Palestine Journal of Mathematics

Vol 15(2)(2026) , 203–224

HARDY-TYPE INEQUALITIES WITH NEGATIVE EXPONENTS IN NABLA CALCULUS ON TIME SCALES

Suket Kumar and Rohit Chauhan

Communicated by: Jaganmohan J.

MSC 2020 Classifications: 26D10; 26D15; 39A10; 34N05; 39A12; 47B38.

Keywords and phrases:Time scales, Dynamic inequality, Nabla integral, Hardy's inequality, Dynamic Hölder's inequality

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Corresponding Author: S. Kumar

Abstract. In this work, we employ the framework of nabla calculus on time scales to develop new dynamic Hardy-type inequalities involving negative exponents. Specifically, we extend Hardy-type inequalities for the exponent $\gamma < 0$ by applying the nabla chain rule, integration by parts, and the nabla version of the reverse Hölder's inequality on time scales. When the time scale $\mathbb{T} = \mathbb{R}$ represents the set of real numbers, our results reduce to well-known integral inequalities previously introduced by Azzouz. In contrast, when the time scale $\mathbb{T} = \mathbb{N}_0$ represents the set of non-negative integers, we derive new discrete inequalities. Furthermore, we present inequalities for other time scales, such as $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, which constitute largely novel contributions.

1 Introduction

Hardy-type inequalities play a fundamental role in mathematical analysis, with numerous applications in differential equations, functional analysis, and mathematical physics. Since the classical Hardy inequality was introduced by G.H. Hardy in the early 20th century, various generalizations and extensions have been developed, including both integral and discrete versions. To unify these cases, time scales calculus offers a framework that integrates differential and difference calculus, making it possible to extend Hardy-type inequalities to dynamic systems on hybrid domains.

In 1920, Hardy [1] proved the following result:

Theorem 1.1. *Let* $\{\lambda_k\}_{k=1}^{\infty}$ *be a sequence of non-negative real numbers. If* $\gamma > 1$, $\{\lambda_k\}_{k=1}^{\infty} \geq 0$ *for all* $k \geq 1$, *and* $\sum_{k=1}^{\infty} \lambda_k^{\gamma} < \infty$, *then*

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^{\gamma} \leq \left(\frac{\gamma}{\gamma - 1} \right)^{\gamma} \sum_{k=1}^{\infty} \lambda_k^{\gamma}. \tag{1.1}$$

In 1925, Hardy [2] also presented the continuous analog of inequality (1.1) as follows:

Theorem 1.2. *If* $\gamma > 1$ *and* $\omega(\zeta) \geq 0)$ *be integrable functions over the finite interval* $(0, \zeta)$ *for every positive* ζ, *then*

$$\int_0^{\infty} \left(\frac{\Omega(\zeta)}{\zeta} \right)^{\gamma} d\zeta \leq \left(\frac{\gamma}{\gamma - 1} \right)^{\gamma} \int_0^{\infty} \omega^{\gamma}(\zeta)\, d\zeta, \tag{1.2}$$

where
$\Omega(\zeta) = \int_0^{\zeta} \omega(\bar{\zeta})\, d\bar{\zeta}$, *for all* $\zeta > 0$.
Furthermore, the constant $\left(\frac{\gamma}{\gamma-1}\right)^{\gamma}$ *in inequalities (1.1) and (1.2) is the best possible.*

In 1927, Hardy and Littlewood [3] established the reverse inequality of (1.2) for $0 < \gamma < 1$, on the condition that the integral $\Omega(\zeta) = \int_0^{\zeta} \omega(\overline{\zeta})\, d\overline{\zeta}$ is replaced with $\Omega(\zeta) = \int_{\zeta}^{\infty} \omega(\overline{\zeta})\, d\overline{\zeta}$.

Theorem 1.3. *Let $\omega(\zeta)$ consisting of non-negative integrable function on $[0, \infty)$. If $0 < \gamma < 1$, then*

$$\int_0^{\infty} \left(\frac{\Omega(\zeta)}{\zeta}\right)^{\gamma} d\zeta \geq \left(\frac{\gamma}{\gamma - 1}\right)^{\gamma} \int_0^{\infty} \omega^{\gamma}(\zeta)\, d\zeta, \tag{1.3}$$

where
$\Omega(\zeta) = \int_{\zeta}^{\infty} \omega(\overline{\zeta})\, d\overline{\zeta}$, *for all* $\zeta > 0$.

In 2007, Yang [8] established the Hardy-type integral inequalities for negative exponent $\gamma < 0$ as follows:

Theorem 1.4. *If $\gamma < 0$, $\omega(\overline{\zeta}) \geq 0$ and $0 < \int_0^{\infty} \zeta^{(\gamma - r)} \omega^{\gamma}(\zeta)\, d(\zeta) < \infty$, then*

(i) For $\gamma < 0$ and $r > 1$,

$$\int_0^{\infty} \frac{1}{\zeta^r} \left(\Omega^{\gamma}(\zeta)\right) d\zeta \leq \left(\frac{\gamma}{1 - r}\right)^{\gamma} \int_0^{\infty} \zeta^{(\gamma - r)} \omega^{\gamma}(\zeta)\, d\zeta, \tag{1.4}$$

where

$\Omega(\zeta) = \int_{\zeta}^{\infty} \omega(\overline{\zeta})\, d\overline{\zeta}$, *for all* $\zeta > 0$.

(ii) For $\gamma < 0$ and $r < 1$,

$$\int_0^{\infty} \frac{1}{\zeta^r} \left(\Omega^{\gamma}(\zeta)\right) d\zeta \leq \left(\frac{\gamma}{r - 1}\right)^{\gamma} \int_0^{\infty} \zeta^{(\gamma - r)} \omega^{\gamma}(\zeta)\, d\zeta, \tag{1.5}$$

where
$\Omega(\zeta) = \int_0^{\zeta} \omega(\overline{\zeta})\, d\overline{\zeta}$, *for all* $\zeta > 0$.

Furthermore, the constants $\left(\frac{\gamma}{1-r}\right)^{\gamma}$ and $\left(\frac{\gamma}{r-1}\right)^{\gamma}$ in inequalities (1.4) and (1.5) are the best possible.

In 2020, Benaissa et al. [12] generalized the inequalities (1.4) and (1.5) by introducing a monotone weighted function $\left(\frac{\zeta}{\omega_2(\zeta)}\right)$ as given by

Theorem 1.5. *Let $\gamma < 0$, $\omega_1, \omega_2 > 0$ and $0 < \int_0^{\infty} \omega_2{}^{-r}(\zeta) \left(\zeta\, \omega_1(\zeta)\right)^{\gamma} d\zeta < \infty$, then*

(i) If $r > 1$ and $\frac{\zeta}{\omega_2(\zeta)}$ is non-decreasing function,

$$\int_0^{\infty} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)}\, d\zeta \leq \left(\frac{\gamma}{1 - r}\right)^{\gamma} \int_0^{\infty} \omega_2{}^{-r}(\zeta) \left(\zeta\, \omega_1(\zeta)\right)^{\gamma} d\zeta, \tag{1.6}$$

where

$\Omega(\zeta) = \int_{\zeta}^{\infty} \omega_1(\overline{\zeta})\, d\overline{\zeta}$, *for all* $\zeta > 0$.

(ii) If $0 \leq r < 1$ and $\frac{\zeta}{\omega_2(\zeta)}$ is non-increasing function,

$$\int_0^{\infty} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)}\, d\zeta \leq \left(\frac{\gamma}{r - 1}\right)^{\gamma} \int_0^{\infty} \omega_2{}^{-r}(\zeta) \left(\zeta\, \omega_1(\zeta)\right)^{\gamma} d\zeta, \tag{1.7}$$

where

$\Omega(\zeta) = \int_0^{\zeta} \omega_1(\overline{\zeta})\, d\overline{\zeta}$, *for all* $\zeta > 0$.

(iii) If $r < 0$ *and* $\frac{\zeta}{\omega_2(\zeta)}$ *is non-decreasing function,*

$$\int_0^\infty \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)}\, d\zeta \leq \left(\frac{\gamma}{r-1}\right)^\gamma \int_0^\infty \omega_2{}^{-r}(\zeta)\, (\zeta\, \omega_1(\zeta))^\gamma\, d\zeta, \tag{1.8}$$

where

$\Omega(\zeta) = \int_0^\zeta \omega_1(\bar{\zeta})\, d\bar{\zeta}$, *for all* $\zeta > 0$.

In 2023, Azzouz et al. [19] extended the inequalities (1.6), (1.7) and (1.8) by replacing $(0,\infty)$ with (τ_1,τ_2), where $0 \leq \tau_1 < \tau_2 \leq \infty$ as follows:

Theorem 1.6. *Let* $\gamma < 0$, $0 \leq \tau_1 < \tau_2 \leq \infty$ *and* ω_1, ω_2 *are positive measurable function on* (τ_1, τ_2)*, then*

(i) If $r > 1$ *and* $\frac{\zeta}{\omega_2(\zeta)}$ *is non-decreasing function,*

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)}\, d\zeta \leq \left(\frac{\gamma}{1-r}\right)^\gamma \int_{\tau_1}^{\tau_2} \omega_2{}^{-r}(\zeta) \left(1 - \left(\frac{\zeta}{\tau_2}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1} (\zeta\, \omega_1(\zeta))^\gamma\, d\zeta, \tag{1.9}$$

where

$\Omega(\zeta) = \int_\zeta^{\tau_2} \omega_1(\bar{\zeta})\, d\bar{\zeta}$, *for all* $\zeta > 0$.

(ii) If $0 \leq r < 1$ *and* $\frac{\zeta}{\omega_2(\zeta)}$ *is non-increasing function,*

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)}\, d\zeta \leq \left(\frac{\gamma}{r-1}\right)^\gamma \int_{\tau_1}^{\tau_2} \omega_2{}^{-r}(\zeta) \left(1 - \left(\frac{\tau_1}{\zeta}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} (\zeta\, \omega_1(\zeta))^\gamma\, d\zeta, \tag{1.10}$$

where

$\Omega(\zeta) = \int_{\tau_1}^\zeta \omega_1(\bar{\zeta})\, d\bar{\zeta}$, *for all* $\zeta > 0$.

