**Palestine Journal of Mathematics**

**Vol 15(2)(2026)** , 1–13 

# THE NEIGHBORHOOD DEGREE ECCENTRICITY BASED TOPOLOGICAL INDICES OF A GRAPH

Veena Mathad and Parvathi

Communicated by: Harikrishnan Panackal

MSC 2020 Classifications: Primary 05C07, 05C12; Secondary 92B10.

Keywords and phrases: Degree-based topological indices, Vertex degree, Distance in graphs, Neighborhood degree sum, Neighborhood topological indices, Graph theory in biology.

**Corresponding Author: Parvathi**

**Abstract**. In this paper, we introduce the neighborhood versions of topological indices, namely neighborhood degree-eccentric connectivity index(ndec index), neighborhood first Zagreb degree-eccentric index(nfzde index), neighborhood second Zagreb degree-eccentric index(nszde index), neighborhood amplified degree-eccentric connectivity index(nadec index), neighborhood harmonic degree-eccentric index(nhde index), neighborhood fourth geometric-arithmetic degree-eccentric index(nfgade index), neighborhood fifth atom bond connectivity degree-eccentric index(nfabcde index), neighborhood Randic degree-eccentric index(nrde index). We compute the general formulae for all these neighborhood versions of topological indices of standard graphs. Moving on, various graph theoretic properties and bounds of these topological indices are studied.

## 1 Introduction

Topological indices are numbers associated with chemical structures derived from their hydrogen -depleted graphs as a tool for compact and effective description of structural formulas which are used to study and predict the structure-property correlations of organic compounds. Molecular descriptors are playing significant role in chemistry, pharmacology, etc. Among them topological indices have a prominent place [2]. In 1947, Harold Wiener introduced the first topological index related to molecular branching and showed that his topological index is closely related to the boiling points of alkane molecules, his QSPR and QSAR analysis showed that it is also related with other quantities such as the parameters of its critical point, density, surface tension, viscosity of its liquid phase. The Wiener index $W(G)$ of $G$ [12] is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v)$$

.

In this paper, all graphs are simple, finite and connected. As usual, we denote the number of vertices and edges of a graph $G$ by $n = |V|$ and $m = |E|$, respectively. Let $v \in V(G)$. The number of edges incident at $v$ in $G$ is called the degree(or valency) of the vertex $v$ in $G$ and is denoted by $d_G(v)$ *or* $d_v$, we denote the maximum degree of $G$ by $\Delta(G)$ or $\Delta$, the minimum degree of $G$ by $\delta(G)$ or $\delta$, where $\Delta(G) = max\{deg(v) \mid v \in V(G)\}$ and $\delta(G) = min\{deg(v) \mid v \in V(G)\}$. The open neighborhood $N(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$ in $G$ and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of $v$ [4, 6]. Let $N(v)$ denotes the set of neighbors of $v$, the sum of degrees of the neighbors of $v$ is called the neighbor degree sum, denoted by $D_v$ and is given by $D_v = \sum_{u \in N(v)} d_u$ [9, 10]. The distance $d(u, v)$ or $d_G(u, v)$ is defined as the length of the shortest path between $u$ and $v$ in $G$. The eccentricity of a vertex $v \in V(G)$ is $e_v = max\{d(u, v) : u \in V(G)\}$. The radius $r(G)$ of $G$ is the minimum eccentricity of the vertices while the diameter $d(G)$ is the maximum eccentricity of all vertices in the graph.

If $G$ is $r$-regular graph with $n$ vertices then, $m = \frac{nr}{2}$ edges [3, 5, 7]. In 1997, the topological descriptor termed the eccentric connectivity index [11] is defined as

$$\xi^c(G) = \sum_{v \in V(G)} d_v e_v.$$

The amplified eccentric connectivity index of a graph $G$ with at least one edge[8] is defined as

$$\xi^{ac}(G) = \sum_{uv \in V(G)} (d_u e_u + d_v e_v).$$

A graph is equieccentric if all of its vertices have the same eccentricity.

**Lemma 1.1.** ***[1]Cauchy-Schwarz Inequality :*** *Let $x_i$ and $y_i$ be real numbers for all $1 \le i \le n$ then*

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

*Equality holds if and only if $x_i = k y_i$ for some constant $k$ for every $1 \le i \le n$.*

**Lemma 1.2.** ***[1]Chebyshev's Inequality :*** *Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be real numbers then we have*

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \le n \sum_{i=1}^{n} a_i b_i$$

*or*

$$\sum_{i=1}^{n} a_i b_i \ge \frac{1}{n} \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

*Equality holds if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.*

**Lemma 1.3.** *[1] Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two sequences of non-negative real numbers and $c_i > 0$, $i = 1, 2, \cdots, n$ such that $\frac{a_1}{c_1} \ge \frac{a_2}{c_2} \ge \cdots \ge \frac{a_n}{c_n}$ and $\frac{b_1}{c_1} \ge \frac{b_2}{c_2} \ge \cdots \ge \frac{b_n}{c_n}$ then*

$$\sum_{i=1}^{n} \frac{a_i b_i}{c_i} \ge \frac{\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i}{\sum_{i=1}^{n} c_i}$$

In this article, We study the graph theoretic properties of these new topological indices for some well known graphs.

## 2 Main Results

For every vertex $v \in V(G)$, the sum of the eccentricities of the neighbors of $v$ is called neighbor eccentric sum, denoted by $E_v = \sum_{u \in N(v)} e_u$.

