkappa_4 + 2\eta(\varkappa_3)\bar{R}(\varkappa_1, \varkappa_2)\varkappa_4,$

(iv) ${}'\bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) + {}'\bar{R}(\varkappa_1, \varkappa_2, \varkappa_4, \varkappa_3) = \kappa\{\alpha(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) - \alpha(\varkappa_2, \varkappa_4)g(\varkappa_1, \varkappa_3)$
$+ \alpha(\varkappa_1, \varkappa_4)g(\varkappa_2, \varkappa_3) - \alpha(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4)\} + 2(\kappa - 1)\mathrm{d}\eta(\varkappa_1, \varkappa_2)g(\varkappa_4, \varkappa_3),$

(v) ${}'\bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) - {}'\bar{R}(\varkappa_3, \varkappa_4, \varkappa_1, \varkappa_2) = \kappa\{\mathrm{d}\eta(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) + \alpha(\varkappa_4, \varkappa_1)g(\varkappa_2, \varkappa_3)$
$- \alpha(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4)\} + (\kappa - 1)\{\mathrm{d}\eta(\varkappa_4, \varkappa_3)g(\varkappa_1, \varkappa_2) + \mathrm{d}\eta(\varkappa_1, \varkappa_2)g(\varkappa_3, \varkappa_4)\}.$

Proof. To obtain (i), we interchange $\varkappa_1$ and $\varkappa_2$ in (3.1) and add with (3.1). Also, the cyclic sum of (3.1) gives

$$\begin{aligned}\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 + \bar{R}(\varkappa_2, \varkappa_3)\varkappa_1 + \bar{R}(\varkappa_3, \varkappa_1)\varkappa_2 = {} & \{\alpha(\varkappa_3, \varkappa_2) - \alpha(\varkappa_2, \varkappa_3)\}\varkappa_1 \\ & + \{\alpha(\varkappa_1, \varkappa_3) - \alpha(\varkappa_3, \varkappa_1)\}\varkappa_2 \\ & + \{\alpha(\varkappa_2, \varkappa_1) - \alpha(\varkappa_1, \varkappa_2)\}\varkappa_3,\end{aligned}$$

which satisfies the first identity of Bianchi if and only if the 1-form η is closed. Hence, (ii) is verified. Also, the Bianchi's second identity for $\bar{\Gamma}$ can be exhibited as follows

$$\begin{aligned}&(\bar{\Gamma}_{\varkappa_1}\bar{R})(\varkappa_2, \varkappa_3)\varkappa_4 + (\bar{\Gamma}_{\varkappa_2}\bar{R})(\varkappa_3, \varkappa_1)\varkappa_4 + (\bar{\Gamma}_{\varkappa_3}\bar{R})(\varkappa_1, \varkappa_2)\varkappa_4 \\ &= -\bar{R}(\bar{T}(\varkappa_1, \varkappa_2), \varkappa_3)\varkappa_4 - \bar{R}(\bar{T}(\varkappa_2, \varkappa_3), \varkappa_1)\varkappa_4 - \bar{R}(\bar{T}(\varkappa_3, \varkappa_1), \varkappa_2)\varkappa_4.\end{aligned}$$

From the above equation with equation (1.1) and (i), we get (iii).
If $g(\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3, \varkappa_4) = {}'\bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)$ and $g(R(\varkappa_1, \varkappa_2)\varkappa_3, \varkappa_4) = {}'R(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)$, then (3.1) becomes

$$\begin{aligned}{}'\bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = {} & {}'R(\varkappa_1, \varkappa_2, \varkappa_4, \varkappa_3) + \kappa\{\alpha(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) \\ & - \alpha(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4)\} + (\kappa - 1)\{\alpha(\varkappa_1, \varkappa_2) \\ & - \alpha(\varkappa_2, \varkappa_1)\}g(\varkappa_3, \varkappa_4).\end{aligned} \tag{3.8}$$

The expressions for (iv) and (v) can be solved by utilizing equations (3.2) and (3.8). Thus, the theorem. □

Theorem 3.4. *Let a Riemannian manifold* (M^n, g) *of dimension* $n \geq 3$ *admits an SSNMC* $\bar{\Gamma}$. *Then the necessary and sufficient condition for the projective curvature tensor with respect to* $\bar{\Gamma}$ *and* Γ *to coincide is that either 1-form* η *is closed or* $\kappa = 1$, *that is, an SSNMC* $\bar{\Gamma}$ *defined in (2.1) reduces to an SSNMC in Agashe and Chafle sense.*

Proof. Suppose η is closed, then from (3.2), we get α is symmetric. Now, equation (3.1) brings

$$\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 = R(\varkappa_1, \varkappa_2)\varkappa_3 + \kappa\{\alpha(\varkappa_1, \varkappa_3)\varkappa_2 - \alpha(\varkappa_2, \varkappa_3)\varkappa_1\}. \tag{3.9}$$

Taking the contraction of equation (3.9) along $\varkappa_1$, we obtain

$$\bar{S}(\varkappa_2, \varkappa_3) = S(\varkappa_2, \varkappa_3) - \kappa(n - 1)\alpha(\varkappa_2, \varkappa_3). \tag{3.10}$$