(iii) If $r < 0$ *and* $\frac{\zeta}{\omega_2(\zeta)}$ *is non-decreasing function,*

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)}\, d\zeta \leq \left(\frac{\gamma}{r-1}\right)^\gamma \int_{\tau_1}^{\tau_2} \omega_2{}^{-r}(\zeta) \left(1 - \left(\frac{\tau_1}{\zeta}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} (\zeta\, \omega_1(\zeta))^\gamma\, d\zeta, \tag{1.11}$$

where

$\Omega(\zeta) = \int_{\tau_1}^\zeta \omega_1(\bar{\zeta})\, d\bar{\zeta}$, *for all* $\zeta > 0$.

Over the past two decades, the theory of time scales introduced to unify continuous and discrete analysis has become a powerful framework for extending classical inequalities to dynamic settings. In this direction, many authors have explored Hardy-type inequalities on time scales using delta and nabla calculus.

In 2005, Rehak [7] was the first to extend the classical Hardy inequality (1.2) within the framework of time scales. Moreover, Rehak showed that if $\gamma > 1$ and $\omega_1 \geq 0$ and the delta integral $\int_{\tau_1}^\zeta \omega_1(\bar{\zeta})\, \Delta\bar{\zeta}$ exists, then

$$\int_{\tau_1}^\infty \left(\frac{\Omega^\sigma(\zeta)}{\sigma(\zeta) - \tau_1}\right)^\gamma \Delta\zeta < \left(\frac{\gamma}{\gamma-1}\right)^\gamma \int_{\tau_1}^\infty \omega_1^\gamma(\zeta)\, \Delta\zeta, \tag{1.12}$$

where

$\Omega(\zeta) = \int_{\tau_1}^\zeta \omega_1(\bar{\zeta})\, \Delta\bar{\zeta}$, for $\zeta \in [\tau_1, \infty)$.

In particular, a number of studies have investigated Hardy-type inequalities using alternative approaches in time scale calculus, notably through the use of the delta integral [11, 14, 15, 18, 25, 26, 27] and the nabla integral [13, 16, 17, 22, 23, 24, 28]. These studies highlight the ongoing interest in dynamic Hardy-type inequalities and demonstrate the potential for further development in this area.

A comparative overview of Hardy-type inequalities on different time scales is presented in Table 1. This summary highlights the types of time scales, definitions, associated inequalities, and their applications.

Time Scale	**Definition**	**Hardy-Type Inequality**	**Applications**
Continuous Time	$\mathbb{R}$	Classical Hardy inequalities involving integrals and derivatives	Analysis of differential equations and continuous integral inequalities
Discrete Time	$\mathbb{Z}$	Discrete Hardy-type inequalities for sums and difference operators	Stability analysis and behavior of discrete dynamic systems
Time Scales	Arbitrary closed subsets of $\mathbb{R}$	Unified generalizations of Hardy-type inequalities for both continuous and discrete cases	Modeling hybrid systems with both continuous and discrete dynamics
Delta and Nabla Time Scales	Based on forward/backward jump operators σ, ρ	Extensions of Hardy-type inequalities using delta or nabla calculus	Systems with forward or backward evolving dynamics, e.g., population models or control systems
Mixed Time Scales	Combination of different time scales	Corresponding Hardy-type inequalities developed for mixed time scales	Systems with components evolving on different time scales

Table 1. Hardy-type inequalities on various time scales

While significant attention has been given to Hardy-type inequalities involving positive exponents, relatively fewer results exist for inequalities with negative exponents, particularly in the context of nabla calculus. Recent studies, such as Aly et al. [21], have partially addressed this gap by establishing certain inequalities with negative exponents within nabla calculus on time scales.

Motivated by these developments, the main objective of this paper is to develop new Hardy-type inequalities with negative exponents in the nabla calculus framework on time scales. Specifically, we aim to generalize the results of Azzouz et al. [19] by using the reverse Hölder's inequality, nabla integration by parts, and the nabla chain rule. Our findings yield dynamic versions of known inequalities when the time scale $\mathbb{T} = \mathbb{R}$, and yield new discrete results for $\mathbb{T} = \mathbb{N}_0$. We also provide new inequalities on quantum time scales such as $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$.

The remainder of the paper is structured as follows. In Section 2, we present the necessary definitions, theorems, and lemmas related to nabla calculus on time scales. Section 3 establishes our main results and derives corresponding corollaries for specific time scales, including $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}_0$, and $\mathbb{T} = q^{\mathbb{N}_0}$.

2 Preliminaries and basic Lemmas

In this section, we introduce the fundamental results and lemmas of nabla calculus. For further details on time scale analysis, we refer the reader to the works by Bohner and Peterson [4, 5, 6] which provide an extensive review of time scale calculus.

Definition 2.1. [4]. A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the set of all real numbers $\mathbb{R}$.

Definition 2.2. [4]. Let $\mathbb{T}$ represent a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined for any $\zeta \in \mathbb{T}$ as follows:

$$\sigma(\zeta) = \inf\{\zeta_1 \in \mathbb{T} : \zeta_1 > \zeta\}.$$

Similarly, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined for each $\zeta \in \mathbb{T}$ as follows:

$$\rho(\zeta) = \sup\{\zeta_2 \in \mathbb{T} : \zeta_2 < \zeta\}.$$

Definition 2.3. [4].

(i) The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(\zeta) = \sigma(\zeta) - \zeta.$$

(ii) The backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ is defined by

$$\nu(\zeta) = \zeta - \rho(\zeta).$$

Definition 2.4. [4]. Let $\omega : \mathbb{T} \to \mathbb{R}$ be a function defined on a time scale $\mathbb{T}$. Then we define the function $\omega^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$\omega^\sigma(\zeta) = (\omega \circ \sigma)(\zeta) = \omega(\sigma(\zeta)), \quad \zeta \in \mathbb{T},$$

and the function $\omega^\rho : \mathbb{T} \to \mathbb{R}$ by

$$\omega^\rho(\zeta) = (\omega \circ \rho)(\zeta) = \omega(\rho(\zeta)), \quad \zeta \in \mathbb{T}.$$

Definition 2.5. [6]. A function $\omega : \mathbb{T} \to \mathbb{R}$ is called the ∇-derivative of ω at a point $\zeta \in \mathbb{T}_\kappa$ if, for any $\epsilon > 0$, there exists a neighbourhood N around ζ such that for all $\zeta_1 \in N$, we have

$$\left| \frac{[\omega(\rho(\zeta)) - \omega(\zeta_1)] - \omega^\nabla(\zeta)[\rho(\zeta) - \zeta_1]}{|\rho(\zeta) - \zeta_1|} \right| \leq \epsilon.$$

In this context $\omega^\nabla(\zeta)$ is referred to as ∇-derivative of $\omega(\zeta)$ at ζ.

Definition 2.6. (See [4], Definition 8.42). A function $\Omega : \mathbb{T} \to \mathbb{R}$ is called a ∇-integral of function $\omega : \mathbb{T} \to \mathbb{R}$ if, $\Omega^\nabla(\zeta) = \omega(\zeta)$ holds for all $\zeta \in \mathbb{T}_\kappa$. We then define the integral of ω by

$$\int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla(\zeta) = \Omega(\tau_2) - \Omega(\tau_1) \quad \text{for all } \zeta \in \mathbb{T}.$$

Definition 2.7. (Ld-Continuity, [see [4], Definition 8.43]). A function $\omega : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous if it is continuous at all left-dense points in $\mathbb{T}$ and possesses finite right-hand limits at the right-dense points. The set of all such left-dense continuous functions is represented by $C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R})$.

Theorem 2.8. *(Existence of a nabla Antiderivative, [see [4], Theorem 8.45]). Every ld-continuous function has a nabla antiderivative.*

Theorem 2.9. *(see [4], Theorem 8.46). If $\zeta \in \mathbb{T}_\kappa$ and $\omega \in C_{ld}(\mathbb{T}, \mathbb{R})$, then*

$$\int_{\rho(\zeta)}^{\zeta} \omega(\bar{\zeta})\, \nabla\bar{\zeta} = \nu(\zeta)\, \omega(\zeta), \tag{2.1}$$

where
$\nu(\zeta) = \zeta - \rho(\zeta)$.

Theorem 2.10. *(see [4], Theorem 8.48). Assume $\tau_1, \tau_2 \in \mathbb{T}$ and $\omega : \mathbb{T} \to \mathbb{R}$ is ld-continuous.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \int_{\tau_1}^{\tau_2} \omega(\zeta)d\zeta,$$

where the integral on the right is the Riemann integral from calculus.

(ii) If $\mathbb{T}$ consists of only isolated points, then

$$\int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \begin{cases} \sum_{\zeta\in(\tau_1,\tau_2]} \omega(\zeta)\nu(\zeta) & \text{if } \tau_1 < \tau_2, \\ 0 & \text{if } \tau_1 = \tau_2, \\ -\sum_{\zeta\in(\tau_2,\tau_1]} \omega(\zeta)\nu(\zeta) & \text{if } \tau_1 > \tau_2. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$*, where* $h > 0$*, then*

$$\int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \begin{cases} \sum_{\bar{\zeta}=\frac{\tau_1+h}{h}}^{\frac{\tau_2}{h}} \omega(\bar{\zeta}h)h & \textit{if } \tau_1 < \tau_2, \\ 0 & \textit{if } \tau_1 = \tau_2, \\ -\sum_{\bar{\zeta}=\frac{\tau_2+h}{h}}^{\frac{\tau_1}{h}} \omega(\bar{\zeta}h)h & \textit{if } \tau_1 > \tau_2. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$*, then*

$$\int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \begin{cases} \sum_{\bar{\zeta}=\tau_1+1}^{\tau_2} \omega(\bar{\zeta}) & \textit{if } \tau_1 < \tau_2, \\ 0 & \textit{if } \tau_1 = \tau_2, \\ -\sum_{\bar{\zeta}=\tau_2+1}^{\tau_1} \omega(\bar{\zeta}) & \textit{if } \tau_1 > \tau_2. \end{cases}$$

Some relationship between time-scale calculus $\mathbb{T}$, continuous calculus $\mathbb{R}$ and discrete calculus $\mathbb{Z}$ on time scales.