We define the following topological indices, which are presented in the below Table.

| Sl. No. | **Topological Indices** | **Formula** |
|---|---|---|
| 1. | Neighborhood degree-eccentric connectivity index(ndec index) | $N_e\xi^c(G) = \sum_{u\in V(G)} D_uE_u$ |
| 2. | Neighborhood first Zagreb degree-eccentric index(nfzde index) | $N_eE_1(G) = \sum_{u\in V(G)} (D_uE_u)^2$ |
| 3. | Neighborhood second Zagreb degree-eccentric index(nszde index) | $N_eE_2(G) = \sum_{uv\in E(G)} (D_uE_uD_vE_v)$ |
| 4. | Neighborhood amplified degree-eccentric connectivity index(nadec index) | $N_e\xi^{ac}(G) = \sum_{uv\in E(G)} (D_uE_u + D_vE_v)$ |
| 5. | Neighborhood harmonic degree-eccentric index(nhde index) | $N_eH_e(G) = \sum_{uv\in E(G)} \frac{2}{D_uE_u + D_vE_v}$ |
| 6. | Neighborhood fourth geometric-arithmetic degree-eccentric index(nfgade index) | $N_eGA_4(G) = \sum_{uv\in E(G)} \frac{2\sqrt{D_uE_uD_vE_v}}{D_uE_u + D_vE_v}$ |
| 7. | Neighborhood fifth atom bond connectivity degree-eccentric index(nfabcde index) | $N_eABC_5(G) = \sum_{uv\in E(G)} \sqrt{\frac{D_uE_u + D_vE_v - 2}{D_uE_uD_vE_v}}$ |
| 8. | Neighborhood Randic degree-eccentric index(nrde index) | $N_eR(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{D_uE_uD_vE_v}}.$ |

In Figure $G_1$, $D_{v_i} = 4+4+4+4 = 16$ for every $1 \leq i \leq 6$ and $E_{v_i} = 2+2+2+2 = 8$ for every $1 \leq i \leq 6$ .

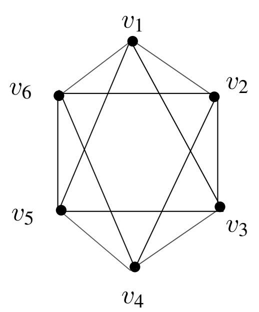


Figure : $G_1$

$$
\begin{aligned}
N_e\xi^c(G_1) =& \sum_{i=1}^{6} D_{v_i}E_{v_i}\\
=& D_{v_1}E_{v_1} + D_{v_2}E_{v_2} + D_{v_3}E_{v_3} + D_{v_4}E_{v_4} + D_{v_5}E_{v_5} + D_{v_6}E_{v_6}\\
=& 16\times 8 + 16\times 8 + 16\times 8 + 16\times 8 + 16\times 8 + 16\times 8\\
=& 6\times 128\\
=& 768.
\end{aligned}
$$

$$\begin{aligned}
N_eE_1(G_1) =& \sum_{i=1}^{6}(D_{v_i}E_{v_i})^2\\
=&(D_{v_1}E_{v_1})^2+(D_{v_2}E_{v_2})^2+(D_{v_3}E_{v_3})^2+(D_{v_4}E_{v_4})^2+(D_{v_5}E_{v_5})^2+(D_{v_6}E_{v_6})^2\\
=&(16\times 8)^2+(16\times 8)^2+(16\times 8)^2+(16\times 8)^2+(16\times 8)^2+(16\times 8)^2\\
=&6\times(128)^2\\
=&98304.
\end{aligned}$$

$$\begin{aligned}
N_eE_2(G_1) =& \sum_{uv\in E(G_1)}(D_uE_uD_vE_v)\\
=&(D_{v_1}E_{v_1}D_{v_2}E_{v_2})+(D_{v_1}E_{v_1}D_{v_3}E_{v_3})+(D_{v_1}E_{v_1}D_{v_5}E_{v_5})+(D_{v_1}E_{v_1}D_{v_6}E_{v_6})\\
&+(D_{v_2}E_{v_2}D_{v_3}E_{v_3})+(D_{v_2}E_{v_2}D_{v_4}E_{v_4})+(D_{v_2}E_{v_2}D_{v_6}E_{v_6})\\
&+(D_{v_3}E_{v_3}D_{v_4}E_{v_4})+(D_{v_3}E_{v_3}D_{v_5}E_{v_5})\\
&+(D_{v_4}E_{v_4}D_{v_5}E_{v_5})+(D_{v_4}E_{v_4}D_{v_6}E_{v_6})\\
&+(D_{v_5}E_{v_5}D_{v_6}E_{v_6})\\
=&4(16\times 8)^2+3(16\times 8)^2+2(16\times 8)^2+2(16\times 8)^2+(16\times 8)^2\\
=&196608.
\end{aligned}$$

$$\begin{aligned}
N_e\xi^{ac}(G_1) =& \sum_{uv\in E(G_1)}(D_uE_u+D_vE_v)\\
=&(D_{v_1}E_{v_1}+D_{v_2}E_{v_2})+(D_{v_1}E_{v_1}\\
&+D_{v_3}E_{v_3})+(D_{v_1}E_{v_1}+D_{v_5}E_{v_5})+(D_{v_1}E_{v_1}+D_{v_6}E_{v_6})\\
&+(D_{v_2}E_{v_2}+D_{v_3}E_{v_3})+(D_{v_2}E_{v_2}+D_{v_4}E_{v_4})+(D_{v_2}E_{v_2}+D_{v_6}E_{v_6})\\
&+(D_{v_3}E_{v_3}+D_{v_4}E_{v_4})+(D_{v_3}E_{v_3}+D_{v_5}E_{v_5})\\
&+(D_{v_4}E_{v_4}+D_{v_5}E_{v_5})+(D_{v_4}E_{v_4}+D_{v_6}E_{v_6})\\
&+(D_{v_5}E_{v_5}+D_{v_6}E_{v_6})\\
=&8(16\times 8)+6(16\times 8)+4(16\times 8)+4(16\times 8)+2(16\times 8)\\
=&3072.
\end{aligned}$$