From the above equation it follows that

$$\bar{Q}\varkappa_2 = Q\varkappa_2 - \kappa(n - 1)A\varkappa_2 \tag{3.11}$$

and

$$\bar{r} = r - \kappa(n - 1)a. \tag{3.12}$$

The projective curvature tensor [15] with respect to an SSNMC $\bar{\Gamma}$ is given by

$$\bar{P}(\varkappa_1, \varkappa_2)\varkappa_3 = \bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 - \frac{1}{n - 1}\big[\bar{S}(\varkappa_2, \varkappa_3)\varkappa_1 - \bar{S}(\varkappa_1, \varkappa_3)\varkappa_2\big]. \tag{3.13}$$

With the help of (1.4), (3.9) and (3.10), the latest equation becomes

$$\bar{P}(\varkappa_1,\varkappa_2)\varkappa_3 = P(\varkappa_1,\varkappa_2)\varkappa_3. \tag{3.14}$$

On the other hand, we suppose that $\kappa = 1$, an SSNMC $\bar{\Gamma}$ defined in (2.1) becomes [18]. Then equation (3.1) takes the form

$$\bar{R}(\varkappa_1,\varkappa_2)\varkappa_3 = R(\varkappa_1,\varkappa_2)\varkappa_3 + \alpha(\varkappa_1,\varkappa_3)\varkappa_2 - \alpha(\varkappa_2,\varkappa_3)\varkappa_1. \tag{3.15}$$

Taking the contraction of equation (3.15) along $\varkappa_1$, we obtain

$$\bar{S}(\varkappa_2,\varkappa_3) = S(\varkappa_2,\varkappa_3) - (n-1)\alpha(\varkappa_2,\varkappa_3). \tag{3.16}$$

Also, from (3.16), it follows that

$$\bar{Q}\varkappa_2 = Q\varkappa_2 - (n-1)A\varkappa_2 \tag{3.17}$$

and

$$\bar{r} = r - (n-1)a. \tag{3.18}$$

In view of equations (1.4), (3.13) and (3.15)-(3.18), we notice that equation (3.14) is satisfied.

Next, we assume that an SSNMC $\bar{\Gamma}$ on (M^n, g) satisfies (3.14). Using equations (1.4), (3.1), (3.4), (3.11) and (3.13) in (3.14), we get

$$\begin{aligned}&\frac{\kappa-1}{n-1}\Big[\big\{\alpha(\varkappa_3,\varkappa_2)-\alpha(\varkappa_2,\varkappa_3)\big\}\varkappa_1+\big\{\alpha(\varkappa_1,\varkappa_3)-\alpha(\varkappa_3,\varkappa_1)\big\}\varkappa_2\Big]\\&+(\kappa-1)\big\{\alpha(\varkappa_1,\varkappa_2)-\alpha(\varkappa_2,\varkappa_1)\big\}\varkappa_3=0.\end{aligned}$$

Contracting the above expression along $\varkappa_1$, turns

$$(\kappa-1)\mathsf{d}\eta(\varkappa_2,\varkappa_3)=0,$$

which shows that either $\kappa = 1$ or its exterior derivative d vanishes. □

Theorem 3.5. *Consider an n-dimensional Riemannian manifold (M^n, g) endowed with an SSNMC $\bar{\Gamma}$ whose curvature tensor $\bar{R}$ vanishes, then*

(i) (M^n, g) is projectively flat,

(ii) the (1,2)-type tensor α is symmetric in its lower indices.

Proof. Let $\bar{R} = 0$ and α is symmetric. Then we can rewrite equation (3.9) as

$$R(\varkappa_1,\varkappa_2)\varkappa_3 = \kappa\big[\alpha(\varkappa_2,\varkappa_3)\varkappa_1 - \alpha(\varkappa_1,\varkappa_3)\varkappa_2\big], \tag{3.19}$$

which gives

$$S(\varkappa_2,\varkappa_3) = \kappa(n-1)\alpha(\varkappa_2,\varkappa_3) \quad \text{and} \quad r = \kappa(n-1)a. \tag{3.20}$$

Using equations (1.4), (3.19) and (3.20), we can conclude that the projective curvature tensor is flat.

Conversely, if $P = 0$ and $\bar{R} = 0$, then equations (1.4) and (3.1) yield

$$R(\varkappa_1,\varkappa_2)\varkappa_3 = \frac{1}{n-1}\Big[S(\varkappa_2,\varkappa_3)\varkappa_1 - S(\varkappa_1,\varkappa_3)\varkappa_2\Big] \tag{3.21}$$

and

$$R(\varkappa_1,\varkappa_2)\varkappa_3 = \kappa\big[\alpha(\varkappa_2,\varkappa_3)\varkappa_1 - \alpha(\varkappa_1,\varkappa_3)\varkappa_2\big] + (\kappa-1)\big[\alpha(\varkappa_2,\varkappa_1) - \alpha(\varkappa_1,\varkappa_2)\big]\varkappa_3. \tag{3.22}$$

