Palestine Journal of Mathematics

Vol 15(2)(2026) , 225–236

A Study of (R, S)-Semimodules over Hemirings

Dian Ariesta Yuwaningsih and Yassin Dwi Cahyo

Communicated by: Ayman Badawi

MSC 2010 Classifications: Primary 16D10; Secondary 16D20, 16D70, 16Y60, 16D80.

Keywords and phrases: hemirings, semimodules, subsemimodules, bisubsemimodules

The authors would like to thank the reviewers for their valuable comments and suggestions. In addition, the first author expresses her gratitude to Universitas Ahmad Dahlan for funding this research through an internal basic research grant with contract number PD-133/SP3/LPPM-UAD/XI/2024.

Corresponding Author: Dian Ariesta Yuwaningsih

Abstract Let R and S be hemirings, and M a commutative additive monoid. In this paper, we present the construction of an (R, S)-semimodule over hemirings as a generalization of the (R, S)-bisemimodule. We use the hemiring structure because, if the semiring structure were used, the (R, S)-semimodule would actually become an (R, S)-bisemimodule. Furthermore, we present the construction of (R, S)-subsemimodules, cyclic (R, S)-semimodules, and factor (R, S)-semimodules over hemirings, along with an investigation of some of their properties.

1 Introduction

Researchers have generalized ring structures to hemiring and semiring structures. According to [1], a nonempty set R is called a hemiring if $(R, +)$ is a commutative monoid with identity element 0_R; $(R, \cdot)$ is a semigroup; it satisfies both the left and right distributive properties; and it satisfies $0_R r = 0_R = r 0_R$ for every $r \in R$. Furthermore, Golan [1] defines a semiring as a nonempty set R that is a hemiring with a multiplicative identity element 1_R. Some related properties of semirings and hemirings are presented in [1, 2, 3, 4].

On the other hand, the module structure has also been generalized to a semimodule structure. According to [1], a left semimodule over a semiring R is a commutative monoid $(M, +)$ with identity element 0_M, equipped with a scalar multiplication $\_\cdot\_ : R \times M \to M$, defined by $\_\cdot\_\,(r, m) = rm$, for every $r \in R$ and $m \in M$, satisfying the following properties:

(i) $r(m + m') = rm + rm'$

(ii) $(r + r')m = rm + r'm$

(iii) $(rr')m = r(r'm)$

(iv) $1_R m = m$

(v) $r 0_M = 0_M = 0_R m$

for all $r, r' \in R$ and $m, m' \in M$. The definition of a right semimodule over a semiring R is given analogously. If, in the definition of a semimodule above, the semiring structure is replaced with a hemiring structure, then property (iv) is not satisfied because a hemiring does not contain an identity element for the multiplication operation. Further concepts of semimodules over semirings have been discussed in [5, 6, 7, 8]. Next, Golan [1] defines the structure of a bisubsemimodule over semirings. Let R and S be semirings. An (R, S)-bisemimodule M is a left R-semimodule and a right S-semimodule that satisfies the compatibility condition $(rm)s = r(ms)$ for all $m \in M$, $r \in R$, and $s \in S$. Some properties of (R, S)-bisemimodules are presented in [1] and [9]. Furthermore, the structure of (R, S)-bisemimodules can be viewed as a generalization of the structure of (R, S)-bimodules.

The (R, S)-bimodule structure has been generalized to the (R, S)-module structure, which was introduced in [10]. It can be shown that every (R, S)-bimodule is an (R, S)-module, but not vice versa. An (R, S)-module is an (R, S)-bimodule if R and S are both rings with central idempotents. Some related properties of (R, S)-modules are presented in [10, 11, 12].

Until now, there has been no research that generalizes the (R, S)-bisemimodule structure to the (R, S)-semimodule structure. Therefore, in this study, we construct (R, S)-semimodules over hemirings as a generalization of (R, S)-bisemimodules. We choose the hemiring structure because, if the semiring structure is used, (R, S)-semimodules will actually become (R, S)-bisemimodules. Furthermore, this paper presents the construction of (R, S)-subsemimodules, cyclic (R, S)-semimodules, factor (R, S)-semimodules, along with an investigation of their properties.

Throughout this paper, R and S are hemirings, unless stated otherwise.

2 The Concept of (R, S)-Semimodules over Hemirings

In this section, we present the definition of an (R, S)-semimodule over a hemiring and some of its properties.

Definition 2.1. Let R and S be hemirings, and $(M, +)$ a commutative monoid. We define the scalar multiplication function $\_ \cdot\_ \bullet\_ : R \times M \times S \longrightarrow M$ by $\_ \cdot\_ \bullet\_(r, m, s) = r \cdot m \bullet s$ for all $r \in R$, $m \in M$, and $s \in S$. A commutative monoid M is called an (R, S)-semimodule under the scalar multiplication operation $\_ \cdot\_ \bullet\_$ if for every $r, r' \in R$, $m, n \in M$, and $s, s' \in S$, the following conditions are satisfies:

(i) $r \cdot (m + n) \bullet s = r \cdot m \bullet s + r \cdot n \bullet s$

(ii) $(r + r') \cdot m \bullet s = r \cdot m \bullet s + r' \cdot m \bullet s$

(iii) $r \cdot m \bullet (s + s') = r \cdot m \bullet s + r \cdot m \bullet s'$

(iv) $r(r' \cdot m \bullet s)s' = (rr') \cdot m \bullet (ss')$

(v) $r \cdot 0_M \bullet s = 0_M = 0_R \cdot m \bullet 0_S$.

Hereafter, throughout this paper, $r \cdot m \bullet s$ is simply written as rms. Based on the definition of M as an (R, S)-semimodule, it can be shown that a hemiring R is an (R, R)-semimodule under multiplication over the hemiring R. The following are some examples of (R, S)-semimodules over a hemiring.