(i) If $\mathbb{T} = \mathbb{R}$, then $\rho(\zeta) = \zeta$, $\nu(\zeta) = 0$,

$$\omega^{\nabla}(\zeta) = \omega'(\zeta), \quad \text{and} \quad \int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \int_{\tau_1}^{\tau_2} \omega(\zeta)\, d\zeta. \tag{2.2}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then $\rho(\zeta) = \zeta - 1$, $\nu(\zeta) = 1$,

$$\omega^{\nabla}(\zeta) = \omega(\zeta) - \omega(\zeta - 1) = \nabla\omega(\zeta), \quad \text{and} \quad \int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \sum_{\bar{\zeta}=\tau_1+1}^{\tau_2} \omega(\bar{\zeta}), \tag{2.3}$$

where ∇ are the backward difference operators.

(iii) If $\mathbb{T} = q^{\mathbb{N}_0} = \{\zeta = q^i : i \in \mathbb{N}\} \cup \{0\}$ where $q > 1$, then $\rho(\zeta) = \frac{\zeta}{q}$, $\nu(\zeta) = \frac{\zeta\,(q-1)}{q}$,

$$\omega^{\nabla}(\zeta) = \frac{q\left(\omega(\zeta) - \omega\left(\frac{\zeta}{q}\right)\right)}{(q-1)\zeta} \quad \text{and} \quad \int_{\tau_1}^{\tau_2} \omega(\zeta)\nabla\zeta = \frac{(q-1)}{q} \sum_{i=\log_q \tau_1+1}^{\log_q \tau_2} q^i\omega(q^i). \tag{2.4}$$

Lemma 2.11. *(Chain Rule, [see [10], Lemma 3.1]). Let* $\omega_2 : \mathbb{R} \to \mathbb{R}$ *be a continuous function,* $\omega_2 : \mathbb{T} \to \mathbb{R}$ *be a* ∇*-differentiable function on* $\mathbb{T}^{\kappa}$*, and* $\omega_1 : \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function. Then, there exists a point* d *in the interval* $[\rho(\zeta), \zeta]$ *such that*

$$(\omega_1 \circ \omega_2)^{\nabla}(\zeta) = \omega_1'(\omega_2(d))\,\omega_2^{\nabla}(\zeta). \tag{2.5}$$

Theorem 2.12. *(Integration By Parts, [see [4], Theorem 8.47]). Let* $\tau_1, \tau_2 \in \mathbb{T}$ *and* $\omega_1, \omega_2 : \mathbb{T} \to \mathbb{R}$ *are ld-continuous functions, then*

$$\int_{\tau_1}^{\tau_2} \omega_1(\zeta)\,\omega_2^{\nabla}(\zeta)\nabla\zeta = \omega_1(\zeta)\,\omega_2(\zeta)|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \omega_1^{\nabla}(\zeta)\,\omega_2^{\rho}(\zeta)\nabla\zeta. \tag{2.6}$$

Theorem 2.13. *(Reverse Hölder's Inequality, [see [9], Theorem 2.3.6]). Let* $\tau_1, \tau_2 \in \mathbb{T}$ *and* $\omega_1, \omega_2 \in C_{ld}([\tau_1, \tau_2], \mathbb{R})$*, we have*

$$\int_{\tau_1}^{\overline{\tau_2}} |\omega_1(\zeta)\omega_2(\zeta)|\,\nabla\zeta \geq \left(\int_{\tau_1}^{\tau_2} |\omega_1(\zeta)|^{\gamma}\,\nabla\zeta\right)^{\frac{1}{\gamma}} \left(\int_{\tau_1}^{\tau_2} |\omega_2(\zeta)|^{\gamma'}\,\nabla\zeta\right)^{\frac{1}{\gamma'}}, \tag{2.7}$$

where $\gamma < 0$ *or* $\gamma' < 0$ *and* $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$.

3 Main Results

In this section, we present several lemmas that are essential for proving the main results of this paper. Throughout this study, we assume that ω_1 and ω_2 are ld-continuous functions on the interval $[\tau_1, \tau_2]_{\mathbb{T}}$. Also, by γ' we denote the conjugate exponent of γ, i.e., $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Furthermore, we assume the existence of positive constants K_1 and K_2 such that

$$\frac{\rho(\zeta) - \tau_1}{\zeta - \tau_1} \geq \frac{1}{K_1}, \quad \zeta > \tau_1, \tag{3.1}$$

and

$$\frac{\rho(\zeta) - \frac{1}{\tau_2}}{\zeta - \frac{1}{\tau_2}} \geq \frac{1}{K_2}, \quad \zeta > \frac{1}{\tau_2}. \tag{3.2}$$

Lemma 3.1. *Let $\tau_1, \tau_2 \in \mathbb{T}$, with $\gamma < 0$ and $r > 1$. Suppose that ω_1 is a ld-continuous and ∇-integrable function on $[\tau_1, \tau_2]_{\mathbb{T}}$. Then*

$$\Omega^{\gamma}(\zeta) \leq \left(\frac{\gamma}{1-r}\right)^{\gamma-1} \left((\zeta - \tau_1)^{\frac{r-1}{\gamma}} - (\tau_2 - \tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \int_{\zeta}^{\tau_2} \left(\rho(\bar{\zeta}) - \tau_1\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\bar{\zeta})\, \nabla\bar{\zeta}, \tag{3.3}$$

where
$\Omega(\zeta) = \int_{\zeta}^{\tau_2} \omega_1(\bar{\zeta})\, \nabla\bar{\zeta}$.

Proof. Let $\gamma < 0$ and $r > 1$. By applying the reverse Hölder's inequality (2.7) for exponents satisfying $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$, we obtain

$$\begin{aligned}
\Omega(\zeta) &= \int_{\zeta}^{\tau_2} \omega_1(\bar{\zeta})\, \nabla\bar{\zeta} \\
&= \int_{\zeta}^{\tau_2} (\rho(\bar{\zeta}) - \tau_1)^{-\frac{1+\gamma-r}{\gamma\gamma'}} (\rho(\bar{\zeta}) - \tau_1)^{\frac{1+\gamma-r}{\gamma\gamma'}} \omega_1(\bar{\zeta})\, \nabla\bar{\zeta} \\
&\geq \left(\int_{\zeta}^{\tau_2} (\rho(\bar{\zeta}) - \tau_1)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\bar{\zeta})\, \nabla\bar{\zeta}\right)^{\frac{1}{\gamma}} \left(\int_{\zeta}^{\tau_2} (\rho(\bar{\zeta}) - \tau_1)^{-\frac{1+\gamma-r}{\gamma}}\, \nabla\bar{\zeta}\right)^{\frac{1}{\gamma'}}.
\end{aligned} \tag{3.4}$$

By applying the chain (2.5) rule to the term $(\bar{\zeta} - \tau_1)^{\frac{r-1}{\gamma}}$, there exists a point $d \in [\rho(\bar{\zeta}), \bar{\zeta}]$ such that

$$\begin{aligned}
\frac{\gamma}{r-1}\left[(\bar{\zeta} - \tau_1)^{\frac{r-1}{\gamma}}\right]^{\nabla} &= (\bar{\zeta} - \tau_1)^{\frac{r-1}{\gamma}-1} \geq (d - \tau_1)^{\frac{r-1}{\gamma}-1} \\
&\geq (\rho(\bar{\zeta}) - \tau_1)^{\frac{r-1}{\gamma}-1}.
\end{aligned} \tag{3.5}$$

Since $\gamma < 0$ and $r > 1$, it follows that $\frac{r-1}{\gamma} - 1 < 0$ and $d \leq \rho(\bar{\zeta})$. Then, by applying inequality (3.5), we obtain

$$\begin{aligned}
\int_{\zeta}^{\tau_2} (\rho(\bar{\zeta}) - \tau_1)^{-\frac{1+\gamma-r}{\gamma}}\, \nabla\bar{\zeta} &\geq \left(\frac{\gamma}{r-1}\right) \int_{\zeta}^{\tau_2} \left[(\bar{\zeta} - \tau_1)^{\frac{r-1}{\gamma}}\right]^{\nabla} \nabla\bar{\zeta} \\
&= \left(\frac{\gamma}{r-1}\right) \left((\tau_2 - \tau_1)^{\frac{r-1}{\gamma}} - (\zeta - \tau_1)^{\frac{r-1}{\gamma}}\right).
\end{aligned} \tag{3.6}$$

By substituting inequality (3.6) into inequality (3.4), we get

$$\Omega(\zeta) \geq \left(\frac{\gamma}{r-1}\right)^{\frac{1}{\gamma'}} \left((\tau_2 - \tau_1)^{\frac{r-1}{\gamma}} - (\zeta - \tau_1)^{\frac{r-1}{\gamma}}\right)^{\frac{1}{\gamma'}} \left(\int_{\zeta}^{\tau_2} (\rho(\bar{\zeta}) - \tau_1)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\bar{\zeta})\, \nabla\bar{\zeta}\right)^{\frac{1}{\gamma}}. \tag{3.7}$$

For $\gamma < 0$, the inequality (3.7) simplifies to

$$\Omega^{\gamma}(\zeta) \leq \left(\frac{\gamma}{1-r}\right)^{\gamma-1} \left((\zeta-\tau_1)^{\frac{r-1}{\gamma}} - (\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\int_{\zeta}^{\tau_2} (\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right),$$

which satisfies inequality (3.3).