$$
\begin{aligned}
N_eH_e(G_1) = &\sum_{uv\in E(G_1)} \frac{2}{D_uE_u + D_vE_v}\\
=&\frac{2}{D_{v_1}E_{v_1} + D_{v_2}E_{v_2}} + \frac{2}{D_{v_1}E_{v_1} + D_{v_3}E_{v_3}}\\
+&\frac{2}{D_{v_1}E_{v_1} + D_{v_5}E_{v_5}} + \frac{2}{D_{v_1}E_{v_1} + D_{v_6}E_{v_6}}\\
+&\frac{2}{D_{v_2}E_{v_2} + D_{v_3}E_{v_3}} + \frac{2}{D_{v_2}E_{v_2} + D_{v_4}E_{v_4}} + \frac{2}{D_{v_2}E_{v_2} + D_{v_6}E_{v_6}}\\
+&\frac{2}{D_{v_3}E_{v_3} + D_{v_4}E_{v_4}} + \frac{2}{D_{v_3}E_{v_3} + D_{v_5}E_{v_5}}\\
+&\frac{2}{D_{v_4}E_{v_4} + D_{v_5}E_{v_5}} + \frac{2}{D_{v_4}E_{v_4} + D_{v_6}E_{v_6}}\\
+&\frac{2}{D_{v_5}E_{v_5} + D_{v_6}E_{v_6}}\\
=&4\left(\frac{1}{16\times 8}\right) + 3\left(\frac{1}{16\times 8}\right) + 2\left(\frac{1}{16\times 8}\right) + 2\left(\frac{1}{16\times 8}\right) + \frac{1}{16\times 8}\\
=&12\left(\frac{1}{16\times 8}\right)\\
=&\frac{3}{32}.
\end{aligned}
$$

$$
\begin{aligned}
N_eGA_4(G_1) = &\sum_{uv\in E(G_1)} \frac{2\sqrt{D_uE_uD_vE_v}}{D_uE_u + D_vE_v}\\
=&\frac{2\sqrt{D_{v_1}E_{v_1}D_{v_2}E_{v_2}}}{D_{v_1}E_{v_1} + D_{v_2}E_{v_2}} + \frac{2\sqrt{D_{v_1}E_{v_1}D_{v_3}E_{v_3}}}{D_{v_1}E_{v_1} + D_{v_3}E_{v_3}}\\
+&\frac{2\sqrt{D_{v_1}E_{v_1}D_{v_5}E_{v_5}}}{D_{v_1}E_{v_1} + D_{v_5}E_{v_5}} + \frac{2\sqrt{D_{v_1}E_{v_1}D_{v_6}E_{v_6}}}{D_{v_1}E_{v_1} + D_{v_6}E_{v_6}}\\
+&\frac{2\sqrt{D_{v_2}E_{v_2}D_{v_3}E_{v_3}}}{D_{v_2}E_{v_2} + D_{v_3}E_{v_3}} + \frac{2\sqrt{D_{v_2}E_{v_2}D_{v_4}E_{v_4}}}{D_{v_2}E_{v_2} + D_{v_4}E_{v_4}} + \frac{2\sqrt{D_{v_2}E_{v_2}D_{v_6}E_{v_6}}}{D_{v_2}E_{v_2} + D_{v_6}E_{v_6}}\\
+&\frac{2\sqrt{D_{v_3}E_{v_3}D_{v_4}E_{v_4}}}{D_{v_3}E_{v_3} + D_{v_4}E_{v_4}} + \frac{2\sqrt{D_{v_3}E_{v_3}D_{v_5}E_{v_5}}}{D_{v_3}E_{v_3} + D_{v_5}E_{v_5}}\\
+&\frac{2\sqrt{D_{v_4}E_{v_4}D_{v_5}E_{v_5}}}{D_{v_4}E_{v_4} + D_{v_5}E_{v_5}} + \frac{2\sqrt{D_{v_4}E_{v_4}D_{v_6}E_{v_6}}}{D_{v_4}E_{v_4} + D_{v_6}E_{v_6}}\\
+&\frac{2\sqrt{D_{v_5}E_{v_5}D_{v_6}E_{v_6}}}{D_{v_5}E_{v_5} + D_{v_6}E_{v_6}}\\
=&4\left(\frac{2\sqrt{(16\times 8)^2}}{16\times 8}\right) + 3\left(\frac{2\sqrt{(16\times 8)^2}}{16\times 8}\right)\\
+&2\left(\frac{2\sqrt{(16\times 8)^2}}{16\times 8}\right) + 2\left(\frac{2\sqrt{(16\times 8)^2}}{16\times 8}\right)\\
+&\frac{2\sqrt{(16\times 8)^2}}{16\times 8}\\
=&24.
\end{aligned}
$$