Example 2.2. Let $(2\mathbb{N}_0, +, \cdot)$ and $(3\mathbb{N}_0, +, \cdot)$ be hemirings, and $(M_{m\times n}(\mathbb{N}_0), +)$ a commutative monoid. We can show that $M_{m\times n}(\mathbb{N}_0)$ is an $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule under the scalar multiplication operation defined as follows:

$$\begin{array}{rcl} \_ \cdot\_ \bullet\_ : 2\mathbb{N}_0 \times M_{m\times n}(\mathbb{N}_0) \times 3\mathbb{N}_0 & \rightarrow & M_{m\times n}(\mathbb{N}_0) \\ (r, [a_{ij}], s) & \mapsto & r \cdot A \bullet s := r[a_{ij}]s = [ra_{ij}s], \end{array}$$

for all $r \in 2\mathbb{N}_0$, $s \in 3\mathbb{N}_0$, and $[a_{ij}] \in M_{m\times n}(\mathbb{N}_0)$.

Example 2.3. Let R and S be hemirings defined by

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in 2\mathbb{N}_0 \right\} \text{ and } S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a, b \in 2\mathbb{N}_0 \right\}.$$

Let $(M_1, +)$ and $(M_2, +)$ be commutative monoids defined by

$$M_1 = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a, b, c \in 2\mathbb{N}_0 \right\} \text{ and } M_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in 2\mathbb{N}_0 \right\}.$$

Both M_1 and M_2 are (R, S)-semimodules under the usual matrix multiplication operation.

Example 2.4. Let $S^\star$ be a hemiring of diagonal 2×2 matrices over $2\mathbb{N}_0$, M a commutative additive monoid of 2×1 vectors over $2\mathbb{N}_0$, and $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a,b \in 2\mathbb{N}_0 \right\}$ a hemiring. Next, we define a hemiring S, derived from the hemiring $S^\star$, by equipping it with the commutative multiplication operation $AB := BA$ for all diagonal matrices $A, B \in S$. Furthermore, we define a scalar multiplication operation $\_ \cdot \_ \bullet \_ : R \times M \times S \to M$ by $\_ \cdot \_ \bullet \_(Y, m, X) = Y \cdot m \bullet X := XYm$ for all matrices $Y \in R$, vectors $m \in M$, and diagonal matrices $X \in S$. Therefore, the commutative monoid M is an (R,S)-semimodule under the scalar multiplication operation $\_ \cdot \_ \bullet \_$ as defined above.

Previously, we explained that (R,S)-semimodules is a generalization of (R,S)-bisemimodules. Thus, every (R,S)-bisemimodule is an (R,S)-semimodule, but the converse is not necessarily true, since there exists an (R,S)-semimodule that is not an (R,S)-bisemimodule.

Example 2.5. Based on Example 2.4, we know that the commutative monoid M is an (R,S)-semimodule. Furthermore, we can show that M is a left R-semimodule under the usual matrix multiplication and a right S-semimodule under the scalar multiplication $\_ * \_ : M \times S \longrightarrow M$ defined by $m * A := Am$ for every vector $m \in M$ and diagonal matrix $A \in S$. Now, let $B \in R$ be any matrix, $m \in M$ a vector, and $A \in S$ a diagonal matrix. We obtain $(Bm) * A = ABm \neq BAm = B(m * A)$. Thus, M is not an (R,S)-bisemimodule.

Example 2.6. Let R and S be hemirings defined by

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a,b \in 2\mathbb{N}_0 \right\} \text{ and } S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a,b \in 2\mathbb{N}_0 \right\}.$$

According to Example 2.3, we have $M_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c \in 2\mathbb{N}_0 \right\}$ which is an (R,S)-semimodule under the usual matrix multiplication operation. We can show that M_2 is a left R-semimodule under the usual matrix multiplication, but it is not a right S-semimodule under the same operation. Let the matrix $\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \in M_2$ and $\begin{pmatrix} 2 & 0 \\ 8 & 0 \end{pmatrix} \in S$. We obtain

$$\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 8 & 0 \end{pmatrix} = \begin{pmatrix} 36 & 0 \\ 16 & 0 \end{pmatrix} \notin M_2$$

Hence, M_2 is not an (R,S)-bisemimodule.

If R is a hemiring, then an element $a \in R$ is called multiplicatively central idempotent if it satisfies $aa = a$ and $ax = xa$ for every element $x \in R$. In order for an (R,S)-semimodule to form an (R,S)-bisemimodule, additional conditions are required, as explained in the following proposition.

Proposition 2.7. *Let M be an (R,S)-semimodule. If the hemirings R and S each have a multiplicatively central idempotents, then M is an (R,S)-bisemimodule.*

Proof. Assume that the hemirings R and S each have a multiplicatively central idempotent. We will show that M is an (R,S)-bisemimodule. The existence of these multiplicatively central idempotents is used to define scalar multiplication operations on M. Let $\alpha \in R$ and $\beta \in S$ be multiplicatively central idempotents. We define the scalar multiplication operation as follows

$$\begin{aligned} \cdot_\alpha \quad : \quad & M \times S \to M \\ & (m,s) \to \cdot_\alpha(m,s) = \alpha m s \end{aligned}$$

and

$$\begin{aligned} \cdot_\beta \quad : \quad & R \times M \to M \\ & (r,m) \to \cdot_\beta(r,m) = r m \beta, \end{aligned}$$

for all $m \in M$, $r \in R$, and $s \in S$.
We will prove that M is a right S-semimodule under the scalar multiplication operation $\cdot_\alpha$. Let $s, s' \in S$ and $m, n \in M$. We obtain

a). $(m+n)s = \alpha(m+n)s = \alpha ms + \alpha ns = ms + ns$

b). $m(s+s') = \alpha m(s+s') = \alpha ms + \alpha ms' = ms + ms'$

c). $(ms)s' = (\alpha ms)s' = \alpha(\alpha ms)s' = \alpha^2 m(ss') = \alpha m(ss') = m(ss')$

d). $0_M s = 0_M = m0_S$.