□

Lemma 3.2. *Let* $\tau_1, \tau_2 \in \mathbb{T}$, *with* $\gamma < 0$ *and* $r < 1$. *Suppose that* ω_1 *is a ld-continuous and* ∇*-integrable function on* $[\tau_1, \tau_2]_{\mathbb{T}}$. *Then*

(i) If $\frac{r-1}{\gamma} \leq 1$, *then*

$$\Omega^{\gamma}(\zeta) \leq \left(\frac{\gamma}{r-1}\right)^{\gamma-1} \left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \int_{\tau_1}^{\zeta} \left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}. \tag{3.8}$$

(ii) If $\frac{r-1}{\gamma} \geq 1$, *then*

$$\Omega^{\gamma}(\zeta) \leq \left(\frac{\gamma}{r-1}\right)^{\gamma-1} \left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \int_{\tau_1}^{\zeta} \left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}, \tag{3.9}$$

where
$\Omega(\zeta) = \int_{\tau_1}^{\zeta} \omega_1(\overline{\zeta})\,\nabla\overline{\zeta}$.

Proof. **Case 1**: For $\frac{r-1}{\gamma} \leq 1$.

Let $\gamma < 0$ and $r < 1$. By applying reverse Hölder's inequality (2.7) for exponents satisfying $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$, we obtain

$$\begin{aligned}\Omega(\zeta) &= \int_{\tau_1}^{\zeta} \omega_1(\overline{\zeta})\,\nabla\overline{\zeta} \\ &\geq \left(\int_{\tau_1}^{\zeta} \left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)^{\frac{1}{\gamma}} \left(\int_{\tau_1}^{\zeta} \left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{-\frac{1+\gamma-r}{\gamma}} \nabla\overline{\zeta}\right)^{\frac{1}{\gamma'}}.\end{aligned} \tag{3.10}$$

By applying the chain rule (2.5) to the term $\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}$, there exists a point $d \in [\rho(\overline{\zeta}), \overline{\zeta}]$ such that

$$\begin{aligned}\frac{\gamma}{r-1}\left[\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right]^{\nabla} &= \left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}-1} \geq \left(d-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}-1} \\ &\geq \left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}-1}.\end{aligned} \tag{3.11}$$

Since $\frac{r-1}{\gamma} - 1 \leq 0$ and $d \leq \rho(\overline{\zeta})$, then from inequality (3.11), we have

$$\begin{aligned}\int_{\tau_1}^{\zeta} \left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{-\frac{1+\gamma-r}{\gamma}} \nabla\overline{\zeta} &\geq \left(\frac{\gamma}{r-1}\right) \int_{\tau_1}^{\zeta} \left[\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right]^{\nabla} \nabla\overline{\zeta} \\ &= \left(\frac{\gamma}{r-1}\right)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right).\end{aligned} \tag{3.12}$$

Substituting inequality (3.12) into inequality (3.10), we get

$$\Omega(\zeta) \geq \left(\frac{\gamma}{r-1}\right)^{\frac{1}{\gamma'}} \left(\left(\zeta - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1 - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\frac{1}{\gamma'}} \left(\int_{\tau_1}^{\zeta} \left(\rho(\overline{\zeta}) - \frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\, \nabla\overline{\zeta}\right)^{\frac{1}{\gamma}}. \tag{3.13}$$

For $\gamma < 0$, the inequality (3.13) simplifies to

$$\Omega^{\gamma}(\zeta) \leq \left(\frac{\gamma}{r-1}\right)^{\gamma-1} \left(\left(\zeta - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1 - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \int_{\tau_1}^{\zeta} \left(\rho(\overline{\zeta}) - \frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\, \nabla\overline{\zeta},$$

which satisfies inequality (3.8).

Case 2: For $\frac{r-1}{\gamma} \geq 1$.

$$\begin{aligned} \Omega(\zeta) &= \int_{\tau_1}^{\zeta} \omega_1(\overline{\zeta})\, \nabla\overline{\zeta} \\ &\geq \left(\int_{\tau_1}^{\zeta} \left(\overline{\zeta} - \frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\, \nabla\overline{\zeta}\right)^{\frac{1}{\gamma}} \left(\int_{\tau_1}^{\zeta} \left(\overline{\zeta} - \frac{1}{\tau_2}\right)^{-\frac{1+\gamma-r}{\gamma}} \nabla\overline{\zeta}\right)^{\frac{1}{\gamma'}}. \end{aligned} \tag{3.14}$$

Since $\frac{r-1}{\gamma} - 1 \geq 0$ and $\overline{\zeta} \geq d$, then from inequality (3.11), we have

$$\begin{aligned} \int_{\tau_1}^{\zeta} \left(\overline{\zeta} - \frac{1}{\tau_2}\right)^{-\frac{1+\gamma-r}{\gamma}} \nabla\overline{\zeta} &\geq \left(\frac{\gamma}{r-1}\right) \int_{\tau_1}^{\zeta} \left[\left(\overline{\zeta} - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right]^{\nabla} \nabla\overline{\zeta} \\ &= \left(\frac{\gamma}{r-1}\right) \left(\left(\zeta - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1 - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right). \end{aligned} \tag{3.15}$$

Substituting inequality (3.15) into inequality (3.14), we get

$$\Omega(\zeta) \geq \left(\frac{\gamma}{r-1}\right)^{\frac{1}{\gamma'}} \left(\left(\zeta - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1 - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\frac{1}{\gamma'}} \left(\int_{\tau_1}^{\zeta} \left(\overline{\zeta} - \frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\, \nabla\overline{\zeta}\right)^{\frac{1}{\gamma}}. \tag{3.16}$$

For $\gamma < 0$, the inequality (3.16) simplifies to

$$\Omega^{\gamma}(\zeta) \leq \left(\frac{\gamma}{r-1}\right)^{\gamma-1} \left(\left(\zeta - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1 - \frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \int_{\tau_1}^{\zeta} \left(\overline{\zeta} - \frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\, \nabla\overline{\zeta},$$

which satisfies inequality (3.9).

□

We now formulate the time scale version of inequality (1.9) in the context of nabla calculus.

Theorem 3.3. *Assume $\tau_1, \tau_2 \in \mathbb{T}$, with $\gamma < 0$, $r > 1$ and ω_1, ω_2 be ld-continuous and ∇-integrable functions such that $\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}$ is non-decreasing on $[\tau_1, \tau_2]_{\mathbb{T}} \subseteq (0, \infty)$. If inequality (3.1) is satisfied, then* *(i) For* $\frac{1-r}{\gamma} \geq 1$,

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)} \nabla\zeta \\ &\leq \left(\frac{\gamma}{1-r}\right)^{\gamma} K_1^{\frac{r-1}{\gamma'}} \int_{\tau_1}^{\tau_2} \omega_2{}^{-r}(\zeta) \left(1-\left(\frac{\zeta-\tau_1}{\tau_2-\tau_1}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1} \left((\rho(\zeta)-\tau_1)\,\omega_1(\zeta)\right)^{\gamma} \nabla\zeta.\end{aligned} \tag{3.17}$$

(ii) For $\frac{1-r}{\gamma} \leq 1$,

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)} \nabla\zeta \\ &\leq \left(\frac{\gamma}{1-r}\right)^{\gamma} K_1^{r} \int_{\tau_1}^{\tau_2} \omega_2{}^{-r}(\zeta) \left(1-\left(\frac{\zeta-\tau_1}{\tau_2-\tau_1}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1} \left((\rho(\zeta)-\tau_1)\,\omega_1(\zeta)\right)^{\gamma} \nabla\zeta,\end{aligned} \tag{3.18}$$

where
$\Omega(\zeta) = \int_{\zeta}^{\tau_2} \omega_1(\overline{\zeta})\,\nabla\overline{\zeta}$.

Proof. By using Lemma 3.1, we obtain that

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)} \nabla\zeta \leq \left(\frac{\gamma}{1-r}\right)^{\gamma-1} &\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta) \left((\zeta-\tau_1)^{\frac{r-1}{\gamma}} - (\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \\ &\left(\int_{\zeta}^{\tau_2} (\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} \omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right) \nabla\zeta.\end{aligned} \tag{3.19}$$

Applying integration by parts (2.6), we get

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta) \left((\zeta-\tau_1)^{\frac{r-1}{\gamma}} - (\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\int_{\zeta}^{\tau_2} (\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} \omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right) \nabla\zeta \\ &= \omega_2(\zeta)\left(\int_{\zeta}^{\tau_2} (\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} \omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\Bigg|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} (\rho(\zeta)-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} \omega_1{}^{\gamma}(\zeta)\,\omega_2^{\rho}(\zeta)\,\nabla\zeta,\end{aligned} \tag{3.20}$$

where

$$\omega_1(\zeta) = \int_{\zeta}^{\tau_2} (\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} \omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}, \quad \omega_1{}^{\nabla}(\zeta) = -\,(\rho(\zeta)-\tau_1)^{\frac{1+\gamma-r}{\gamma'}} \omega_1{}^{\gamma}(\zeta),$$

$$\omega_2{}^{\nabla}(\zeta) = \omega_2^{-r}(\zeta)\left((\zeta-\tau_1)^{\frac{r-1}{\gamma}} - (\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}$$

and

$$\omega_2(\zeta) = \int_{\infty}^{\zeta} \omega_2{}^{-r}(\overline{\zeta})\left((\overline{\zeta}-\tau_1)^{\frac{r-1}{\gamma}} - (\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \nabla\overline{\zeta}.$$

Since $\omega_1(\tau_2) = 0$, then inequality (3.20) becomes

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left((\zeta-\tau_1)^{\frac{r-1}{\gamma}}-(\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\zeta}^{\tau_2}(\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}\,\omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\
&=\int_{\tau_1}^{\tau_2}(\rho(\zeta)-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}\omega_1{}^{\gamma}(\zeta)\left(\int_{\infty}^{\rho(\zeta)}\omega_2^{-r}(\overline{\zeta})\left((\overline{\zeta}-\tau_1)^{\frac{r-1}{\gamma}}-(\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\right)\nabla\zeta \\
&=\int_{\tau_1}^{\tau_2}(\rho(\zeta)-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}\omega_1{}^{\gamma}(\zeta)\left(\int_{\infty}^{\rho(\zeta)}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\right)\nabla\zeta.
\end{aligned} \tag{3.21}$$