$$
\begin{aligned}
N_eABC_5(G_1) = &\sum_{uv\in E(G_1)} \sqrt{\frac{D_uE_u + D_vE_v - 2}{D_uE_uD_vE_v}}\\
=&\sqrt{\frac{D_{v_1}E_{v_1} + D_{v_2}E_{v_2} - 2}{D_{v_1}E_{v_1}D_{v_2}E_{v_2}}} + \sqrt{\frac{D_{v_1}E_{v_1} + D_{v_3}E_{v_3} - 2}{D_{v_1}E_{v_1}D_{v_3}E_{v_3}}}\\
+&\sqrt{\frac{D_{v_1}E_{v_1} + D_{v_5}E_{v_5} - 2}{D_{v_1}E_{v_1}D_{v_5}E_{v_5}}}\\
+&\sqrt{\frac{D_{v_1}E_{v_1} + D_{v_6}E_{v_6} - 2}{D_{v_1}E_{v_1}D_{v_6}E_{v_6}}}\\
+&\sqrt{\frac{D_{v_2}E_{v_2} + D_{v_3}E_{v_3} - 2}{D_{v_2}E_{v_2}D_{v_3}E_{v_3}}} + \sqrt{\frac{D_{v_2}E_{v_2} + D_{v_4}E_{v_4} - 2}{D_{v_2}E_{v_2}D_{v_4}E_{v_4}}}\\
+&\sqrt{\frac{D_{v_2}E_{v_2} + D_{v_6}E_{v_6} - 2}{D_{v_2}E_{v_2}D_{v_6}E_{v_6}}}\\
+&\sqrt{\frac{D_{v_3}E_{v_3} + D_{v_4}E_{v_4} - 2}{D_{v_3}E_{v_3}D_{v_4}E_{v_4}}} + \sqrt{\frac{D_{v_3}E_{v_3} + D_{v_5}E_{v_5} - 2}{D_{v_3}E_{v_3}D_{v_5}E_{v_5}}}\\
+&\sqrt{\frac{D_{v_4}E_{v_4} + D_{v_5}E_{v_5} - 2}{D_{v_4}E_{v_4}D_{v_5}E_{v_5}}} + \sqrt{\frac{D_{v_4}E_{v_4} + D_{v_6}E_{v_6} - 2}{D_{v_4}E_{v_4}D_{v_6}E_{v_6}}}\\
+&\sqrt{\frac{D_{v_5}E_{v_5} + D_{v_6}E_{v_6} - 2}{D_{v_5}E_{v_5}D_{v_6}E_{v_6}}}\\
=&4\left(\sqrt{\frac{2(16\times 8) - 2}{(16\times 8)^2}}\right) + 3\left(\sqrt{\frac{2(16\times 8) - 2}{(16\times 8)^2}}\right)\\
+&2\left(\sqrt{\frac{2(16\times 8) - 2}{(16\times 8)^2}}\right)\\
+&2\left(\sqrt{\frac{2(16\times 8) - 2}{(16\times 8)^2}}\right) + \sqrt{\frac{2(16\times 8) - 2}{(16\times 8)^2}}\\
=&12\left(\sqrt{\frac{2(16\times 8) - 2}{(16\times 8)^2}}\right)\\
=&\frac{3\sqrt{254}}{32}.
\end{aligned}
$$

$$
\begin{aligned}
N_eR(G_1) &= \sum_{uv\in E(G_1)} \frac{1}{\sqrt{D_uE_uD_vE_v}}\\
&=\frac{1}{\sqrt{D_{v_1}E_{v_1}D_{v_2}E_{v_2}}}+\frac{1}{\sqrt{D_{v_1}E_{v_1}D_{v_3}E_{v_3}}}\\
&+\frac{1}{\sqrt{D_{v_1}E_{v_1}D_{v_5}E_{v_5}}}+\frac{1}{\sqrt{D_{v_1}E_{v_1}D_{v_6}E_{v_6}}}\\
&+\frac{1}{\sqrt{D_{v_2}E_{v_2}D_{v_3}E_{v_3}}}+\frac{1}{\sqrt{D_{v_2}E_{v_2}D_{v_4}E_{v_4}}}+\frac{1}{\sqrt{D_{v_2}E_{v_2}D_{v_6}E_{v_6}}}\\
&+\frac{1}{\sqrt{D_{v_3}E_{v_3}D_{v_4}E_{v_4}}}+\frac{1}{\sqrt{D_{v_3}E_{v_3}D_{v_5}E_{v_5}}}\\
&+\frac{1}{\sqrt{D_{v_4}E_{v_4}D_{v_5}E_{v_5}}}+\frac{1}{\sqrt{D_{v_4}E_{v_4}D_{v_6}E_{v_6}}}\\
&+\frac{1}{\sqrt{D_{v_5}E_{v_5}D_{v_6}E_{v_6}}}\\
&=4\left(\frac{1}{\sqrt{(16\times 8)^2}}\right)+3\left(\frac{1}{\sqrt{(16\times 8)^2}}\right)+2\left(\frac{1}{\sqrt{(16\times 8)^2}}\right)\\
&+2\left(\frac{1}{\sqrt{(16\times 8)^2}}\right)+\frac{1}{\sqrt{(16\times 8)^2}}\\
&=12\left(\frac{1}{\sqrt{(16\times 8)^2}}\right)\\
&=\frac{3}{32}.
\end{aligned}
$$

**Theorem 2.1.** *Let $G = K_n$, then*

*(i)* $N_e\xi^c(K_n) = n(n-1)^3$.

*(ii)* $N_eE_1(K_n) = n(n-1)^6$.

*(iii)* $N_eE_2(K_n) = \frac{n(n-1)^7}{2}$.

*(iv)* $N_e\xi^{ac}(K_n) = n(n-1)^4$.

*(v)* $N_eH_e(K_n) = \frac{n}{2(n-1)^2}$.

*(vi)* $N_eGA_4(K_n) = \frac{n(n-1)}{2}$.

*(vii)* $N_eABC_5(K_n) = \frac{n\sqrt{(n-1)^3-1}}{\sqrt{2}(n-1)^2}$.

*(viii)* $N_eR(K_n) = \frac{n}{2(n-1)^2}$.

**Theorem 2.2.** *Let $G = K_{m,n}$ be the complete bipartite graph, then*

*(i)* $N_e\xi^c(K_{m,n}) = 4m^2n^2$.

*(ii)* $N_eE_1(K_{m,n}) = 4m^3n^3(m+n)$.