Comparing (3.21) and (3.22), along with the use of (3.4), yields

$$\begin{aligned}(\kappa n-1)\big[\alpha(\varkappa_3,\varkappa_2)-\alpha(\varkappa_2,\varkappa_3)\big]\varkappa_1+(\kappa n-1)\big[\alpha(\varkappa_1,\varkappa_3)-\alpha(\varkappa_3,\varkappa_1)\big]\varkappa_2&\\-(\kappa-1)(n-1)[\alpha(\varkappa_2,\varkappa_1)-\alpha(\varkappa_1,\varkappa_2)]\varkappa_3=0.&\end{aligned}$$

Taking the contraction of the above equation with respect to $\varkappa_1$, we obtain

$$\alpha(\varkappa_3,\varkappa_2) = \alpha(\varkappa_2,\varkappa_3),$$

provided $\kappa \neq \frac{2}{n+1}$. Therefore, the proof is concluded. □

Theorem 3.6. *Let (M^n, g) be a Riemannian manifold endowed with an SSNMC $\bar{\Gamma}$ whose curvature tensor $\bar{R}$ vanishes. Then the tensor field α is symmetric if and only if the following identity holds for all vector fields $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$:*

$$(n-2)\Big[\,'C(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) + \,'W(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4)\Big] = -2\,'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4).$$

Proof. Taking the inner product of (1.5) with the vector field $\varkappa_4$, we get

$$\begin{aligned}'C(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = {}& 'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) - \frac{1}{n-2}\Big[S(\varkappa_2,\varkappa_3)g(\varkappa_1,\varkappa_4) \\ & - S(\varkappa_1,\varkappa_3)g(\varkappa_2,\varkappa_4) + g(\varkappa_2,\varkappa_3)S(\varkappa_1,\varkappa_4) - g(\varkappa_1,\varkappa_3)S(\varkappa_2,\varkappa_4)\Big] \\ & + \frac{r}{(n-1)(n-2)}\Big[g(\varkappa_2,\varkappa_3)g(\varkappa_1,\varkappa_4) - g(\varkappa_1,\varkappa_3)g(\varkappa_2,\varkappa_4)\Big],\end{aligned} \tag{3.23}$$

where $'C(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = g(C(\varkappa_1,\varkappa_2)\varkappa_3,\varkappa_4)$. Assume that the tensor field α is symmetric and $\bar{R} = 0$.
By substituting (3.19) and (3.20) in (3.23), we have

$$\begin{aligned}'C(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = {}& -\frac{n}{(n-2)}\,'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) + \frac{2\kappa a}{(n-2)}\Big[g(\varkappa_2,\varkappa_3)g(\varkappa_1,\varkappa_4) \\ & - g(\varkappa_1,\varkappa_3)g(\varkappa_2,\varkappa_4)\Big].\end{aligned} \tag{3.24}$$

The concircular curvature tensor [11] of type (0,4) with respect to the LCC Γ is defined by

$$\begin{aligned}'W(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = {}& 'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) - \frac{r}{(n-1)(n-2)}\Big[g(\varkappa_2,\varkappa_3)g(\varkappa_1,\varkappa_4) \\ & - g(\varkappa_1,\varkappa_3)g(\varkappa_2,\varkappa_4)\Big],\end{aligned} \tag{3.25}$$

where $'W(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = g(W(\varkappa_1,\varkappa_2)\varkappa_3,\varkappa_4)$. Inserting (3.20) and (3.25) in (3.24), we get

$$(n-2)\Big[\,'C(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) + \,'W(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4)\Big] = -2\,'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4). \tag{3.26}$$

Conversely, from equations (3.1), (3.23), (3.25) and (3.26), we get

$$\begin{aligned}& S(\varkappa_2,\varkappa_3)\varkappa_1 - S(\varkappa_1,\varkappa_3)\varkappa_2 - g(\varkappa_1,\varkappa_3)Q\varkappa_2 + g(\varkappa_2,\varkappa_3)Q\varkappa_1 \\ & = 2(n-1)\big[\kappa\{\alpha(\varkappa_2,\varkappa_3)\varkappa_1 - \alpha(\varkappa_1,\varkappa_3)\varkappa_2\} + (\kappa-1)\{\alpha(\varkappa_2,\varkappa_1) - \alpha(\varkappa_1,\varkappa_2)\}\varkappa_3\big].\end{aligned} \tag{3.27}$$

On taking the contraction of equation (3.27) with respect to $\varkappa_3$, we have

$$\alpha(\varkappa_1,\varkappa_2) = \alpha(\varkappa_2,\varkappa_1),$$

which shows that α is symmetric. This completes the proof. □

Corollary 3.7. *Let (M^n, g) be an n-dimensional Riemannian manifold equipped with an SSNMC $\bar{\Gamma}$, whose curvature tensor $\bar{R}$ vanishes identically, then*

$$(n-2)\,'L(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) + n\,'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = 0.$$

Proof. We know that the Riemannian manifold of (M^n, g) of n-dimension satisfies

$$'C(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) + \,'W(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = \,'L(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) + \,'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4). \tag{3.28}$$