Thus, M is a right S-semimodule under the scalar multiplication operation $\cdot_\alpha$. Next, we prove that M is a left R-semimodule under the scalar multiplication operation $\cdot_\beta$. Let $r, r' \in R$ and $m, n \in M$. We get

a). $r(m+n) = r(m+n)\beta = rm\beta + rn\beta = rm + rn$

b). $(r+r')m = (r+r')m\beta = rm\beta + r'm\beta = rm + r'm$

c). $r(r'm) = r(r'm\beta) = r(r'm\beta)\beta = (rr')m\beta^2 = (rr')m\beta = (rr')m$

d). $r0_M = 0_M = 0_R m$.

Thus, M is a left R-semimodule under the scalar multiplication operation $\cdot_\beta$.
Now, let $r \in R$, $m \in M$, and $s \in S$. We compute

$$\begin{aligned} r(ms) &= r(\alpha ms) \\ &= r(\alpha ms)\beta \\ &= (r\alpha)m(s\beta) \\ &= (\alpha r)m(\beta s) \\ &= \alpha(rm\beta)s \\ &= \alpha(rm)s \\ &= (rm)s. \end{aligned}$$

Hence, M is an (R, S)-bisemimodule. □

Next, we present the definitions of subtractive and strong sets in an (R, S)-semimodule.

Definition 2.8. Let M be an (R, S)-semimodule and $N \subseteq M$ a nonempty set. The set N is called

(i) a subtractive if, for all $m, m' \in M$, the conditions $m, m+m' \in N$ and $m' \in M$ imply that $m' \in N$.

(ii) a strong if, for all $m, m' \in M$, the condition $m + m' \in N$ implies that $m, m' \in N$.

Below, we provide several examples of subtractive and strong sets of an (R, S)-semimodule.

Example 2.9. Let $6\mathbb{N}_0$ be an $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule. The set $12\mathbb{N}_0$ is both substractive and strong in $6\mathbb{N}_0$.

Example 2.10. Let $M_{m\times n}(\mathbb{N}_0)$ be an $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule. The set $M_{m\times n}(2\mathbb{N}_0)$ is both subtractive and strong in $M_{m\times n}(\mathbb{N}_0)$.

3 On (R, S)-Subsemimodules over Hemirings

In this section, we present the definition of (R, S)-subsemimodules over hemirings, along with some of their properties.

Definition 3.1. Let M be an (R, S)-semimodule. A nonempty subset $N \subseteq M$ is called a subsemimodule of M if N is closed under addition and scalar multiplication in M.

Based on the definition above, a nonempty set $N \subseteq M$ is called a subsemimodule of an (R, S)-semimodule M if N is itself an (R, S)-semimodule under the same addition and scalar multiplication operations as M. In the following, we provide necessary and sufficient conditions for an (R, S)-subsemimodule.

Proposition 3.2. *Let M be an (R, S)-semimodule and $N \subseteq M$ a nonempty set. The set N is a subsemimodule of the (R, S)-semimodule M if and only if it satisfies the following conditions*

(i) $a + b \in N$, for all $a, b \in N$.

(ii) $ras \in N$, for all $r \in R$, $a \in N$, and $s \in S$.

Proof. Assume that N is a subsemimodule of the (R, S)-semimodule M. This means that N is closed under addition and scalar multiplication in M. Consequently, for every $r \in R$, $a, b \in N$, and $s \in S$, we have $a + b \in N$ and $ras \in N$. Conversely, suppose that for every $r \in R$, $a, b \in N$, and $s \in S$ we have $a + b \in N$ and $ras \in N$. Then N is closed under addition and scalar multiplication in M. Furthermore, since $N \subseteq M$, it follows that for all $r, r' \in R$, $m, n \in N$, and $s, s' \in S$, the following properties hold

(i) $r(m + n)s = rms + rns$

(ii) $(r + r')ms = rms + r'ms$

(iii) $rm(s + s') = rms + rms'$

(iv) $r(r'ms)s' = (rr')m(ss')$

(v) $r0_Ms = 0_M = 0_Rm0_S$.

Thus, N is an (R, S)-semimodule under the same scalar multiplication operation as M, and hence N is an (R, S)-subsemimodule of M. □

In the following, we provide several examples of subsemimodules of an (R, S)-semimodule over a hemiring.

Example 3.3. Let M be an (R, S)-semimodule. The sets $\{0_M\}$ and M are the trivial (R, S)-subsemimodules of M.

Example 3.4. Let R and S be hemirings defined by

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in 2\mathbb{N}_0 \right\} \text{ and } S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a, b \in 2\mathbb{N}_0 \right\}.$$

According to Example 2.3, we have $M_1 = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a, b, c \in 2\mathbb{N}_0 \right\}$ which is an (R, S)-semimodule under the usual matrix multiplication. We can show that the nonempty set K of M, with $K = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in 2\mathbb{N}_0 \right\}$, is an (R, S)-subsemimodule of M_1. Furthermore, let $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix} \in K$, $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R$, and $\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \in S$. We compute

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix} = \begin{pmatrix} a + a' & 0 \\ 0 & b + b' \end{pmatrix} \in K$$

and

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} xac + ybd & 0 \\ 0 & 0 \end{pmatrix} \in K.$$

Thus, K is an (R, S)-subsemimodule of M_1.