*(iii)* $N_eE_2(K_{m,n}) = 4m^4n^4$.

*(iv)* $N_e\xi^{ac}(K_{m,n}) = 2m^2n^2(m+n)$.

*(v)* $N_eH_e(K_{m,n}) = \frac{1}{(m+n)}$.

*(vi)* $N_eGA_4(K_{m,n}) = \frac{2mn\sqrt{mn}}{(m+n)}$.

*(vii)* $N_eABC_5(K_{m,n}) = \sqrt{\frac{n^2m+m^2n-1}{2mn}}$.

*(viii)* $N_eR(K_{m,n}) = \frac{1}{2\sqrt{mn}}$.

**Theorem 2.3.** *Let $G = K_{1,n}$ be the star graph, then*

*(i)* $N_e\xi^c(K_{1,n}) = 3n^2$.

*(ii)* $N_eE_1(K_{1,n}) = n^3(1 + 4n)$.

*(iii)* $N_eE_2(K_{1,n}) = 2n^4$.

*(iv)* $N_e\xi^{ac}(K_{1,n}) = n^2(2n + 1)$.

*(v)* $N_eH_e(K_{1,n}) = \frac{2}{2n+1}$.

*(vi)* $N_eGA_4(K_{1,n}) = \frac{2\sqrt{2n^3}}{2n+1}$.

*(vii)* $N_eABC_5(K_{1,n}) = \sqrt{\frac{2n^2+n-2}{2n}}$.

*(viii)* $N_eR(K_{1,n}) = \frac{1}{\sqrt{2n}}$.

**Theorem 2.4.** *Let $G = C_n$ be the cycle graph, then*

*(i)* $N_e\xi^c(C_n) = \begin{cases} 4n^2, & \text{if } n \text{ is even} \\ 4n(n-1), & \text{if } n \text{ is odd.} \end{cases}$

*(ii)* $N_eE_1(C_n) = \begin{cases} 16n^3, & \text{if } n \text{ is even} \\ 16n(n-1)^2, & \text{if } n \text{ is odd.} \end{cases}$

*(iii)* $N_eE_2(C_n) = \begin{cases} 16n^3, & \text{if } n \text{ is even} \\ 16n(n-1)^2, & \text{if } n \text{ is odd.} \end{cases}$

*(iv)* $N_e\xi^{ac}(C_n) = \begin{cases} 8n^2, & \text{if } n \text{ is even} \\ 8n(n-1), & \text{if } n \text{ is odd.} \end{cases}$

*(v)* $N_eH_e(C_n) = \begin{cases} \frac{1}{4}, & \text{if } n \text{ is even} \\ \frac{n}{4(n-1)}, & \text{if } n \text{ is odd.} \end{cases}$

*(vi)* $N_eGA_4(C_n) = \begin{cases} n, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd.} \end{cases}$

*(vii)* $N_eABC_5(C_n) = \begin{cases} \frac{\sqrt{8n-2}}{4}, & \text{if } n \text{ is even} \\ \frac{n}{2(n-1)}\sqrt{\frac{4n-5}{2}}, & \text{if } n \text{ is odd.} \end{cases}$

*(viii)* $N_eR(C_n) = \begin{cases} \frac{1}{4}, & \text{if } n \text{ is even} \\ \frac{n}{4(n-1)}, & \text{if } n \text{ is odd.} \end{cases}$

**Theorem 2.5.** *Let $G = P_n$, then*

*(i)* $N_e\xi^c(P_n) = \begin{cases} 18n - 12 + 8\sum\limits_{i=3}^{\frac{n-2}{2}}(2n-2i), & \text{if } n \text{ is even} \\ 20n - 28 + 8\sum\limits_{i=3}^{\frac{n-1}{2}}(2n-2i), & \text{if } n \text{ is odd.} \end{cases}$

*(ii)* $N_eE_1(P_n) = \begin{cases} 2\left(56n^2 - 128n + 176\right) + 32\sum\limits_{i=3}^{\frac{n-2}{2}}(2n-2i)^2, & \text{if } n \text{ is even} \\ 96n^2 - 288n + 336 + 32\sum\limits_{i=3}^{\frac{n-1}{2}}(2n-2i)^2, & \text{if } n \text{ is odd.} \end{cases}$

$$(iii)\ N_eE_2(P_n) = \begin{cases} 168n^2 - 448n + 752 + 32\sum_{i=3}^{\frac{n-4}{2}}(2n-2i)(2n-2i-2), & \text{if } n \text{ is even} \\ 12\left(2n^2 - 8n + 8\right) + 24(4n^2 - 20n + 24) + 32(n+1)^2 \\ +\sum_{i=3}^{\frac{n-3}{2}}(2n-2i)(2n-2i-2), & \text{if } n \text{ is odd.} \end{cases}$$

$$(iv)\ N_e\xi^{ac}(P_n) = \begin{cases} 44n - 40 + 16\sum_{i=3}^{\frac{n-2}{2}}(2n-2i), & \text{if } n \text{ is even} \\ 36n - 48 + 16\sum_{i=3}^{\frac{n-1}{2}}(2n-2i), & \text{if } n \text{ is odd.} \end{cases}$$

$$(v)\ N_eH_e(P_n) = \begin{cases} \frac{4}{8n-16} + \frac{4}{14n-36} + \frac{4}{8n+12} + \frac{2}{8n+8} \\ +\sum_{i=3}^{\frac{n-4}{2}} \frac{1}{(2n-2i)+(2n-2i-2)}, & \text{if } n \text{ is even} \\ \frac{4}{8n-16} + \frac{4}{14n-36} + \frac{1}{2(n+1)} + \sum_{i=3}^{\frac{n-3}{2}} \frac{1}{(2n-2i)+(2n-2i-2)}, & \text{if } n \text{ is odd.} \end{cases}$$