Note that by using (2.1), we get

$$\begin{aligned}
&\int_{\infty}^{\rho(\zeta)}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta} \\
&=\int_{\infty}^{\zeta}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}-\int_{\rho(\zeta)}^{\zeta}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1} \\
&\quad\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta} \\
&=\int_{\infty}^{\zeta}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}-\eta(\zeta)\left(\frac{(\zeta-\tau_1)}{\omega_2(\zeta)}\right)^{r}(\zeta-\tau_1)^{\frac{1-r}{\gamma}-1} \\
&\quad\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}.
\end{aligned} \tag{3.22}$$

Since $\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}$ and $\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}$ are non-decreasing functions, then $\overline{\zeta}\leq\zeta$, it follows that

$$\begin{aligned}
\int_{\infty}^{\zeta}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\leq\left(\frac{(\zeta-\tau_1)}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \\
\int_{\infty}^{\zeta}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta}.
\end{aligned} \tag{3.23}$$

From inequalities (3.22) and (3.23), we have

$$\begin{aligned}
&\int_{\infty}^{\rho(\zeta)}\left(\frac{(\overline{\zeta}-\tau_1)}{\omega_2(\overline{\zeta})}\right)^{r}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\left(1-\left(\frac{\tau_2-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta} \\
&\leq\left(\frac{(\zeta-\tau_1)}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\int_{\infty}^{\zeta}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta}-\eta(\zeta)\left(\frac{(\zeta-\tau_1)}{\omega_2(\zeta)}\right)^{r}(\zeta-\tau_1)^{\frac{1-r}{\gamma}-1} \\
&\quad\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \\
&=\left(\frac{(\zeta-\tau_1)}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\infty}^{\rho(\zeta)}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta}\right).
\end{aligned} \tag{3.24}$$

By substituting inequality (3.24) into inequality (3.21), we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left((\zeta-\tau_1)^{\frac{r-1}{\gamma}}-(\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\zeta}^{\tau_2}(\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}\,{\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\
&\le \int_{\tau_1}^{\tau_2}(\rho(\zeta)-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}{\omega_1}^{\gamma}(\zeta)(\zeta-\tau_1)^{r}{\omega_2}^{-r}(\zeta)\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \\
&\quad\left(\int_{\infty}^{\rho(\zeta)}(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\,\nabla\overline{\zeta}\right)\nabla\zeta. \qquad (3.25)
\end{aligned}$$

To complete proof, we have two cases:

Case 1: If $\frac{1-r}{\gamma}\ge 1$.

Applying the chain rule (2.5) to the term $(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}}$, there exists a point $d\in[\rho(\overline{\zeta}),\overline{\zeta}]$, such that

$$\begin{aligned}
\frac{\gamma}{1-r}\left[(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}}\right]^{\nabla} &= (\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1} \ge (d-\tau_1)^{\frac{1-r}{\gamma}-1} \\
&\ge (\rho(\overline{\zeta})-\tau_1)^{\frac{1-r}{\gamma}-1}. \qquad (3.26)
\end{aligned}$$

Since $\frac{1-r}{\gamma}\ge 1$, it follows that $\frac{1-r}{\gamma}-1\ge 0$ and $d\ge\rho(\overline{\zeta})$. Then by applying inequality (3.26), we obtain

$$\begin{aligned}
\int_{\infty}^{\rho(\zeta)}(\rho(\overline{\zeta})-\tau_1)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta} &\le \left(\frac{\gamma}{1-r}\right)\int_{\infty}^{\rho(\zeta)}\left[(\overline{\zeta}-\tau_1)^{\frac{1-r}{\gamma}}\right]^{\nabla}\nabla\overline{\zeta} \\
&= \left(\frac{\gamma}{1-r}\right)\frac{1}{(\rho(\zeta)-\tau_1)^{\frac{r-1}{\gamma}}}. \qquad (3.27)
\end{aligned}$$

Multiplying and dividing on right hand side of inequality (3.25), by the term $\left(\frac{\rho(\overline{\zeta})-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{1-r}{\gamma}-1}$, we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left((\zeta-\tau_1)^{\frac{r-1}{\gamma}}-(\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\zeta}^{\tau_2}(\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}\,{\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\
&\le \left(\frac{\rho(\overline{\zeta})-\tau_1}{\overline{\zeta}-\tau_1}\right)^{\frac{1-r}{\gamma}-1}\int_{\tau_1}^{\tau_2}(\rho(\zeta)-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}{\omega_1}^{\gamma}(\zeta)(\zeta-\tau_1)^{r}{\omega_2}^{-r}(\zeta)\left(1-\left(\frac{\tau_2-\tau_1}{\zeta-\tau_1}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \\
&\quad\left(\int_{\infty}^{\rho(\zeta)}(\rho(\overline{\zeta})-\tau_1)^{\frac{1-r}{\gamma}-1}\,\nabla\overline{\zeta}\right)\nabla\zeta. \qquad (3.28)
\end{aligned}$$

Using inequalities (3.27), (3.28) and (3.1), we have

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left((\zeta-\tau_1)^{\frac{r-1}{\gamma}}-(\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\zeta}^{\tau_2}(\rho(\overline{\zeta})-\tau_1)^{\frac{1+\gamma-r}{\gamma'}}\,{\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\
&\le \left(\frac{\gamma}{1-r}\right)K_1^{\frac{r-1}{\gamma'}}\int_{\tau_1}^{\tau_2}{\omega_2}^{-r}(\zeta)\left(1-\left(\frac{\zeta-\tau_1}{\tau_2-\tau_1}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1}\left((\rho(\zeta)-\tau_1)\,\omega_1(\zeta)\right)^{\gamma}\nabla\zeta. \qquad (3.29)
\end{aligned}$$

Substituting inequality (3.29) into inequality (3.19), we get

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)}\,\nabla\zeta\\ &\leq \left(\frac{\gamma}{1-r}\right)^{\gamma} K_1{}^{\frac{r-1}{\gamma'}} \int_{\tau_1}^{\tau_2} \omega_2(\zeta)^{-r}\left(1-\left(\frac{\zeta-\tau_1}{\tau_2-\tau_1}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta)-\tau_1\right)\omega_1(\zeta)\right)^{\gamma}\,\nabla\zeta,\end{aligned}$$

which satisfies inequality (3.17).

Case 2: If $\frac{1-r}{\gamma} \leq 1$.

Since $\gamma < 0$ and $\frac{1-r}{\gamma} \leq 1$, it follows that $\frac{1-r}{\gamma} - 1 \leq 0$ and $d \geq \bar{\zeta}$. Then by applying inequality (3.26), we obtain

$$\begin{aligned}\int_{\infty}^{\rho(\zeta)} (\bar{\zeta}-\tau_1)^{\frac{1-r}{\gamma}-1}\,\nabla\bar{\zeta} &\leq \left(\frac{\gamma}{1-r}\right)\int_{\infty}^{\rho(\zeta)}\left[(\bar{\zeta}-\tau_1)^{\frac{1-r}{\gamma}}\right]^{\nabla}\nabla\bar{\zeta}\\ &= \left(\frac{\gamma}{1-r}\right)\frac{1}{\left(\rho(\zeta)-\tau_1\right)^{\frac{r-1}{\gamma}}}.\end{aligned} \tag{3.30}$$

Using inequalities (3.30) and (3.25), we have

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left((\zeta-\tau_1)^{\frac{r-1}{\gamma}} - (\tau_2-\tau_1)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\zeta}^{\tau_2}\left(\rho(\bar{\zeta})-\tau_1\right)^{\frac{1+\gamma-r}{\gamma'}}\omega_1{}^{\gamma}(\bar{\zeta})\,\nabla\bar{\zeta}\right)\nabla\zeta\\ &\leq \left(\frac{\gamma}{1-r}\right)\left(\frac{\zeta-\tau_1}{\rho(\zeta)-\tau_1}\right)^{r}\int_{\tau_1}^{\tau_2}\omega_2{}^{-r}(\zeta)\left(1-\left(\frac{\zeta-\tau_1}{\tau_2-\tau_1}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1}\left(\left(\rho(\zeta)-\tau_1\right)\omega_1(\zeta)\right)^{\gamma}\,\nabla\zeta.\end{aligned} \tag{3.31}$$

Substituting inequality (3.31) into inequality (3.19), we get

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)}\,\nabla\zeta\\ &\leq \left(\frac{\gamma}{1-r}\right)^{\gamma} K_1{}^{r} \int_{\tau_1}^{\tau_2} \omega_2{}^{-r}(\zeta)\left(1-\left(\frac{\zeta-\tau_1}{\tau_2-\tau_1}\right)^{\frac{1-r}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta)-\tau_1\right)\omega_1(\zeta)\right)^{\gamma}\,\nabla\zeta,\end{aligned}$$

which satisfies inequality (3.18). □

Remark 3.4. If the time scale $\mathbb{T} = \mathbb{R}$ in Theorem 3.3 and $\tau_1 = 0$, then $\rho(\zeta) = \zeta$. Consequently, we see that inequality (3.1) holds with $K_1 = 1$. As a result, inequalities (3.17) and (3.18) simplifies to inequality (1.9), and for $\tau_2 = \infty$, we obtain inequality (1.6).

Corollary 3.5. *If* $\mathbb{T} = \mathbb{N}_0$ *in Theorem* 3.3 *with* $\tau_1 = 0$, $\tau_2 = \infty$ *and* ω_1, ω_2 *are positive sequences such that* $\frac{\zeta}{\omega_2(\zeta)}$ *is a non-decreasing on* $[\tau_1, \infty)$ *then, by using relation (2.3), we have*

(i) If $\frac{1-r}{\gamma} \geq 1$, *then*

$$\sum_{\zeta=1}^{\infty} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)} \leq \left(\frac{\gamma}{1-r}\right)^{\gamma} 2^{\frac{r-1}{\gamma'}} \sum_{\zeta=1}^{\infty} (\zeta-1)^{\gamma}\,\omega_2{}^{-r}(\zeta)\,\omega_1{}^{\gamma}(\zeta). \tag{3.32}$$

(ii) If $\frac{1-r}{\gamma} \leq 1$, *then*

$$\sum_{\zeta=1}^{\infty} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)} \leq \left(\frac{\gamma}{1-r}\right)^{\gamma} 2^{r} \sum_{\zeta=1}^{\infty} (\zeta-1)^{\gamma}\,\omega_2{}^{-r}(\zeta)\,\omega_1{}^{\gamma}(\zeta). \tag{3.33}$$

where

$$\Omega(\zeta) = \sum_{\bar{\zeta}=\zeta+1}^{\infty} \omega_1(\bar{\zeta}) \quad \textit{and} \quad \frac{\rho(\zeta)-\tau_1}{\zeta-\tau_1} = \frac{\zeta-1}{\zeta} = 1-\frac{1}{\zeta} \geq \frac{1}{2}, \quad \zeta > \tau_1.$$

Thus inequality (3.1), holds with $K_1 = 2$ *and* $\zeta \geq 2$.