Every ideal of the hemiring of R is an (R, R)-subsemimodule of R. However, an (R, R)-subsemimodule of R is not necessarily an ideal of the hemiring R. A subsemimodule of the (R, R)-semimodule R is an ideal of the hemiring R if R is a semiring.

Furthermore, we present some properties of (R, S)-subsemimodules over a hemiring. The first property shows that the intersection of an arbitrary (possibly infinite) family of (R, S)-subsemimodules of M is also an (R, S)-subsemimodule of M.

Proposition 3.5. *Let M be an (R, S)-semimodule. If $\{N_i\}_{i\in I}$ is a family of (R, S)-subsemimodules of M, then $\bigcap\limits_{i\in I} N_i$ is also an (R, S)-subsemimodule of M.*

Proof. Since N_i is an (R, S)-subsemimodule of M for every $i \in I$, it follows that $0_M \in N_i$ for all $i \in I$. Thus, $0_M \in \bigcap\limits_{i\in I} N_i$ and consequently $\bigcap\limits_{i\in I} N_i \neq \emptyset$. Let $a, b \in \bigcap\limits_{i\in I} N_i$. Then $a, b \in N_i$ for all $i \in I$. Since each N_i is an (R, S)-subsemimodule of M, it follows that $a + b \in N_i$ for every $i \in I$. Consequently, $a + b \in \bigcap\limits_{i\in I} N_i$. Next, let $r \in R$ and $s \in S$. Again, since each N_i is an (R, S)-subsemimodule of M, we have $ras \in N_i$ for every $i \in I$. Hence, $ras \in \bigcap\limits_{i\in I} N_i$. Therefore, $\bigcap\limits_{i\in I} N_i$ is an (R, S)-subsemimodule of M. □

Next, we show that the sum of an arbitrary (possibly infinite) family of (R, S)-subsemimodules of M is also an (R, S)-subsemimodule of M.

Proposition 3.6. *Let M be an (R, S)-semimodule. If $\{N_i\}_{i\in I}$ is a family of (R, S)-subsemimodule of M, then $\sum\limits_{i\in I} N_i$ is also an (R, S)-subsemimodule of M.*

Proof. Since N_i is an (R, S)-subsemimodule of M for all $i \in I$, we have $0_M \in N_i$ for all $i \in I$. Thus, $0_M \in \sum\limits_{i\in I} N_i$, and consequently $\sum\limits_{i\in I} N_i \neq \emptyset$. Let $a, b \in \sum\limits_{i\in I} N_i$. Then there exist elements $n_i, m_i \in N_i$ for each $i \in I$ such that $a = \sum\limits_{i\in I} n_i$ and $b = \sum\limits_{i\in I} m_i$. Since N_i is an (R, S)-subsemimodule of M, it follows that

$$a + b = \sum_{i\in I} n_i + \sum_{i\in I} m_i = \sum_{i\in I} (n_i + m_i) \in \sum_{i\in I} N_i.$$

Next, let $r \in R$ and $s \in S$. Since each N_i is an (R, S)-subsemimodule of M, we have

$$ras = r\Big(\sum_{i\in I} n_i\Big)s = \sum_{i\in I} rn_is \in \sum_{i\in I} N_i.$$

Therefore, $\sum\limits_{i\in I} N_i$ is an (R, S)-subsemimodule of M. □

In the following, we provide the definition of an k-subsemimodule of (R, S)-semimodules over hemirings, as a generalization of the k-subsemimodule of R-semimodules defined in [6].

Definition 3.7. Let M be an (R, S)-semimodule. An (R, S)-subsemimodule N of M is called an k-subsemimodule if N is a subtractive subsemimodule; that is, if $x, x + y \in N$ and $y \in M$, then $y \in N$.

Example 3.8. According to Example 2.9, we know that the set $12\mathbb{N}_0$ is a subtractive set of the $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule $6\mathbb{N}_0$. Moreover, since $12\mathbb{N}_0$ is also an $(2\mathbb{N}_0, 3\mathbb{N}_0)$-subsemimodule of $6\mathbb{N}_0$, it follows that $12\mathbb{N}_0$ is a k-subsemimodule of $6\mathbb{N}_0$.

Example 3.9. Referring to Example 2.10, we have that the set $M_{m\times n}(2\mathbb{N}_0)$ is a subtractive set of the $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule $M_{m\times n}(\mathbb{N}_0)$. Moreover, since $M_{m\times n}(2\mathbb{N}_0)$ is also an $(2\mathbb{N}_0, 3\mathbb{N}_0)$-subsemimodule of $M_{m\times n}(\mathbb{N}_0)$, it follows that $M_{m\times n}(2\mathbb{N}_0)$ is an k-subsemimodule of $M_{m\times n}(\mathbb{N}_0)$.

Next, we introduce the definition of the multiplication between nonempty subsets of the hemiring R, the hemiring S, and the (R, S)-semimodule M.

Definition 3.10. Let M be an (R, S)-semimodule and $X \subseteq R$, $Y \subseteq M$, and $Z \subseteq S$ be nonempty subsets. The multiplication of the sets X, Y, and Z is defined by

$$XYZ = \left\{ \sum_{i=1}^{n} x_i m_i z_i \mid x_i \in R, m_i \in Y, z_i \in Z, (\forall i = 1, 2, ..., n) \right\}.$$

Since the structure of an (R, S)-semimodule involves two hemirings, the concept of the annihilator is divided into two parts, namely the annihilator over the hemiring R and the annihilator over the hemiring S. In the following, we provide the definition of the annihilator of a nonempty subset of an (R, S)-semimodule M.