$$(vi)\ N_eGA_4(P_n) = \begin{cases} \frac{4\sqrt{6(n-2)(2n-4)}}{8n-16} + \frac{8\sqrt{3(2n-4)(2n-6)}}{14n-36} + \frac{4\sqrt{(n+2)(n+1)}}{2n+3} + 1 \\ +16\sum_{i=3}^{\frac{n-4}{2}} \frac{\sqrt{(2n-2i)(2n-2i-2)}}{(2n-2i)+(2n-2i-2)}, & \text{if } n \text{ is even} \\ \frac{4\sqrt{6(n-2)(2n-4)}}{8n-16} + \frac{8\sqrt{3(2n-4)(2n-6)}}{14n-36} + 2 \\ +4\sum_{i=3}^{\frac{n-3}{2}} \frac{\sqrt{(2n-2i)(2n-2i-2)}}{(2n-2i)+(2n-2i-2)}, & \text{if } n \text{ is odd.} \end{cases}$$

$$(vii)\ N_eABC_5(P_n) = \begin{cases} 2\sqrt{\frac{8n-18}{6(n-2)(2n-4)}} + \sqrt{\frac{14n-38}{3(2n-4)(2n-6)}} \\ +2\sqrt{\frac{4(n+2)+4(n+1)-2}{4(n+2)4(n+1)}} + \frac{\sqrt{8(n+1)-2}}{4(n+1)} \\ +2\sum_{i=3}^{\frac{n-4}{2}} \sqrt{\frac{4(2n-2i)+4(2n-2i-2)-2}{16(2n-2i)(2n-2i-2)}}, & \text{if } n \text{ is even} \\ 2\sqrt{\frac{8n-18}{6(n-2)(2n-4)}} + \sqrt{\frac{7n-16}{3(2n-4)(2n-6)}} + \frac{\sqrt{8n+6}}{2(n+1)} \\ +\frac{1}{2}\sum_{i=3}^{\frac{n-3}{2}} \sqrt{\frac{4(2n-2i)+4(2n-2i-2)-2}{(2n-2i)(2n-2i-2)}}, & \text{if } n \text{ is odd.} \end{cases}$$

$$(viii)\ N_eR(P_n) = \begin{cases} \frac{2}{\sqrt{12n^2-48n+48}} + \frac{2}{\sqrt{48n^2-240n+288}} + \frac{1}{2\sqrt{n^2+3n+2}} + \frac{1}{4(n+1)} \\ +\sum_{i=3}^{\frac{n-4}{2}} \frac{1}{\sqrt{(2n-2i)(2n-2i-2)}}, & \text{if } n \text{ is even} \\ \frac{2}{\sqrt{6(n-2)(2n-4)}} + \frac{1}{\sqrt{3(2n-4)(2n-6)}} + \frac{1}{2(n+1)} \\ +\frac{1}{2}\sum_{i=3}^{\frac{n-3}{2}} \frac{1}{\sqrt{(2n-2i)(2n-2i-2)}}, & \text{if } n \text{ is odd.} \end{cases}$$

.

**Theorem 2.6.** *Let* $G = W_n (n \geq 5)$*, then*

*(i)* $N_e\xi^c(W_n) = 6(n-1)^2 + 5(n^2 + 4n - 5)$.

*(ii)* $N_eE_1(W_n) = (n-1)\left(36n^3 - 83n^2 + 358n + 589\right)$.

*(iii)* $N_eE_2(W_n) = (n-1)\left(30n^3 + 115n^2 - 20n + 775\right)$.

*(iv)* $N_e\xi^{ac}(W_n) = (n-1)(6n^2 + 3n + 81)$.

*(v)* $N_eH_e(W_n) = 2(n-1)\left(\frac{6n^2+3n+81}{60n^3+230n^2-40n+1550}\right)$

*(vi)* $N_eGA_4(W_n) = (n-1)\left(\frac{2\sqrt{30n^3+90n^2-270n+150}}{6n^2-7n+31} + 1\right)$.

*(vii)* $N_eABC_5(W_n) = (n-1)\sqrt{\frac{6n^2-7n+29}{30n^3+90n^2-270n+150}} + \frac{(n-1)\sqrt{48+10n}}{25+5n}$.

*(viii)* $N_eR(W_n) = (n-1)\left(\frac{1}{\sqrt{30n^3+90n^2-270n+150}} + \frac{1}{25+5n}\right)$.

**Remark 2.7.** (i) If $G$ is regular graph then $D_u = D_v$ and we denote $max(D_u) = D$.

(ii) If $G$ is a regular equieccentric graph then $D_uE_u = D_vE_v$ and $D_uE_uD_vE_v(D_uE_u+D_vE_v-2) = 2D^2E_u^2(DE_u-1)$.