The inner product of (1.7) with $\varkappa_4$ is given by

$$\begin{aligned}'L(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) = {}& 'R(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4) - \frac{1}{n-2}\Big[S(\varkappa_2,\varkappa_3)g(\varkappa_1,\varkappa_4) \\ & - S(\varkappa_1,\varkappa_3)g(\varkappa_2,\varkappa_4) + g(\varkappa_2,\varkappa_3)S(\varkappa_1,\varkappa_4) \\ & - g(\varkappa_1,\varkappa_3)S(\varkappa_2,\varkappa_4)\Big],\end{aligned} \tag{3.29}$$

where $'L$ is the conharmonic curvature tensor and $'L(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = g(L(\varkappa_1, \varkappa_2)\varkappa_3, \varkappa_4)$. From (3.26) and (3.28), we get

$$(n-2)\,'L(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) + n\,'R(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = 0.$$

Thus, the proof completes. □

4 Group manifolds with respect to a semi-symmetric non-metric connection

An n-dimensional Riemannian manifold (M^n, g) endowed with an SSNMC $\bar{\Gamma}$ satisfies the conditions of being a group manifold [13], then it must satisfy the following equations

$$\bar{\Gamma}_{\varkappa_1}\bar{T} = 0 \quad \text{and} \quad \bar{R} = 0. \tag{4.1}$$

From (2.15) and (4.1), we get

$$(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_3)\varkappa_2 - (\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2)\varkappa_3 = 0.$$

By using (2.10) the above equation yields

$$(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2) = 0 \iff (\Gamma_{\varkappa_1}\eta)(\varkappa_2) = (2\kappa - 1)\eta(\varkappa_1)\eta(\varkappa_2). \tag{4.2}$$

With the help of (3.1) and (4.1), we obtain

$$R(\varkappa_1, \varkappa_2)\varkappa_3 = -\kappa(\kappa - 1)\big[\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 - \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1\big]. \tag{4.3}$$

Contracting (4.3) with respect to $\varkappa_1$, we have

$$S(\varkappa_2, \varkappa_3) = \kappa(\kappa - 1)(n-1)\eta(\varkappa_2)\eta(\varkappa_3), \tag{4.4}$$

which implies

$$Q\varkappa_2 = \kappa(\kappa - 1)(n-1)\eta(\varkappa_2)\xi. \tag{4.5}$$

Replacing $\varkappa_3$ by ξ in equation (4.4) and with the help of using (1.2), we have

$$S(\varkappa_2, \xi) = \kappa(\kappa - 1)(n-1)\eta(\xi)g(\varkappa_2, \xi).$$

This leads to the following key result:

Theorem 4.1. *If a Riemannian manifold (M^n, g) $(n \geq 3)$ be equipped with an SSNMC $\bar{\Gamma}$, then the eigenvalue of the Ricci tensor S with respect to the eigenvector ξ is $\kappa(\kappa - 1)(n-1)\eta(\xi)$.*

Proposition 4.2. *If a Riemannian manifold (M^n, g) $(n \geq 3)$ be equipped with an SSNMC $\bar{\Gamma}$, then the eigenvalue of the Ricci tensor S with respect to the eigenvector ξ is zero if and only if either $\kappa = 0$ or $\kappa = 1$.*

Again, contracting (4.5) corresponds to $\varkappa_2$, gives

$$r = \kappa(\kappa - 1)(n-1)\eta(\xi). \tag{4.6}$$

From equations (4.3) and (4.4), substituted into (1.4), we obtain $P = 0$. Hence, the theorem below holds:

Theorem 4.3. *A group manifold (M^n, g) endowed with an SSNMC $\bar{\Gamma}$ is projectively flat.*

Additionally, we will prove the following theorems.

Theorem 4.4. *A group manifold (M^n, g) equipped with an SSNMC $\bar{\Gamma}$ is ξ-conformally flat.*

Proof. Applying equations (4.3)–(4.6) in equation (1.5), we get

$$\begin{aligned} C(\varkappa_1,\varkappa_2)\varkappa_3 &= \frac{\kappa(\kappa-1)}{(n-2)}\Big[\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1-\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2\Big] \\ &\quad+\frac{\kappa(\kappa-1)(n-1)}{(n-2)}\Big[g(\varkappa_2,\varkappa_3)\eta(\varkappa_1)-g(\varkappa_1,\varkappa_3)\eta(\varkappa_2)\Big]\xi \\ &\quad-\frac{\kappa(\kappa-1)}{(n-2)}\eta(\xi)\Big[g(\varkappa_2,\varkappa_3)\varkappa_1-g(\varkappa_1,\varkappa_3)\varkappa_2\Big]. \end{aligned} \tag{4.7}$$

If the conformal curvature tensor C satisfies $C(\varkappa_1,\varkappa_2)\xi=0$, then an n-dimensional Riemannian manifold (M^n,g) is said to be ξ-conformally flat [3].

On replacing $\varkappa_3$ by ξ in equation (4.7), we get

$$C(\varkappa_1,\varkappa_2)\xi=0.$$

Thus, the proof completes. □

Theorem 4.5. *A group manifold (M^n,g) of dimension n $(n\geq 3)$, endowed with an SSNMC $\bar\Gamma$ is Ricci-symmetric if and only if $\bar\Gamma$ is either*

(i) a semi-symmetric recurrent metric connection introduced by Andonie and Smaranda [21], and Liang [30] or

(ii) an SSNMC defined by Agashe and Chafle [18] or

(iii) a new type of SSNMC defined by Chaubey and Yildiz [14].