Definition 3.11. Let M be an (R, S)-semimodule and X a nonempty subset of M. The annihilator of X over the hemiring R is defined as $(0_M :_R X) = \{r \in R \mid rXS = 0_M\}$. The annihilator of X over the hemiring S is defined as $(0_M :_S X) = \{s \in S \mid RXs = 0_M\}$.

We can show that the annihilator of a set in the (R, S)-semimodule M is generally a subhemiring of each respective hemiring, but not necessarily an ideal. If the hemiring S satisfies $S^2 = S$, then $(0_M :_R X)$ forms a left ideal of R. Likewise, $(0_M :_S X)$ will form a right ideal of S if the hemiring R satisfies $R^2 = R$.

Furthermore, if X is an (R, S)-subsemimodule of M, then $(0_M :_R X)$ and $(0_M :_S X)$ can form ideals in R and S, respectively, under certain additional conditions, as explained in the following proposition.

Proposition 3.12. *Let N be an (R, S)-subsemimodule of M.*

(i) If the hemiring S satisfies $S^2 = S$, then $(0_M :_R N)$ is an ideal of R.

(ii) If the hemiring R satisfies $R^2 = R$, then $(0_M :_S N)$ is an ideal of S.

Proof. (i) Clearly, $(0_M :_R N)$ is a submonoid of R. Next, let $r \in R$ and $a \in (0_M :_R N)$. Then $aNS = 0_M$. Since $S^2 = S$, it follows that $raNS = raNSS = r(aNS)S = 0_M$, so $ra \in (0_M :_R N)$. Furthermore, since N is an (R, S)-subsemimodule and $S^2 = S$, we have $arNS = arNSS = a(rNS)S \subseteq aNS = 0_M$, so $ar \in (0_M :_R N)$. Thus, $(0_M :_R N)$ is an ideal of R.

(ii) Similarly, $(0_M :_S N)$ is a submonoid of S. Next, let $s \in S$ and $a \in (0_M :_S N)$. Then $RNa = 0_M$. Since $R^2 = R$, it follows that $RNas = RRNas = R(RNa)s = 0_M$, so $as \in (0_M :_S N)$. Furthermore, since N is an (R, S)-subsemimodule and $R^2 = R$, we have $RNsa = RRNsa = R(RNs)a \subseteq RNa = 0_M$, so $sa \in (0_M :_S N)$. Thus, $(0_M :_S N)$ is an ideal of S.

□

Furthermore, let M be an (R, S)-semimodule. Then the annihilator of M over the hemiring R is $(0_M :_R M) = \{r \in R \mid rMS = 0_M\}$ and the annihilator of M over the hemiring S is $(0_M :_S M) = \{s \in S \mid RMs = 0_M\}$.

Example 3.13. Let $M_{m\times n}(\mathbb{N}_0)$ be a $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule. The annihilator of $M_{m\times n}(\mathbb{N}_0)$ over the hemiring $2\mathbb{N}_0$ is

$$\left(0_{M_{m\times n}(\mathbb{N}_0)} :_{2\mathbb{N}_0} M_{m\times n}(\mathbb{N}_0)\right) = \left\{x \in 2\mathbb{N}_0 \mid x(M_{m\times n}(\mathbb{N}_0))(3\mathbb{N}_0) = 0_{M_{m\times n}(\mathbb{N}_0)}\right\} = \{0_{2\mathbb{N}_0}\}.$$

Similarly, the annihilator of $M_{m\times n}(\mathbb{N}_0)$ over the hemiring $3\mathbb{N}_0$ is

$$\left(0_{M_{m\times n}(\mathbb{N}_0)} :_{3\mathbb{N}_0} M_{m\times n}(\mathbb{N}_0)\right) = \left\{y \in 3\mathbb{N}_0 \mid (2\mathbb{N}_0)(M_{m\times n}(\mathbb{N}_0))y = 0_{M_{m\times n}(\mathbb{N}_0)}\right\} = \{0_{3\mathbb{N}_0}\}.$$

4 The Construction of Cyclic (R, S)-Semimodules

According to Proposition 3.5, we conclude that the intersection of an arbitrary (possibly infinite) family of (R, S)-subsemimodules of M is itself an (R, S)-subsemimodule of M. This result motivates the definition of an (R, S)-subsemimodule of M generated by a subset of M.

Definition 4.1. Let M be an (R,S)-semimodule and $Y \subseteq M$. We define the set $\langle Y\rangle$ as the intersection of all (R,S)-subsemimodules of M that contain Y. Moreover, $\langle Y\rangle$ is called the (R,S)-subsemimodules of M generated by Y.

Based on the definition above, we know that $\langle Y\rangle$ is an (R,S)-subsemimodule of M that contains Y and is contained in every (R,S)-subsemimodule of M that contains Y. In other words, $\langle Y\rangle$ is the smallest (R,S)-subsemimodule of M containing Y. Next, we present a proposition that describes the structure of the elements in the (R,S)-subsemimodule $\langle Y\rangle$.