**Theorem 2.8.** *Let $G$ be a regular equieccentric graph with $m$ edges then, $N_eGA_4(G) = m$.*

*Proof.* We have, $N_eGA_4(G) = \sum_{uv\in E(G)} \frac{2\sqrt{D_uE_uD_vE_v}}{D_uE_u + D_vE_v}$. From Remark 2.7(ii),

$$N_eGA_4(G) = \sum_{uv\in E(G)} \frac{2\sqrt{(D_uE_u)^2}}{2(D_uE_u)} = m.$$

□

## 3 Bounds

**Theorem 3.1.** *Let $G$ be a regular equieccentric graph with $m$ edges then, $4m^3 \leq DN_eGA_4(G) \times N_eR(G) \times N_e\xi^{ac}(G)$.*

*Proof.* We know that, $N_eGA_4(G) = \sum_{uv\in E(G)} \frac{2\sqrt{D_uE_uD_vE_v}}{D_uE_u + D_vE_v}$,
$N_eR(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{D_uE_uD_vE_v}}$, and $N_e\xi^{ac}(G) = \sum_{uv\in E(G)} (D_uE_u + D_vE_v)$.
If $a = \sqrt{D_uE_uD_vE_v}$ and $b = \frac{D_uE_u+D_vE_v}{2}$ then,

$$\sum_{uv\in E(G)} 1 = \sum_{uv\in E(G)} \left(\sqrt{\frac{a}{b}} \times \frac{1}{\sqrt{\frac{a}{b}}}\right).$$

Squaring and applying Lemma1.1 and Lemma1.2, we have

$$\left(\sum_{uv\in E(G)} 1\right)^2 \leq \sum_{uv\in E(G)} \left(\sqrt{\frac{a}{b}}\right)^2 \times \sum_{uv\in E(G)} \left(\frac{1}{\sqrt{\frac{a}{b}}}\right)^2$$

$$m^2 \leq \sum_{uv\in E(G)} \frac{2\sqrt{D_uE_uD_vE_v}}{D_uE_u + D_vE_v} \times \sum_{uv\in E(G)} \frac{\frac{D_uE_u+D_vE_v}{2}}{\sqrt{D_uE_uD_vE_v}}$$

$$\leq N_eGA_4(G) \times \sum_{uv\in E(G)} \frac{D_uE_u + D_vE_v}{2\sqrt{D_uE_uD_vE_v}}$$

$$
\begin{aligned}
2m^2 \leq & N_eGA_4(G) \times \sum_{uv\in E(G)} \frac{1}{\sqrt{D_uE_uD_vE_v}} \sum_{uv\in E(G)} (D_uE_u + D_vE_v) \\
2m^2 \leq & N_eGA_4(G) \times \left(\frac{D}{2m}\right) N_eR(G) \times N_e\xi^{ac}(G) \\
4m^3 \leq & DN_eGA_4(G) \times N_eR(G) \times N_e\xi^{ac}(G).
\end{aligned}
$$

□

**Theorem 3.2.** *Let G be a regular equieccentric graph then,*

$$2m(N_eGA_4(G))^2 \leq D(N_eH_e(G))^2 N_eE_2(G)$$

.

*Proof.* We know that, $N_eGA_4(G) = \sum_{uv\in E(G)} \frac{2\sqrt{D_uE_uD_vE_v}}{D_uE_u + D_vE_v}$,

$$N_eH_e(G) = \sum_{uv\in E(G)} \frac{2}{D_uE_u + D_vE_v}$$

and $N_eE_2(G) = \sum_{uv\in E(G)} D_uE_uD_vE_v$. Let $a = \frac{2}{D_uE_u+D_vE_v}$ and $b = \sqrt{D_uE_uD_vE_v}$, by squaring and applying Lemma1.1 and Lemma1.2, we have,

$$
\begin{aligned}
(N_eGA_4(G))^2 = & \left(\sum_{uv\in E(G)} \frac{2}{D_uE_u + D_vE_v} \sqrt{D_uE_uD_vE_v}\right)^2 \\
\leq & \sum_{uv\in E(G)} \left(\frac{2}{D_uE_u + D_vE_v}\right)^2 \sum_{uv\in E(G)} \left(\sqrt{D_uE_uD_vE_v}\right)^2 \\
(N_eGA_4(G))^2 = & \sum_{uv\in E(G)} \left(\frac{2}{D_uE_u + D_vE_v}\right)^2 \sum_{uv\in E(G)} (D_uE_uD_vE_v) \\
(N_eGA_4(G))^2 \leq & \frac{D}{2m}(N_eH_e(G))^2 N_eE_2(G) \\
2m(N_eGA_4(G))^2 \leq & D(N_eH_e(G))^2 N_eE_2(G).
\end{aligned}
$$

□

**Theorem 3.3.** *Let G be a regular equieccentric graph then,* $D_uE_u + D_vE_v - 2 \geq \frac{2mD^2E_u^2(DE_u-1)}{N_eE_2(G)}$.

*Proof.* From Remark 2.7(ii), we have

$$D_uE_u + D_vE_v - 2 = \frac{2D^2E_u^2(DE_u - 1)}{D_uE_uD_vE_v}$$

$$\sum_{uv\in E(G)} D_uE_u + D_vE_v - 2 = \sum_{uv\in E(G)} \frac{2D^2E_u^2(DE_u - 1)}{D_uE_uD_vE_v}.$$

Using Lemma1.3,

$$
\begin{aligned}
\sum_{uv\in E(G)} \frac{2D^2E_u^2(DE_u - 1) \times 1}{D_uE_uD_vE_v} \geq & \frac{2 \sum_{uv\in E(G)} D^2E_u^2(DE_u - 1) \times \sum_{uv\in E(G)} 1}{\sum_{uv\in E(G)} D_uE_uD_vE_v} \\
\geq & \frac{2m^2D^2E_u^2(DE_u - 1)}{N_eE_2(G)}.
\end{aligned}
$$

Therefore,

$$D_uE_u + D_vE_v - 2 \geq \frac{2mD^2E_u^2(DE_u - 1)}{N_eE_2(G)}.$$

□

**Theorem 3.4.** *Let $G$ be a regular equieccentric graph then* $N_eABC_5(G) \leq m\sqrt{\frac{2m(DE_u-1)}{N_eE_2(G)}}$.