Proof. The covariant derivative of (4.4) yields

$$(\Gamma_{\varkappa_1}S)(\varkappa_2,\varkappa_3)=\kappa(\kappa-1)(n-1)\big[(\Gamma_{\varkappa_1}\eta)(\varkappa_2)\eta(\varkappa_3)+\eta(\varkappa_2)(\Gamma_{\varkappa_1}\eta)(\varkappa_3)\big]. \tag{4.8}$$

Using (4.2) in (4.8), we have

$$(\Gamma_{\varkappa_1}S)(\varkappa_2,\varkappa_3)=2\kappa(\kappa-1)(2\kappa-1)(n-1)\eta(\varkappa_1)\eta(\varkappa_2)\eta(\varkappa_3). \tag{4.9}$$

Note that a Riemannian manifold (M^n,g) is Ricci symmetric if and only if $\Gamma S=0$. Since $\eta(\varkappa_1)\neq 0$ and therefore (4.9) shows that the group manifold is Ricci symmetric if and only if $\kappa=0,1,\frac{1}{2}$. □

Theorem 4.6. *Let (g,ξ,ψ) be a Ricci soliton on an n-dimensional group manifold (M^n,g) equipped with an SSNMC $\bar\Gamma$. Then the Ricci soliton is*

(i) shrinking if $\psi<0$,

(ii) steady if $\psi=0$,

(iii) expanding if $\psi>0$.

Proof. If (M^n,g) is a group manifold with respect to $\bar\Gamma$, then equations (2.17) and (4.2) give

$$(\pounds_\xi g)(\varkappa_1,\varkappa_2)=2(2\kappa-1)\eta(\varkappa_1)\eta(\varkappa_2). \tag{4.10}$$

A Ricci soliton (g,J,ψ) on a Riemannian manifold (M^n,g) satisfies [24]

$$\pounds_J g+2S+2\psi g=0, \tag{4.11}$$

where g,J,ψ are the metric tensor, a soliton vector field and a soliton constant, respectively.

The value of ψ determines the type of soliton; the soliton is shrinking when $\psi<0$, the Ricci soliton is called a steady soliton when $\psi=0$ and the Ricci soliton is called an expanding soliton when $\psi>0$. If we interchange the soliton vector field J with ξ in equation (4.11) and use equations (4.4) and (4.10), we can get a new expression as

$$\big[(2\kappa-1)+\kappa(\kappa-1)(n-1)\big]\eta(\varkappa_1)\eta(\varkappa_2)+\lambda g(\varkappa_1,\varkappa_2)=0. \tag{4.12}$$

If we replace the vector field $\varkappa_2$ with ξ in equation (4.12), it becomes

$$\left[\psi - \{(1-2\kappa) + \kappa(1-\kappa)(n-1)\}\eta(\xi)\right]\eta(\varkappa_1) = 0, \tag{4.13}$$

which implies that

$$\psi = \left[1 + \kappa(n-3) - \kappa^2(n-1)\right]\eta(\xi),$$

since $\eta(\varkappa_1) \neq 0$ generally holds on (M^n, g). Thus, this completes the proof. □

Following the above theorem, we present the following corollary:

Corollary 4.7. *Let a group manifold (M^n, g) be equipped with an SSNMC $\bar{\Gamma}$ admits a Ricci soliton (g, ξ, ψ). Then we have the following data:*

Value of κ	**Linear connection**	**Value of ψ**	**Soliton is**
0	*SSRMC*	$\psi = \eta(\xi)$	*expanding*
1	*SSNMC by Agashe and Chafle*	$\psi = -\eta(\xi)$	*shrinking*
$\frac{1}{2}$	*SSNMC by Chaubey and Yildiz*	$\psi = \frac{n-1}{4}\eta(\xi)$	*expanding*

5 Example

Let $M^3 = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ be a 3-dimensional Riemannian manifold, where (x, y, z) are regarded as the standard coordinates in $\mathbb{R}^3$.

Let s_1, s_2 and s_3 be vector fields defined on M^3 that are linearly independent at each point of M^3 and are expressed as follows:

$$s_1 = e^{\beta z}\frac{\partial}{\partial x}, \qquad s_2 = e^{\beta z}\frac{\partial}{\partial y}, \qquad s_3 = \frac{\partial}{\partial z},$$

where β is a positive real constant. A Riemannian metric g on M^3 can be defined as

$$g_{ij} = 2\beta, \quad \text{for} \quad i = j, \quad g_{ij} = 0, \quad \text{for} \quad i \neq j,$$

where $i, j = 1, 2, 3$.