Corollary 3.6. *If* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$. $\tau_1 = 1$, $\tau_2 = \infty$ *and* ω_1, ω_2 *are positive sequences such that* $\frac{\zeta-1}{\omega_2(\zeta)}$ *is a non-increasing on* $[\tau_1, \infty)$ *then, by using relation (2.4) , we have*

(i) If $\frac{1-r}{\gamma} \geq 1$, *then*

$$\sum_{i=log_q\tau_1+1}^{\infty} q^i \frac{\Omega^{\gamma}(q^i)}{\omega_2^r(q^i)} \leq \left(\frac{\gamma}{1-r}\right)^{\gamma} q^{\frac{r-1}{\gamma'}} \sum_{i=log_q\tau_1+1}^{\infty} q^i \, {\omega_2}^{-r}(q^i) \left((q^{i-1}-1)\,\omega_1(q^i)\right)^{\gamma}. \tag{3.34}$$

(ii) If $\frac{1-r}{\gamma} \leq 1$, *then*

$$\sum_{i=log_q\tau_1+1}^{\infty} q^i \frac{\Omega^{\gamma}(q^i)}{\omega_2^r(q^i)} \leq \left(\frac{\gamma}{1-r}\right)^{\gamma} q^r \sum_{i=log_q\tau_1+1}^{\infty} q^i \, {\omega_2}^{-r}(q^i) \left((q^{i-1}-1)\,\omega_1(q^i)\right)^{\gamma}, \tag{3.35}$$

where

$$\Omega(q^i) = \frac{(q-1)}{q} \sum_{i=log_q\zeta+1}^{\infty} q^i \, \omega_1(q^i) \quad \textit{and} \quad \frac{\rho(\zeta)-\tau_1}{\zeta-\tau_1} = \frac{q^{i-1}-1}{q^i-1} = \lim_{i\to\infty} \frac{q^{i-1}-1}{q^i-1} \geq \frac{1}{q}.$$

Thus the inequality (3.1) holds with $K_1 = q$, *where* $q > 1$.

We now formulate the time scale version of inequality (1.10) in the context of nabla calculus.

Theorem 3.7. *Assume* $\tau_1, \tau_2 \in \mathbb{T}$, *with* $\gamma < 0$, $0 \leq r < 1$ *and* ω_1, ω_2 *be ld-continuous and* ∇-*integrable functions such that* $\frac{\left(\bar{\zeta}-\frac{1}{\tau_2}\right)}{\omega_2(\bar{\zeta})}$ *is non-increasing on* $[\tau_1, \tau_2]_{\mathbb{T}} \subseteq (0, \infty)$. *If inequality (3.2) is satisfied, then*

(i) For $\frac{r-1}{\gamma} \leq 1$,

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)} \nabla\zeta \leq \left(\frac{\gamma}{r-1}\right)^{\gamma} {K_2}^r \int_{\tau_1}^{\tau_2} {\omega_2}^{-r}(\zeta) \left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta)-\frac{1}{\tau_2}\right)\omega_1(\zeta)\right)^{\gamma} \nabla\zeta. \tag{3.36}$$

(ii) For $\frac{r-1}{\gamma} \geq 1$,

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^r(\zeta)} \nabla\zeta \leq \left(\frac{\gamma}{r-1}\right)^{\gamma} {K_2}^{\frac{r-1}{\gamma}} \int_{\tau_1}^{\tau_2} {\omega_2}^{-r}(\zeta) \left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta)-\frac{1}{\tau_2}\right)\omega_1(\zeta)\right)^{\gamma} \nabla\zeta, \tag{3.37}$$

where
$\Omega(\zeta) = \int_{\tau_1}^{\zeta} \omega_1(\bar{\zeta})\,\nabla\bar{\zeta}$.

Proof. For $\frac{r-1}{\gamma} \leq 1$.

By using Lemma 3.2, we obtain that

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)}\,\nabla\zeta \leq \left(\frac{\gamma}{r-1}\right)^{\gamma-1} \int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta. \tag{3.38}$$

Applying integration by parts (2.6), we get

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\ &= \omega_2(\zeta)\left(\int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\Bigg|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2}\left(\rho(\zeta)-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\,\omega_2^{\rho}(\zeta)\,\nabla\zeta,\end{aligned} \tag{3.39}$$

where

$$\omega_1(\zeta) = \int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}, \quad {\omega_1}^{\nabla}(\zeta) = \left(\rho(\zeta)-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta),$$

$${\omega_2}^{\nabla}(\zeta) = \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}$$

and

$$\omega_2(\zeta) = -\int_{\zeta}^{\infty} \omega_2^{-r}(\overline{\zeta})\left(\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}.$$

Since $\omega_1(\tau_1) = 0$, then inequality (3.39) becomes

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\ &= \int_{\tau_1}^{\tau_2}\left(\rho(\zeta)-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\left(\int_{\rho(\zeta)}^{\infty}\omega_2^{-r}(\overline{\zeta})\left(\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}} - \left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\right)\nabla\zeta \\ &= \int_{\tau_1}^{\tau_2}\left(\rho(\zeta)-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\left(\int_{\rho(\zeta)}^{\infty}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\right)\nabla\zeta.\end{aligned} \tag{3.40}$$

Note that by using (2.1), we get

$$\begin{aligned}
&\int_{\rho(\zeta)}^{\infty}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\\
=&\int_{\rho(\zeta)}^{\zeta}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}+\int_{\zeta}^{\infty}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\\
&\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\\
=&\eta(\zeta)\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}+\int_{\zeta}^{\infty}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\\
&\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}.
\end{aligned}$$

Since the function $\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}$ and $\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}$ are non-increasing functions, then $\overline{\zeta}\geq\zeta$, it follows that

$$\begin{aligned}
&\int_{\rho(\zeta)}^{\infty}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\\
\leq&\,\eta(\zeta)\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}+\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\\
&\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\int_{\zeta}^{\infty}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\nabla\overline{\zeta}\\
=&\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\rho(\zeta)}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\nabla\overline{\zeta}+\int_{\zeta}^{\infty}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\nabla\overline{\zeta}\right)\\
=&\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\int_{\rho(\zeta)}^{\infty}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta}.
\end{aligned}\tag{3.41}$$

Using inequalities (3.40) and (3.41), we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}}{\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta\\
\leq&\int_{\tau_1}^{\tau_2}\left(\rho(\zeta)-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}}{\omega_1}^{\gamma}(\zeta)\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\\
&\left(\int_{\rho(\zeta)}^{\infty}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta}\right)\nabla\zeta.
\end{aligned}\tag{3.42}$$

Apply the chain rule (2.5) to the term $(\overline{\zeta}-\frac{1}{\tau_2})^{\frac{1-r}{\gamma}}$, there exists a point $d\in[\rho(\overline{\zeta}),\overline{\zeta}]$, such that

$$\begin{aligned}\frac{\gamma}{1-r}\left[\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}}\right]^{\nabla} &= \left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1} \geq \left(d-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1} \\ &\geq \left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1}. \end{aligned} \tag{3.43}$$

Since $\gamma<0$ and $0\leq r<1$, it follows that $\frac{1-r}{\gamma}-1\leq 0$, $d\leq\overline{\zeta}$ and the term $\frac{\gamma}{1-r}<0$, then from inequality (3.43), that

$$\begin{aligned}\int_{\rho(\zeta)}^{\infty}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta} &\leq \left(\frac{\gamma}{1-r}\right)\int_{\rho(\zeta)}^{\infty}\left[\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}}\right]^{\nabla}\nabla\overline{\zeta} \\ &= \left(\frac{\gamma}{r-1}\right)\frac{1}{(\rho(\zeta)-\frac{1}{\tau_2})^{\frac{r-1}{\gamma}}}. \end{aligned} \tag{3.44}$$

Using inequalities (3.42) and (3.44), we have

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\tau_1}^{\zeta}\left(\rho(\overline{\zeta})-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}}\omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta \\ &\leq \frac{\gamma}{r-1}\left(\frac{\zeta-\frac{1}{\tau_2}}{\rho(\zeta)-\frac{1}{\tau_2}}\right)^{r}\int_{\tau_1}^{\tau_2}\omega_2{}^{-r}(\zeta)\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\left(\rho(\zeta)-\frac{1}{\tau_2}\right)\omega_1(\zeta)\right)^{\gamma}\nabla\zeta. \end{aligned} \tag{3.45}$$

Substituting inequality (3.45) into inequality (3.38), we get

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)}\nabla\zeta \\ &\leq\left(\frac{\gamma}{r-1}\right)^{\gamma}K_2{}^{r}\int_{\tau_1}^{\tau_2}\omega_2{}^{-r}(\zeta)\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\left(\rho(\zeta)-\frac{1}{\tau_2}\right)\omega_1(\zeta)\right)^{\gamma}\nabla\zeta,\end{aligned}$$

which is inequality (3.36).

Case 2: For $\frac{r-1}{\gamma}\geq 1$.