*Proof.* We Know that, $N_eABC_5(G) = \sum_{uv\in E(G)} \sqrt{\frac{D_uE_u + D_vE_v - 2}{D_uE_uD_vE_v}} \longrightarrow (*)$.
Squaring $(*)$ and applying Lemma 1.1, we have

$$\left(\sum_{uv\in E(G)} \sqrt{\frac{D_uE_u + D_vE_v - 2}{D_uE_uD_vE_v}}\right)^2 \leq \sum_{uv\in E(G)} \left(\sqrt{D_uE_u + D_vE_v - 2}\right)^2 \sum_{uv\in E(G)} \left(\frac{1}{\sqrt{D_uE_uD_vE_v}}\right)^2$$

$$\leq \sum_{uv\in E(G)} D_uE_u + D_vE_v - 2 \sum_{uv\in E(G)} \left(\frac{1}{\sqrt{D_uE_uD_vE_v}}\right)^2.$$

From Remark 2.7(ii) and Lemma 1.3. Since $G$ is regular equieccentric, $D = D_u = D_v$ and $E_u = E_v$, we have

$$\left(\sum_{uv\in E(G)} \sqrt{\frac{D_uE_u + D_vE_v - 2}{D_uE_uD_vE_v}}\right)^2 \leq \frac{2m^2D^2E_u^2(DE_u - 1)}{N_eE_2(G)} \times \frac{m}{D^2E_u^2}$$

$$\sum_{uv\in E(G)} \sqrt{\frac{D_uE_u + D_vE_v - 2}{D_uE_uD_vE_v}} \leq m\sqrt{\frac{2m(DE_u - 1)}{N_eE_2(G)}}$$

$$N_eABC_5(G) \leq m\sqrt{\frac{2m(DE_u - 1)}{N_eE_2(G)}}.$$

□

**Remark 3.5.** We observe some of the following bounds :

(i) $N_e\xi^c(G) \geq \xi^c(G)$. Equality holds if and only if $G = K_2$.

(ii) $\xi^c(G) \leq \xi^{ac}(G)$. Equality holds if and only if $G = K_2$.

(iii) $\xi^c(G) \leq \xi^{ac}(G) \leq N_e\xi^{ac}(G)$. Equality holds if and only if $G = K_2$.

(iv) If $D_uE_u = D_vE_v = t$, for every $u, v \in V(G)$. Then, $N_eH_e(G) = N_eR(G)$ and if $G$ is a unicyclic graph then, $N_eE_1(G) = N_eE_2(G) > N_eGA_4(G)$.
For $t = 1$ in Remark 3.5(iv), $N_eABC_5(G) < l$ and $N_eABC_5(G) < N_e\xi^{ac}(G)$.
If $G$ is a unicyclic graph then, $N_e\xi^{ac}(G) > l$ and $N_eE_1(G) = N_eE_2(G) = N_eH_e(G) = N_eGA_4(G) = N_eR(G) = l$.
For $t > 1$ in Remark 3.5(iv), $N_e\xi^{ac}(G) < N_eE_2(G)$ and if $G$ is a unicyclic graph then, $N_e\xi^{ac}(G) > N_eE_1(G)$, $N_e\xi^{ac}(G) > N_eH_e(G)$ and $N_e\xi^{ac}(G) > N_eR(G)$.

## 4 Conclusion remarks

In this paper, we have proposed eight new topological indices based on neighborhood concept and we have formulated the neighborhood degree eccentricity of some standard graphs. Further, we have obtained various results and bounds for the neighborhood degree eccentricity index of a graph.

## References

[1] Z. Cvetkovski, *Inequalities Theorems, Techniques and Selected Problems*, Springer : Berlin/Heidelberg, Germany, **(107)**, (2010).

[2] K. C. Das, D.-W. Lee, A. Graovac, *Some properties of the Zagreb eccentricity indices*, Ars Mathematica Contemporanea, **6**, 117-125,(2013),(2013).

[3] F. Harary, *Graph theory*, Addison Wesley, Reading Mass, (1969).

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc, 1998.

[5] V. R. Kulli, *Graph Theory*, Vishwa International Publications, Gulbarga, India, (2012).

[6] V. Mathad and S. S. Mahde, *The minimum hub energy of a graph*, Palestine Journal of Mathematics, **6(1)**, 247-256, (2017).

[7] V. Mathad, P. Padmapriya and M. A. Sangamesha, *Some topological indices of phosphorus containing dendrimers*, Palestine Journal of Mathematics, **10**, 151-160, (2021).

[8] V. Mathad, H. N. Sujatha and Puneeth S., *Amplified Eccentric Connectivity Index of Graphs*, TWMS J. App. and Eng. Math., **12(4)**, 1469-1479, (2022).

[9] S. Mondal, N. De and A. Pal, *On Some New Neighborhood Degree - Based Indices for Some Oxide and Silicate Networks*, Multidisciplinary Scientific Journal, **2**, 384-409,(2019).

[10] S. Mondal, N. De and A. Pal, QSPR Analysis of Some Novel Neighbourhood Degree-Based Topological Descriptors, *Complex Intell. Syst*, **7**, 977-996,(2021).

[11] V. Sharma, R. Goswami and A. K. Madan, *Eccentric Connectivity Index : A Novel Highly Discriminating Topological Descriptor for Structure-Property and Structure-Activity Studies*, J. Chem. Inf. Comput. Sci., **37**, 273-282,(1997).

[12] H. Wiener, *Structural determination of paraffin boiling points*, J. Amer. Chem. Soc., **69**, 17-20, (1947).

## Author information

Veena Mathad, Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru - 570 006, India.
E-mail: veena_mathad@rediffmail.com

Parvathi, Assistant Professor, Shah Sogmal Peeraji Oswal Government First Grade College, Muddebihal-586212, India.
E-mail: parvathi3403@gmail.com

Received: 2023-07-29
Accepted: 2026-02-12