Let $\xi = s_3$ and the 1-form $\eta(\varkappa_1) = g(\varkappa_1, s_3)$ be defined on M^3, then the Lie brackets for the vector fields s_1, s_2, s_3 are given by

$$[s_1, s_2] = 0, \qquad [s_1, s_3] = -\beta s_1, \qquad [s_2, s_3] = -\beta s_2.$$

The LCC Γ of the metric tensor g is given by Koszul's formula which is determined by

$$\begin{aligned} 2g(\Gamma_{\varkappa_1}\varkappa_2, \varkappa_3) = &\varkappa_1 g(\varkappa_2, \varkappa_3) + \varkappa_2 g(\varkappa_3, \varkappa_1) - \varkappa_3 g(\varkappa_1, \varkappa_2) - g(\varkappa_1, [\varkappa_2, \varkappa_3]) \\ &- g(\varkappa_2, [\varkappa_1, \varkappa_3]) + g(\varkappa_3, [\varkappa_1, \varkappa_2]). \end{aligned}$$

Using the above formula, we find

$$\begin{aligned} &\Gamma_{s_1}s_1 = \beta s_3, && \Gamma_{s_1}s_2 = 0, && \Gamma_{s_1}s_3 = -\beta s_1, \\ &\Gamma_{s_2}s_1 = 0, && \Gamma_{s_2}s_2 = \beta s_3, && \Gamma_{s_2}s_3 = -\beta s_2, \\ &\Gamma_{s_3}s_1 = 0, && \Gamma_{s_3}s_2 = 0, && \Gamma_{s_3}s_3 = 0, \end{aligned}$$

where Γ is the LCC. Also, by using the above results and (2.1), we get

$$\begin{aligned} &\bar{\Gamma}_{s_1}s_1 = \beta s_3, && \bar{\Gamma}_{s_1}s_2 = 0, && \bar{\Gamma}_{s_1}s_3 = (2\kappa - 1)\beta s_1, \\ &\bar{\Gamma}_{s_2}s_1 = 0, && \bar{\Gamma}_{s_2}s_2 = \beta s_3, && \bar{\Gamma}_{s_2}s_3 = (2\kappa - 1)\beta s_2, \\ &\bar{\Gamma}_{s_3}s_1 = 2(\kappa - 1)\beta s_1, && \bar{\Gamma}_{s_3}s_2 = 2(\kappa - 1)\beta s_2, && \bar{\Gamma}_{s_3}s_3 = 2(2\kappa - 1)\beta s_3. \end{aligned}$$

The non-vanishing components of the Riemannian curvature tensor R and the Ricci tensor S, respectively, are obtained as follow:

$$\begin{aligned} R(s_1,s_2)s_1 &= 2\kappa\beta^2 s_2, & R(s_1,s_3)s_1 &= 2\kappa\beta^2 s_3, & R(s_1,s_2)s_2 &= -2\kappa\beta^2 s_1,\\ R(s_2,s_3)s_2 &= 2\kappa\beta^2 s_3, & R(s_1,s_3)s_3 &= -4\kappa^2\beta^2 s_1, & R(s_2,s_3)s_3 &= -4\kappa^2\beta^2 s_2,\\ S(s_1,s_1) &= -4\kappa\beta^2, & S(s_2,s_2) &= -4\kappa\beta^2, & S(s_3,s_3) &= -8\kappa^2\beta^2. \end{aligned}$$

Accordingly, the other values can be obtained by using symmetric properties. The scalar curvature is given by $r = -8\kappa\beta^2(\kappa+1)$. From the above results, it can be easily shown that $\bar{T}$ defined in (1.3) is valid for all s_i, $(i = 1,2,3)$, i.e.,

$$\bar{T}(s_1,s_3) = 2\beta s_1$$

and

$$\eta(s_3)s_1 - \eta(s_1)s_3 = 2\beta s_1.$$

It can be concluded that $\bar{\Gamma}$ defined in equation (2.1) is a type of connection known as a semi-symmetric connection on the manifold (M^3,g). Also, it can be seen that

$$(\bar{\Gamma}_{s_1} g)(s_1,s_3) = -4\kappa\beta^2 \neq 0.$$

This shows that the above equation is an SSNMC $\bar{\Gamma}$ on (M^3,g). Similarly, we can also verify for other components. It is easy to prove that Γ satisfies

$$(\Gamma_{\varkappa_1}\eta)(\varkappa_2) = (\Gamma_{\varkappa_2}\eta)(\varkappa_1).$$

Thus, if (M^3,g) is equipped with Γ, then it is said to be closed under Γ if and only if Γ is defined for all points in M^3.

Suppose $(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2) = \varkappa_1\eta(\varkappa_2) - g(\bar{\Gamma}_{\varkappa_1}\varkappa_2,\xi)$, we get

$$(\bar{\Gamma}_{\varkappa_1}\eta)(\varkappa_2) = (\bar{\Gamma}_{\varkappa_2}\eta)(\varkappa_1).$$

This verifies the statement of Proposition 2.3. Let $\varkappa_1,\varkappa_2,\varkappa_3 \in M^3$ and the linear combination of s_1, s_2 and s_3 can be expressed by

$$\begin{aligned} \varkappa_1 &= {\varkappa_1}^1 s_1 + {\varkappa_1}^2 s_2 + {\varkappa_1}^3 s_3,\\ \varkappa_2 &= {\varkappa_2}^1 s_1 + {\varkappa_2}^2 s_2 + {\varkappa_2}^3 s_3,\\ \varkappa_3 &= {\varkappa_3}^1 s_1 + {\varkappa_3}^2 s_2 + {\varkappa_3}^3 s_3. \end{aligned}$$