By using Lemma 3.2, we obtain that

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)}\nabla\zeta &\leq\left(\frac{\gamma}{r-1}\right)^{\gamma-1}\int_{\tau_1}^{\tau_2}\omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \\ &\quad\left(\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}}\omega_1{}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta. \end{aligned} \tag{3.46}$$

Applying integration by parts (2.6), we get

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta\\ &= \omega_2(\zeta)\left(\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\Bigg|_{\tau_1}^{\tau_2}-\int_{\tau_1}^{\tau_2}\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\,\omega_2^{\rho}(\zeta)\,\nabla\zeta,\end{aligned} \tag{3.47}$$

where

$$\omega_1(\zeta)=\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta},\quad {\omega_1}^{\nabla}(\zeta)=\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta),$$

$${\omega_2}^{\nabla}(\zeta)=\omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}$$

and

$$\omega_2(\zeta)=-\int_{\zeta}^{\infty}\omega_2^{-r}(\overline{\zeta})\left(\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}.$$

Since $\omega_1(\tau_1)=0$, then inequality (3.47) becomes

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta\\ &=\int_{\tau_1}^{\tau_2}\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\left(\int_{\rho(\zeta)}^{\infty}\omega_2^{-r}(\overline{\zeta})\left(\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\right)\nabla\zeta\\ &=\int_{\tau_1}^{\tau_2}\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\left(\int_{\rho(\zeta)}^{\infty}\left(\frac{\overline{\zeta}-\frac{1}{\tau_2}}{\omega_2(\overline{\zeta})}\right)^{r}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma'}-r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\overline{\zeta}-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\nabla\overline{\zeta}\right)\nabla\zeta.\end{aligned} \tag{3.48}$$

By applying inequality (3.41) to inequality (3.48), we obtain

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta\\ &\leq\int_{\tau_1}^{\tau_2}\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\zeta)\left(\frac{\zeta-\frac{1}{\tau_2}}{\omega_2(\zeta)}\right)^{r}\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\rho(\zeta)}^{\infty}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1-r}{\gamma}-1}\nabla\overline{\zeta}\right)\nabla\zeta.\end{aligned} \tag{3.49}$$

Using inequalities (3.44), (3.49) and (3.2), we have

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \omega_2^{-r}(\zeta)\left(\left(\zeta-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}-\left(\tau_1-\frac{1}{\tau_2}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\int_{\tau_1}^{\zeta}\left(\overline{\zeta}-\frac{1}{\tau_2}\right)^{\frac{1+\gamma-r}{\gamma'}} {\omega_1}^{\gamma}(\overline{\zeta})\,\nabla\overline{\zeta}\right)\nabla\zeta\\ &\leq\frac{\gamma}{r-1}\,K_2^{\frac{r-1}{\gamma}}\int_{\tau_1}^{\tau_2}{\omega_2}^{-r}(\zeta)\left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1}\left(\left(\zeta-\frac{1}{\tau_2}\right)\omega_1(\zeta)\right)^{\gamma}\nabla\zeta.\end{aligned} \tag{3.50}$$

By substituting inequality (3.50) into inequality (3.46) and using $\rho(\zeta) \leq \zeta$, we get

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)} \nabla\zeta \\ &\leq \left(\frac{\gamma}{r-1}\right)^{\gamma} K_2^{\frac{r-1}{\gamma}} \int_{\tau_1}^{\tau_2} {\omega_2}^{-r}(\zeta) \left(1-\left(\frac{\tau_1-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta)-\frac{1}{\tau_2}\right)\omega_1(\zeta)\right)^{\gamma} \nabla\zeta, \end{aligned}$$

which is inequality (3.37).

□

Remark 3.8. If the time scale $\mathbb{T} = \mathbb{R}$ in Theorem 3.7 and $\tau_1 = 0$, then $\rho(\zeta) = \zeta$. Consequently, we see that inequality (3.2) holds with $K_2 = 1$. As a result, inequalities (3.36) and (3.37) simplifies to inequality (1.10), and for $\tau_2 = \infty$, we obtain inequality (1.7).

Corollary 3.9. *If* $\mathbb{T} = \mathbb{N}_0$ *in Theorem* 3.7 *with* $\tau_1 = 0$, $\tau_2 = \infty$ *and* ω_1, ω_2 *are positive sequences such that* $\frac{\zeta}{\omega_2(\zeta)}$ *is non-increasing then, by using relation* *(2.3), we have*

(i) If $\frac{r-1}{\gamma} \leq 1$, *then*

$$\sum_{\zeta=1}^{\infty} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)} \leq \left(\frac{\gamma}{r-1}\right)^{\gamma} 2^{r} \sum_{\zeta=1}^{\infty} (\zeta-1)^{\gamma}\, {\omega_2}^{-r}(\zeta)\, {\omega_1}^{\gamma}(\zeta). \tag{3.51}$$

(ii) If $\frac{r-1}{\gamma} \geq 1$, *then*

$$\sum_{\zeta=1}^{\infty} \frac{\Omega^{\gamma}(\zeta)}{\omega_2^{r}(\zeta)} \leq \left(\frac{\gamma}{r-1}\right)^{\gamma} 2^{\frac{r-1}{\gamma}} \sum_{\zeta=1}^{\infty} (\zeta-1)^{\gamma}\, {\omega_2}^{-r}(\zeta)\, {\omega_1}^{\gamma}(\zeta), \tag{3.52}$$

where

$$\Omega(\zeta) = \sum_{\bar{\zeta}=1}^{\zeta} \omega_1(\bar{\zeta}) \quad \textit{and} \quad \frac{\rho(\zeta)-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}} = \frac{\zeta-1}{\zeta} = 1-\frac{1}{\zeta} \geq \frac{1}{2}, \quad \rho(\zeta) > \frac{1}{\tau_2}.$$

Thus the inequality (3.2), holds with $K_2 = 2$ *and* $\zeta \geq 2$.

Corollary 3.10. *If* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$. $\tau_1 = 1$, $\tau_2 = \infty$ *and* ω_1, ω_2 *are positive sequences such that* $\frac{\zeta-1}{\omega_2(\zeta)}$ *is a non-increasing on* $[\tau_1, \infty)$ *then, by using relation (2.4), we have*

(i) If $\frac{r-1}{\gamma} \geq 1$, *then*

$$\sum_{i=log_q\tau_1+1}^{\infty} q^{i} \frac{\Omega^{\gamma}(q^{i})}{\omega_2^{r}(q^{i})} \leq \left(\frac{\gamma}{r-1}\right)^{\gamma} q^{r} \sum_{i=log_q\tau_1+1}^{\infty} q^{i}\, {\omega_2}^{-r}(q^{i}) \left(\left(q^{i-1}-1\right)\omega_1(q^{i})\right)^{\gamma}. \tag{3.53}$$

(ii) If $\frac{r-1}{\gamma} \leq 1$, *then*

$$\sum_{i=log_q\tau_1+1}^{\infty} q^{i} \frac{\Omega^{\gamma}(q^{i})}{\omega_2^{r}(q^{i})} \leq \left(\frac{\gamma}{1-r}\right)^{\gamma} q^{\frac{r-1}{\gamma}} \sum_{i=log_q\tau_1+1}^{\infty} q^{i}\, {\omega_2}^{-r}(q^{i}) \left(\left(q^{i-1}-1\right)\omega_1(q^{i})\right)^{\gamma}, \tag{3.54}$$

where

$$\Omega(q^{i}) = \frac{(q-1)}{q} \sum_{i=1}^{log_q\zeta} q^{i}\, \omega_1(q^{i}) \quad \textit{and} \quad \frac{\rho(\zeta)-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}} = \frac{q^{i-1}}{q^{i}} \geq \frac{1}{q}.$$

Thus the inequality (3.2) holds with $K_2 = q$*, where* $q > 1$.

We now formulate the time scale version of inequality (1.11) in the context of nabla calculus.

Theorem 3.11. *Assume* $\tau_1, \tau_2 \in \mathbb{T}$*, with* $\gamma < 0$*,* $r < 0$ *and* ω_1, ω_2 *be ld-continuous and* ∇*-integrable functions such that* $\frac{\left(\bar{\zeta} - \frac{1}{\tau_2}\right)}{\omega_2(\bar{\zeta})}$ *is non-decreasing on* $[\tau_1, \tau_2]_{\mathbb{T}} \subseteq (0, \infty)$*. If inequality (3.2) is satisfied, then*

(i) If $\frac{1-r}{\gamma} \leq 1$,

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)} \nabla\zeta$$
$$\leq \left(\frac{\gamma}{r-1}\right)^\gamma \int_{\tau_1}^{\tau_2} {\omega_2}^{-r}(\zeta) \left(1 - \left(\frac{\tau_1 - \frac{1}{\tau_2}}{\zeta - \frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta) - \frac{1}{\tau_2}\right) \omega_1(\zeta)\right)^\gamma \nabla\zeta. \tag{3.55}$$

(ii) If $\frac{1-r}{\gamma} \geq 1$,

$$\int_{\tau_1}^{\tau_2} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)} \nabla\zeta$$
$$\leq \left(\frac{\gamma}{r-1}\right)^\gamma {K_2}^{\frac{r-1}{\gamma}} \int_{\tau_1}^{\tau_2} {\omega_2}^{-r}(\zeta) \left(1 - \left(\frac{\tau_1 - \frac{1}{\tau_2}}{\zeta - \frac{1}{\tau_2}}\right)^{\frac{r-1}{\gamma}}\right)^{\gamma-1} \left(\left(\rho(\zeta) - \frac{1}{\tau_2}\right) \omega_1(\zeta)\right)^\gamma \nabla\zeta, \tag{3.56}$$

where
$\Omega(\zeta) = \int_{\tau_1}^{\zeta} \omega_1(\bar{\zeta}) \nabla\bar{\zeta}$

Proof. The proof of Theorem 3.11 is analogous to that of Theorem 3.7, so we omit the details. □

Remark 3.12. If the time scale $\mathbb{T} = \mathbb{R}$ in Theorem 3.11 and $\tau_1 = 0$, then $\rho(\zeta) = \zeta$. Consequently, we see that inequality (3.2) holds with $K_2 = 1$. As a result, inequalities (3.55) and (3.56) simplifies to inequality (1.11), and for $\tau_2 = \infty$, we obtain inequality (1.8).