Proposition 4.2. *Let M be an (R,S)-semimodule and $Y \subseteq M$. If $Y=\emptyset$, then $\langle Y\rangle=\{0\}$. If $Y\neq\emptyset$, then we have*

$$\langle Y\rangle = \Big\{ \sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j \Big| r_i\in R, y_i, y'_j \in Y, s_i \in S, n_j \in \mathbb{N}_0, (\forall i=1,2,...,t)(\forall j=1,2,...,k)\text{ for some } t,k\in\mathbb{N}\Big\}.$$

Proof. Suppose $Y\neq\emptyset$, and define

$$A = \Big\{ \sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j \mid r_i\in R, y_i, y'_j \in Y, s_i \in S, n_j \in \mathbb{N}_0, (\forall i=1,2,...,t)(\forall j=1,2,...,k)\text{ for some } t,k\in\mathbb{N}\Big\}.$$

We will prove that $\langle Y\rangle = A$. Since $\langle Y\rangle$ is the intersection of all (R,S)-subsemimodules of M containing Y, it is clear that $Y\subseteq\langle Y\rangle$. Because $\langle Y\rangle$ is closed under addition and scalar multiplication by elements of the hemiring R and S, we have $A\subseteq Y\subseteq\langle Y\rangle$. Next, we will show that $\langle Y\rangle\subseteq A$. This is equivalent to showing that A is an (R,S)-subsemimodule of M containing Y. Let $y\in Y$. Then we have $y = 0y0+1y\in A$, so $Y\subseteq A$. Let $\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big), \Big(\sum_{i=1}^{q} r'_i y'_i s'_i + \sum_{j=1}^{l} n'_j y"_j\Big) \in A$. Then

$$\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big) + \Big(\sum_{i=1}^{q} r'_i y'_i s'_i + \sum_{j=1}^{l} n'_j y"_j\Big)$$
$$= \Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{i=1}^{q} r'_i y'_i s'_i\Big) + \Big(\sum_{j=1}^{k} n_j y'_j + \sum_{j=1}^{l} n'_j y"_j\Big) \in A.$$

Next, let $r\in R$ and $s\in S$. Then

$$r\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big)s = \sum_{i=1}^{t} (rr_i) y_i (s_i s) + \sum_{j=1}^{k} (rn_j) y'_j s \in A.$$

Thus, A is an (R,S)-subsemimodule of M containing Y. Hence, $\langle Y\rangle\subseteq A$, and therefore $\langle Y\rangle = A$. □

If we take a singleton set $\{x\}$ for some $x\in M$, then (R,S)-subsemimodules of M generated by $x\in M$ is

$$\langle x\rangle = \{rxs+nx \mid r\in R, s\in S, n\in\mathbb{N}_0\}.$$

Below, we provide the definitions of cyclic (R,S)-semimodules and finitely generated (R,S)-semimodules.

Definition 4.3. An (R,S)-semimodule M is called a cyclic (R,S)-semimodule if there exists an element $x\in M$ such that $M=\langle x\rangle$.

Definition 4.4. An (R, S)-semimodule M is called a finitely generated (R, S)-semimodule if there exists a finite subset $Y \subseteq M$ such that $M = \langle Y \rangle$.

Next, we present several examples of cyclic and finitely generated (R, S)-semimodules.

Example 4.5. Let $6\mathbb{N}_0$ be a $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule. It follows that $6\mathbb{N}_0$ is a cyclic $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule with generator $6 \in 6\mathbb{N}_0$. Hence, $6\mathbb{N}_0 = \langle 6 \rangle$.

Example 4.6. Let $M_2(4\mathbb{N}_0)$ be an $(2\mathbb{N}_0, 2\mathbb{N}_0)$-semimodule. Consider the set $Y \subseteq M_2(4\mathbb{N}_0)$ with $Y = \left\{ \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \right\}$. Then, $M_2(4\mathbb{N}_0)$ is a finitely generated $(2\mathbb{N}_0, 2\mathbb{N}_0)$-semimodule generated by the set Y. Hence, $M_2(4\mathbb{N}_0) = \langle Y \rangle$.

5 The Construction of Factor (R, S)-Semimodules

In this section, we present the definition of a factor (R, S)-semimodule over a hemiring and some of its properties. We begin by defining a relation "$\sim$" on an (R, S)-semimodule.

Definition 5.1. Let M be an (R, S)-semimodule and N an (R, S)-subsemimodule of M. We define a relation "$\sim$" on M by

$$m_1 \sim m_2 \quad \text{if and only if there exist } n_1, n_2 \in N \quad \text{such that } m_1 + n_1 = m_2 + n_2.$$

It can be shown that relation "$\sim$" on an (R, S)-semimodule is an equivalence relation.

Proposition 5.2. *Let M be an (R, S)-semimodule and N an (R, S)-subsemimodule of M. The relation "$\sim$" on M is an equivalence relation.*

Proof. (i) It can be shown that "$\sim$" is reflexive. Let $m \in M$. We prove that $m \sim m$. Since $m + 0_M = m + 0_M$ for some $0_M \in N$, we have $m \sim m$.

(ii) It can be shown that "$\sim$" is symmetric. Let $m_1, m_2 \in M$ with $m_1 \sim m_2$. We prove that $m_2 \sim m_1$. By definition, there exist $n_1, n_2 \in N$ such that $m_1 + n_1 = m_2 + n_2$. Rewriting, $m_2 + n_2 = m_1 + n_1$. Hence, it follows that $m_2 \sim m_1$.

(iii) It can be shown that "$\sim$" is transitive. Let $m_1, m_2, m_3 \in M$ with $m_1 \sim m_2$ and $m_2 \sim m_3$. We prove that $m_1 \sim m_3$. By definition, there exist $n_1, n_2, n_3, n_4 \in N$ such that $m_1 + n_1 = m_2 + n_2$ and $m_2 + n_3 = m_3 + n_4$. Adding n_3 to both sides of the first equation gives $m_1 + n_1 + n_3 = m_2 + n_2 + n_3$. Substituting the second equation, we get $m_1 + n_1 + n_3 = m_3 + n_2 + n_4$. Since $n_1 + n_3, n_2 + n_4 \in N$, it follows that $m_1 \sim m_3$.