Then

$$\begin{aligned} {}'\bar{T}(\varkappa_1,\varkappa_2,\varkappa_3) &= g(\bar{T}(\varkappa_1,\varkappa_2),\varkappa_3)\\ &= 4\beta^2\big[({\varkappa_2}^3{\varkappa_1}^1 - {\varkappa_1}^3{\varkappa_2}^1){\varkappa_3}^1 + ({\varkappa_2}^3{\varkappa_1}^2 - {\varkappa_1}^3{\varkappa_2}^2){\varkappa_3}^2\big] \end{aligned}$$

and

$${}'\bar{T}(\varkappa_2,\varkappa_1,\varkappa_3) = 4\beta^2\big[({\varkappa_1}^3{\varkappa_2}^1 - {\varkappa_2}^3{\varkappa_1}^1){\varkappa_3}^1 + ({\varkappa_1}^3{\varkappa_2}^2 - {\varkappa_2}^3{\varkappa_1}^2){\varkappa_3}^2\big].$$

Thus, the above results confirm the statement of Theorem 2.6.

Now, through direct calculations considering the above facts and $\bar{R}(s_i,s_j)s_k = 0$, for all $i,j,k = 1,2,3$, it follows that the Riemannian manifold (M^3,g) equipped with an SSNMC $\bar{\Gamma}$ is flat. Consequently, the curvature tensor with respect to the LCC Γ is given by

$$\begin{aligned} R(\varkappa_1,\varkappa_2)\varkappa_3 = 2\kappa\beta^2\big[&(-{\varkappa_1}^1{\varkappa_2}^2{\varkappa_3}^2 - 2\kappa{\varkappa_1}^1{\varkappa_2}^3{\varkappa_3}^3 + {\varkappa_1}^2{\varkappa_2}^1{\varkappa_3}^2\\ &+ 2\kappa{\varkappa_1}^3{\varkappa_2}^1{\varkappa_3}^3)s_1 - ({\varkappa_1}^2{\varkappa_2}^1{\varkappa_3}^1 + 2\kappa{\varkappa_1}^2{\varkappa_2}^3{\varkappa_3}^3\\ &- {\varkappa_1}^1{\varkappa_2}^2{\varkappa_3}^1 - 2\kappa{\varkappa_1}^3{\varkappa_2}^2{\varkappa_3}^3)s_2 - ({\varkappa_1}^3{\varkappa_2}^1{\varkappa_3}^1\\ &+ {\varkappa_1}^3{\varkappa_2}^2{\varkappa_3}^2 - {\varkappa_1}^1{\varkappa_2}^3{\varkappa_3}^1 - {\varkappa_1}^2{\varkappa_2}^3{\varkappa_3}^2)s_3\big]. \end{aligned}$$

Using the previously derived components of the Ricci tensor S associated with the LCC Γ, the following result follows:

$$\begin{aligned} S(\varkappa_2,\varkappa_3)\varkappa_1 - S(\varkappa_1,\varkappa_3)\varkappa_2 = 4\kappa\beta^2 \big[& (-{\varkappa_1}^1{\varkappa_2}^2{\varkappa_3}^2 - 2\kappa{\varkappa_1}^1{\varkappa_2}^3{\varkappa_3}^3 + {\varkappa_1}^2{\varkappa_2}^1{\varkappa_3}^2 \\ & + 2\kappa{\varkappa_1}^3{\varkappa_2}^1{\varkappa_3}^3)s_1 - ({\varkappa_1}^2{\varkappa_2}^1{\varkappa_3}^1 + 2\kappa{\varkappa_1}^2{\varkappa_2}^3{\varkappa_3}^3 \\ & - {\varkappa_1}^1{\varkappa_2}^2{\varkappa_3}^1 - 2\kappa{\varkappa_1}^3{\varkappa_2}^2{\varkappa_3}^3)s_2 - ({\varkappa_1}^3{\varkappa_2}^1{\varkappa_3}^1 \\ & + {\varkappa_1}^3{\varkappa_2}^2{\varkappa_3}^2 - {\varkappa_1}^1{\varkappa_2}^3{\varkappa_3}^1 - {\varkappa_1}^2{\varkappa_2}^3{\varkappa_3}^2)s_3 \big]. \end{aligned}$$

Based on the above arguments and using equation (1.4), we can conclude that $P(\varkappa_1,\varkappa_2)\varkappa_3 = 0$, i.e., the Riemannian manifold (M^3, g) is projectively flat. Therefore, Theorem 3.5 is verified.

References

[1] A. Friedmann and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragung*, Math. Zeitschrift, **21**, 211-223, (1924).

[2] L. Bianchi, *Sui simboli a quattro indici e sulla curvatura di Riemann*, Rend. Mat. Acc. Naz. Lincei, **(5) 11**, 3-7, (1902).

[3] G. Zhen, J. L. Cabrerizo and L. M. Fernandez, *On ξ-conformally flat contact metric manifolds*, Indian J. Pure Appl. Math., **28**, 725-734, (1997).

[4] E. M. E. I. Mohamed, A. S. Diallo and L. A. S. Ahmed, *A new semi-symmetric non-metric connection on warped products*, Adv. Pure Appl. Math., **15(3)**, 1-11, (2024).