Corollary 3.13. *If* $\mathbb{T} = \mathbb{N}_0$ *in Theorem* 3.11 *with* $\tau_1 = 0$*,* $\tau_2 = \infty$ *and* ω_1, ω_2 *are positive sequences such that* $\frac{\zeta}{\omega_2(\zeta)}$ *is a non-decreasing then, by using relation (2.3), we have*

(i) If $\frac{r-1}{\gamma} \leq 1$*, then*

$$\sum_{\zeta=1}^{\infty} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)} \leq \left(\frac{\gamma}{r-1}\right)^\gamma \sum_{\zeta=1}^{\infty} (\zeta - 1)^\gamma\, {\omega_2}^{-r}(\zeta)\, {\omega_1}^\gamma(\zeta). \tag{3.57}$$

(ii) If $\frac{r-1}{\gamma} \geq 1$*, then*

$$\sum_{\zeta=1}^{\infty} \frac{\Omega^\gamma(\zeta)}{\omega_2^r(\zeta)} \leq \left(\frac{\gamma}{r-1}\right)^\gamma 2^{\frac{r-1}{\gamma}} \sum_{\zeta=1}^{\infty} (\zeta - 1)^\gamma\, {\omega_2}^{-r}(\zeta)\, {\omega_1}^\gamma(\zeta), \tag{3.58}$$

where

$$\Omega(\zeta) = \sum_{\bar{\zeta}=1}^{\zeta} \omega_1(\bar{\zeta}) \quad \textit{and} \quad \frac{\rho(\zeta) - \frac{1}{\tau_2}}{\zeta - \frac{1}{\tau_2}} = \frac{\zeta - 1}{\zeta} = 1 - \frac{1}{\zeta} \geq \frac{1}{2}.$$

Thus the inequality (3.2), holds with $K_2 = 2$ *and* $\zeta \ge 2$.

Corollary 3.14. *If* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$. $\tau_1 = 1$, $\tau_2 = \infty$ *and* ω_1, ω_2 *are positive sequences such that* $\frac{\zeta-1}{\omega_2(\zeta)}$ *is a non-increasing on* $[\tau_1, \infty)$ *then, by using relation (2.4), we have*

(i) If $\frac{r-1}{\gamma} \ge 1$, *then*

$$\sum_{i=log_q\tau_1+1}^{\infty} q^i \frac{\Omega^\gamma(q^i)}{\omega_2^r(q^i)} \le \left(\frac{\gamma}{r-1}\right)^\gamma \sum_{i=log_q\tau_1+1}^{\infty} q^i \,\omega_2{}^{-r}(q^i)\left(\left(q^{i-1}-1\right)\omega_1(q^i)\right)^\gamma. \tag{3.59}$$

(ii) If $\frac{r-1}{\gamma} \le 1$, *then*

$$\sum_{i=log_q\tau_1+1}^{\infty} q^i \frac{\Omega^\gamma(q^i)}{\omega_2^r(q^i)} \le \left(\frac{\gamma}{1-r}\right)^\gamma q^{\frac{r-1}{\gamma}} \sum_{i=log_q\tau_1+1}^{\infty} q^i \,\omega_2{}^{-r}(q^i)\left(\left(q^{i-1}-1\right)\omega_1(q^i)\right)^\gamma, \tag{3.60}$$

where

$$\Omega(q^i) = \frac{(q-1)}{q}\sum_{i=1}^{log_q\zeta} q^i\,\omega_1(q^i) \quad \textit{and} \quad \frac{\rho(\zeta)-\frac{1}{\tau_2}}{\zeta-\frac{1}{\tau_2}} = \frac{q^{i-1}}{q^i} \ge \frac{1}{q}.$$

Thus the inequality (3.2), holds with $K_2 = q$, *where* $q > 1$.

4 Conclusion

In this paper, we have established new dynamic Hardy-type inequalities with negative exponents using the framework of nabla calculus on time scales. By employing key tools such as the nabla chain rule, integration by parts, and the reverse Hölder's inequality, we extended previously known results in the continuous setting to a broader time scales context. Our findings generalize classical inequalities given by Yang, Benaissa and Azzouz et al., and introduce discrete and quantum analogues for time scales $\mathbb{T} = \mathbb{N}_0$ and $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, which to the best of our knowledge, have not been reported in existing literature.

References

[1] G.H. Hardy, *Note on a theorem of Hilbert*, Math. Z., **6**(3), 314–317, (1920).

[2] G.H. Hardy, *An inequality between integrals*, Messenger of Math, **54**, 150–156, (1925).

[3] J.E. Littlewood and G.H. Hardy, *Elementary theorems concerning power series with positive coefficients and moment constants of positive functions*, J. Reine Angew. Math., **157**, 141–158, (1927).

[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, MA, (2001).

[5] M. Bohner, G. Guseinov and A. Peterson, *Introduction to the Time Scales Calculus*, in *Advances in Dynamic Equations on Time Scales*, Springer, Boston, MA, 1–15, (2003).

[6] D. Anderson, J. Bullock, L. Erbe, A. Peterson and H. Tran, *Nabla dynamic equations*, in *Advances in Dynamic Equations on Time Scales*, Springer, Boston, MA, 47–83, (2003).

[7] P. Řehák, *Hardy inequality on time scales and its application to half-linear dynamic equations*, J. Inequal. Appl., **2005**, 1–13, (2005).

[8] B. Yang, *On a new Hardy-type integral inequality*, Int. Math. Forum, **2**(67), 3317–3322, (2007).

[9] R. Agarwal, D. O'Regan and S. Saker, *Dynamic inequalities on time scales*, Springer, New York, (2014).

[10] A.F. Güvenilir, B. Kaymakçalan and N.N. Pelen, *Constantin's inequality for nabla and diamond-alpha derivative*, J. Inequal. Appl., **2015**, 1–17, (2015).

[11] S.H. Saker, R.R. Mahmoud, M.M. Osman, R.P. Agarwal et al., *Some new generalized forms of Hardy's-type inequality on time scales*, Math. Inequal. Appl., **20**(2), 459–481, (2017).

[12] B. Benaissa and M.Z. Sarikaya, *Generalization of some Hardy-type integral inequality with negative parameter*, Bull. Transilv. Univ. Brasov Ser. III, 69–76, (2020).

[13] A.A. El-Deeb, S.D. Makharesh, E.R. Nwaeze, O.S. Iyiola and D. Baleanu, *On nabla conformable fractional Hardy-type inequalities on arbitrary time scales*, J. Inequal. Appl., **2021**, 1–23, (2021).

[14] E. Awwad and A.I. Saied, *Some new multidimensional Hardy-type inequalities with general kernels on time scales*, J. Math. Inequal., **16**(1), 393–412, (2022).

[15] H.M. Rezk, W. Albalawi, H.A. Abd El-Hamid, A.I. Saied, O. Bazighifan, M.S. Mohamed and M. Zakarya, *Hardy–Leindler-type inequalities via conformable delta fractional calculus*, J. Funct. Spaces, **2022**(1), 2399182, (2022).

[16] A.A. El-Deeb, O. Bazighifan and C. Cesarano, *Important study on the nabla dynamic Hardy–Hilbert-type inequalities on time scales with applications*, Symmetry, **14**(2), 428, (2022).

[17] A.I. Saied, G. AlNemer, M. Zakarya, C. Cesarano and H.M. Rezk, *Some new generalized inequalities of Hardy-type involving several functions on time scale nabla calculus*, Axioms, **11**(12), 662, (2022).

[18] A.A. El-Deeb, *Novel dynamic Hardy-type inequalities on time scales*, Math. Methods Appl. Sci., **46**(5), 5299–5313, (2023).

[19] N. Azzouz, B. Benaissa and A. Senouci, *Some Hardy-type integral inequalities with two parameters of summation*, Ann. Univ. Paedagog. Crac. Stud. Math., **22**(1), 99–109, (2023).

[20] M. Zakarya, A.I. Saied, M. Ali, H.M. Rezk and M.R. Kenawy, *Novel integral inequalities on nabla time scales with C-monotonic functions*, Symmetry, **15**(6), 1248, (2023).

[21] E.S. Aly, Y.A. Madani, F. Gassem, A.I. Saied, H.M. Rezk and W.W. Mohammed, *Some dynamic Hardy-type inequalities with negative parameters on time scales nabla calculus*, AIMS Math., **9**(2), 5147–5170, (2024).

[22] H.M. Rezk, O.S. Balogun and M.E. Bakr, *Unified generalizations of Hardy-type inequalities through the nabla framework on time scales*, Axioms, **13**(10), 723, (2024).

[23] N.Kh. Al-Oushoush, L.E. Azar, E. Awwad, M. Krnić and A.I. Saied, *Some new dynamic inequalities for B-monotone functions with respect to time scales nabla calculus*, J. Inequal. Appl., **2024**(1), 122, (2024).

[24] E. Awwad and A. Saied, *Some weighted dynamic inequalities of Hardy-type with kernels on time scales nabla calculus*, J. Math. Inequal., **18**, 457–475, (2024).

[25] W.M. Hasan, H.M. El-Owaidy, A.A. El-Deeb and H.M. Rezk, *A generalization of some integral inequalities similar to Hardy inequality on time scales*, Kuwait J. Sci., **51**(1), 100–124, (2024).

[26] H.M. Rezk, H.M. El-Owaidy, W.M. Hasan and A.A. El-Deeb, *Dynamic Hardy-type inequalities with convex and non-convex function on time scales*, J. Math. Sci., 1–16, (2025).

[27] M. Bohner, I. Jadlovská and A.I. Saied, *Some new Hardy-type inequalities with negative parameters on time scales*, Qual. Theory Dyn. Syst., **24**(2), 72, (2025).

[28] R. Chauhan and S. Kumar, *Dynamic Hardy-type integral inequalities on time scales via nabla calculus*, Poincaré J. Anal. Appl., **12**(2), 85–101, (2025).

Author information

Suket Kumar, Department of Mathematics and Scientific computing, National Institute of Technology Hamirpur, Hamirpur, India.
E-mail: `suket@nith.ac.in`

Rohit Chauhan, Department of Mathematics and Scientific computing, National Institute of Technology Hamirpur, Hamirpur, India.
E-mail: `chauhanrohit9994@gmail.com`

Received: 2025-06-18
Accepted: 2025-08-14