Thus, "$\sim$" is an equivalence relation. □

Since "$\sim$" is an equivalence relation on the (R, S)-semimodule M, M is partitioned into equivalence classes. The equivalence class of an element $m \in M$ is the set

$$\begin{aligned} [m] &= \{x \in M \mid x \sim m\} \\ &= \{x \in M \mid x + n_1 = m + n_2, \ \text{for some } n_1, n_2 \in N\}. \end{aligned}$$

Furthermore, for every $m_1, m_2 \in M$, we have $m_1 \sim m_2$ if and only if $[m_1] = [m_2]$. The set of all equivalence classes in M is defined as

$$M/N = \{[m] \mid m \in M\} = \{m + N \mid m \in M\},$$

which we call the partition of the (R, S)-semimodule M.

Since N is a commutative additive submonoid of M, we define the factor monoid M/N under the addition by

$$(a + N) + (b + N) := (a + b) + N,$$

for all $a + N, b + N \in M/N$. The zero element (identity) in M/N is $0 + N$.

Next, we state a theorem showing that the commutative additive monoid M/N forms an (R, S)-semimodule under a suitably defined scalar multiplication.

Theorem 5.3. *Let N be a subsemumodule of the (R, S)-semimodule M. The commutative additive monoid M/N is an (R, S)-semimodule under the scalar multiplication defined by*

$$r(a+N)s := (ras) + N,$$

for all $r \in R$, $s \in S$, and $a + N \in M/N$.
Moreover, M/N is called the factor (R, S)-semimodule of M with respect to N.

Proof. First we show that scalar multiplication is a binary operation. Let $r \in R$, $s \in S$, and $a + N \in M/N$. Then $r(a + N)s = (ras) + N \in M/N$, showing that the scalar multiplication is closed. Next, let $a + N, b + N \in M/N$ with $a + N = b + N$. Then there exist $n_1, n_2 \in N$ such that $a + n_1 = b + n_2$. For any $r \in R$ and $s \in S$, we have $ras + rn_1s = rbs + rn_2s$. Since $rn_1s, rn_2s \in N$, it follows that $ras + N = rbs + N$. Hence, scalar multiplication is well-defined and thus a binary operation. Next, we verify that M/N satisfies the axioms of an (R, S)-semimodule:

(i) For all $a + N, b + N \in M/N$, $r \in R$, and $s \in S$,

$$\begin{aligned} r((a+N)+(b+N))s &= r((a+b)+N)s \\ &= (r(a+b)s)+N \\ &= (ras+rbs)+N \\ &= ((ras)+N)+((rbs)+N) \\ &= (r(a+N)s)+(r(b+N)s). \end{aligned}$$

(ii) For all $a + N \in M/N$, $r, r' \in R$, and $s \in S$,

$$\begin{aligned} (r+r')(a+N)s &= ((r+r')as)+N \\ &= (ras+r'as)+N \\ &= ((ras)+N)+((r'as)+N) \\ &= (r(a+N)s)+(r'(a+N)s). \end{aligned}$$

(iii) For all $a + N \in M/N$, $r \in R$, and $s, s' \in S$,

$$\begin{aligned} r(a+N)(s+s') &= (ra(s+s'))+N \\ &= (ras+ras')+N \\ &= ((ras)+N)+((ras')+N) \\ &= (r(a+N)s)+(r(a+N)s'). \end{aligned}$$

(iv) For all $a + N \in M/N$, $r, r' \in R$, and $s, s' \in S$,

$$\begin{aligned} r(r'(a+N)s)s' &= r((r'as)+N)s' \\ &= (rr'ass')+N \\ &= (rr')(a+N)(ss'). \end{aligned}$$

(v) For all $m \in M$, $r \in R$, and $s \in S$,

$$r(0_M+N)s = (r0_Ms)+N = 0_M+N = (0_Rm0_S)+N = 0_R(m+N)0_S.$$

Thus, by (i)-(v), M/N is an (R, S)-semimodule. □

In the following, we present several examples of factor (R, S)-semimodules.

Example 5.4. Let $6\mathbb{N}_0$ be an $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule and $12\mathbb{N}_0$ a subsemimodule of $6\mathbb{N}_0$. Then, the factor $(2\mathbb{N}_0, 3\mathbb{N}_0)$-semimodule of $6\mathbb{N}_0$ with respect to $12\mathbb{N}_0$ is

$$6\mathbb{N}_0/12\mathbb{N}_0 = \{m + 12\mathbb{N}_0 \mid m \in 6\mathbb{N}_0\} = \{0 + 12\mathbb{N}_0, 6 + 12\mathbb{N}_0\}.$$

Example 5.5. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a,b \in 2\mathbb{N}_0 \right\}$ and $S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a,b \in 2\mathbb{N}_0 \right\}$ be hemirings, $M_1 = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a,b,c \in 2\mathbb{N}_0 \right\}$ an (R,S)-semimodule, and K an (R,S)-subsemimodule of M_1 given by $K = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x,y \in 2\mathbb{N}_0 \right\}$. Then, the factor (R,S)-module of M_1 with respect to K is

$$M_1/K = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + K \mid \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_1 \right\} = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} + K \mid b \in 2\mathbb{N}_0 \right\}.$$

Next, we define the annihilator of a factor (R,S)-semimodule.

Definition 5.6. Let P be a proper (R,S)-subsemimodule of M. The annihilator of M/P over the hemiring R is $(P :_R M) = \{r \in R \mid rMS \subseteq P\}$. Similarly, the annihilator of M/P over the hemiring S is $(P :_S M) = \{s \in S \mid RMs \subseteq P\}$.

Based on Proposition 3.12, it follows that $(P :_R M)$ forms an ideal in R if the hemiring S satisfies the property $S^2 = S$. Likewise, $(P :_S M)$ forms an ideal in S if the hemiring R satisfies the property $R^2 = R$.

Next, we present several properties of factor (R,S)-semimodules over hemirings.

Proposition 5.7. *Let M be an (R,S)-semimodule, and N an (R,S)-subsemimodule of M.*

(i) If $a \in N$, then $a + N = N$.