[5] H. A. Hayden, *Sub-spaces of a space with torsion*, Proc. London Math. Soc., **34**, 27-50, (1932).

[6] V. Melhrotra, *On a type of semi symmetric non-metric connection on a Riemannian manifold*, JPS., **3(1)**, 130-133, (2012).

[7] A. Barman, *On Lorentzian α-Sasakian manifolds admitting a type of semi-symmetric non-metric connection*, Palest. J. Math., **9(2)**, 848-857, (2020).

[8] U. C. De, Y. L. Han and P. B. Zhao, *A special type of semi-symmetric non-metric connection on a Riemannian manifold*, Facta Universitatis, **31**, 529-541, (2016).

[9] H. Weyl, *Reine infinitesimalgeometric*, Mathematische Zeitschrift, **2**, 384-411, (1918).

[10] J. Sengupta, U. C. De and T. Q. Binh, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math., **31**, 1659-1670, (2000).

[11] K. Yano, *Concircular geometry I. Concircular transformations*, Proc. Imp. Acad. Tokyo, **16**, 195-200, (1940).

[12] A. Gupta, B. Prasad and R. P. S. Yadav, *Riemannian manifold admitting a new type of semi-symmetric non-metric connection*, J.T.S., **17**, 34-53, (2023).

[13] K. Yano, *On semi-symmetric metric connections*, Revue Roumaine de Mathematiques Pures et Apppliquees, **15**, 1579-1586, (1970).

[14] S. K. Chaubey and A. Yildiz, *Riemannian manifolds admitting a new type of semisymmetric nonmetric connection*, Turk J. Math., **43**, 1887-1904, (2019).

[15] L. P. Eisenhart, *Riemannian geometry*, Princeton. N.J. USA, Princeton University Press, 1949.

[16] M. N. I. Khan, *Submanifolds of a Riemannian manifold endowed with a new type of semi-symmetric non-metric connection in the tangent bundle*, Int. J. Math. Comput. Sci., **17(1)**, 265-275, (2022).

[17] A. S. Diallo and F. Massamba, *Conjugate semi-symmetric non-metric connections*, Facta Univ. Ser. Math. Inform., **34(5)**, 823-836, (2019).

[18] N. S. Agashe and M. R. Chafle, *A semi-symmetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math., **23**, 399-409, (1992).

[19] R. N. Singh and M. K. Pandey, *On semi-symmetric non-metric connection I*, Ganita, **58**, 47-59, (2007).

[20] U. C. De and M. N. I. Khan, *Complete lifts of a semi-symmetric non-metric connection from a Riemannian manifold to its tangent bundles*, Commun. Korean Math. Soc., **38(4)**, 1233-1247, (2023).

[21] O. C. Andonie and D. Smaranda, *Certain connexions semi-symetriques*, Tensor, **31**, 8-12, (1977).

[22] El M. El Id Mohamed, A. S. Diallo and L. A. Sid'Ahmed, *On semi-symmetric recurrent-metric conjugate connection*, Palest. J. Math., **13(4)**, 825-833, (2024).

[23] B. Prasad, R. Kumar and R. Singh, *Semi-symmetric non metric connection on a Riemannian manifold*, JETIR., **8(3)**, 100-108, (2021).

[24] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986)*, Contemp. Math., **71**, 237-262, (1988).

[25] S. K. Chaubey and R. H. Ojha, *On a semi-symmetric non-metric connection*, Filomat, **26**, 63-69, (2012).

[26] Y. Ishii, *On conharmonic transformations*, Tensor, **7**, 73-80, (1957).

[27] A. Singh, L. S. Das, R. Prasad and L. Kumar, *Some properties of Kenmotsu manifolds admitting a new type of semi-symmetric non-metric connection*, Communications in Mathematics and Applications., **15(1)**, 145-160, (2024).

[28] A. Singh, L. S. Das, Pankaj and S. Patel, *Hyperbolic Kenmotsu manifold admitting a new type of semi-symmetric non-metric connection*, Ser. Math. Inform., **39(1)**, 123-139, (2024).

[29] A. Barman, *On Para-Sasakian manifolds admitting a special type of semi-symmetric non-metric η-connection*, Palest. J. Math., **8(2)**, 266-274, (2019).

[30] Y. Liang, *On semi-symmetric recurrent-metric connection*, Tensor N.S., **55**, 107-112, (1994).

Author information

R. Kumar, Department of Mathematics, Pachhunga University College, Mizoram University, Aizawl-796001, Mizoram, India.
E-mail: rajesh_mzu@yahoo.com

S. K. Chaubey, Section of Mathematics, IT Department, University of Technology and Applied Sciences, Shinas, P.O. Box 77, Postal Code 324, Oman.
E-mail: sudhakar.chaubey@utas.edu.com

M. A. Khan, Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box-65892, Riyadh 11566, Saudi Arabia.
E-mail: mskhan@imamu.edu.sa

L. Chawngthu, Department of Mathematics and Computer Science, Mizoram University, Tanhril, Aizawl-796004, Mizoram, India.
E-mail: tluangkimact@gmail.com

Received: 2025-06-17
Accepted: 2025-10-06