(ii) If N is an k-subsemimodule of M and $a \in N$, then, for any $b \in M$, $a + N = b + N$ if and only if $b \in N$.

(iii) If N is an k-subsemimodule of M, then, for any $c \in M$, $c + N = N$ if and only if $c \in N$.

Proof. Let M be an (R,S)-semimodule, and N an (R,S)-subsemimodule of M.

(i) Let $a \in N$. Since $a + 0_M = 0_M + a$, we have $a \sim 0_M$. Hence, $a + N = 0_M + N = N$.

(ii) Let $b \in M$ and assume that $a + N = b + N$. Then, there exist $x, y \in N$ such that $a + x = b + y$. Since N is an k-subsemimodule, it follows that $b \in N$. Conversely, if $b \in N$, then $b + N = N$. Since $a \in N$, we also have $a + N = N$, and thus $a + N = b + N$.

(iii) Let $c \in M$ and assume that $c + N = N$. Then, there exist $x, y \in N$ such that $c + x = 0_M + y$. Since N is an k-subsemimodule of M, it follows that $c \in N$. Conversely, if $c \in N$, then clearly $c + N = N$.

□

Proposition 5.8. *Let M be an (R,S)-semimodule.*

(i) If N and K are (R,S)-subsemimodules of M with $N \subseteq K$, then $K/N = \{k + N \mid k \in K\}$ forms an (R,S)-subsemimodule of M/N.

(ii) If K is an k-subsemimodule of an (R,S)-semimodule M, then K/N forms an k-subsemimodule of the (R,S)-semimodule M/N.

Proof. Let M be an (R,S)-semimodule.

(i) Since K is an (R,S)-subsemimodule of M, we have $0_M + N \in K/N$, so $K/N \neq \emptyset$. Let $m + N, m' + N \in K/N$, $r \in R$, and $s \in S$. Then

$$(m + N) + (m' + N) = (m + m') + N \in K/N$$

and

$$r(m + N)s = (rms) + N \in K/N.$$

Hence, K/N is an (R,S)-subsemimodule of M/N.

(ii) Let K be an k-subsemimodule of M. Since K/N is an (R,S)-subsemimodule of M/N, it remains to show that K/N satisfies the k-subsemimodule condition. Let $u+N, v+N \in M/N$ with $u+N, (u+N)+(v+N) \in K/N$. We prove that $v+N \in K/N$. Since $(u+N)+(v+N) \in K/N$, it follows that $(u+v)+N \in K/N$. This means there exists $k \in K$ such that $(u+v)+N = k+N$. Hence, there exist $x, y \in N$ such that $u+v+x = k+y$. Since K is a k-subsemimodule and $N \subseteq K$, it follows that $v \in K$, and therefore $v+N \in K/N$. Thus, K/N is a k-subsemimodule of M/N.

□

6 Conclusion remarks

This paper presents the structure of (R,S)-semimodules over hemirings as a generalizaion of (R,S)-bisemimodules and their properties. This study introduces a new algebraic structure, expected to provide a reference for future researchers to explore its specific properties.

References

[1] J.S. Golan, *Semirings and their Applications*, Springer, Netherlands, (1999).

[2] Md.Y. Ali, K.R. Chowdhury, A. Sultana, N.K. Mitra, *Some Structures of Hemirings*, Pure and Applied Mathematics Journal, **6**(1), 45–50, (2017).

[3] M.B. Baghirath, K.F. Pawar, and R.P. Deore, *On Radicals of Polynomial Semirings*, Palestine Journal of Mathematics, **11**(3), 289–294, (2022).

[4] C. Abdiolu, K. Charrabi, and A. Mamouni, *Differential identities on weakly left cancellative semirings with derivations*, Palestine Journal of Mathematics, **13**(3), 186–193, (2024).

[5] R.E. Atani, *Prime Subsemimodules of Semimodules*, International Journal of Algebra, **4**(26), 1299–1306, (2010).

[6] F. Fatah and R. Safakish, *ϕ-prime Subsemimodules of Semimodules over Commutative Semirings*, Kyungpook Math. J, **60**, 445–453, (2020).

[7] B. Pareigis and H. Rohl, *Remarks on Semimodules*, arXiv:1305.5531, 1–33, (2013), https://doi.org/10.48550/arXiv.1305.5531.

[8] S. Swomin, *k-Content Semimodule*, Palestine Journal of Mathematics, **13**(3), 138–146, (2024).

[9] J.S. Golan, *Semirings and Affine Equations over Them: Theory and Applications*, Springer Science+Business Media Dordrecht, Netherlands, (2003).

[10] T. Khumprapussorn, S. Pianskool, and M. Hall, *(R,S)-Modules and Their Fully and Jointly Prime Submodules*, International Mathematical Forum, **7**(33), 1631–1643, (2013).

[11] Yuwaningsih, D.A., Wijayanti, I.E., and Prasetyo, P.W., *On (R,S)-Module Homomorphisms*, Journal of Physics: Conference Series, **1188**(1), 012114, (2019).

[12] Yuwaningsih, D.A. and Wijayanti, I.E., *On Left Weakly Jointly Prime (R,S)-Modules*, Algebraic Structures and Their Applications, **11**(4), 273–285, (2024).

Author information

Dian Ariesta Yuwaningsih, Department of Mathematics Education, Universitas Ahmad Dahlan, Bantul 55166, Indonesia.
E-mail: `dian.ariesta@pmat.uad.ac.id`

Yassin Dwi Cahyo, Department of Mathematics, Universitas Diponegoro, Semarang 50275, Indonesia.
E-mail: `yassindwicahyo@gmail.com`

Received: 2025-06-22
Accepted: 2025-10-27