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Cubes-difference factor absorbing ideals of a commutative ring

Faranak Farshadifar

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Corresponding Author: F. Farshadifar

Abstract. Let R be a commutative ring with $1 \neq 0$. The purpose of this paper is to introduce and investigate the cubes-difference factor absorbing ideals of R as a generalization of prime ideals. We say that a proper ideal I of R is a *cubes-difference factor absorbing ideal* (*cdf-absorbing ideal*) of R if whenever $a^3 - b^3 \in I$ for $a, b \in R$, then $a - b \in I$ or $a^2 + ab + b^2 \in I$.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity $1 \neq 0$ and $\mathbb{Z}$ will denote the ring of integers. Also, $\mathbb{Z}_n$ will denote the ring of integers modulo n for positive integer n.

A proper ideal P of R is said to be a *prime ideal* if $ab \in P$ for some $a, b \in R$, then either $a \in P$ or $b \in P$ [9]. Theory of prime ideals is an important tool in classical algebraic geometry. In development of algebraic geometry, some generalizations for the concept of prime ideals has arisen for some examples see [1, 2, 3, 5, 6, 7, 10, 12]. In [5], the authors introduced and studied the notion of square-difference factor absorbing ideals of a commutative ring. A proper ideal I of R is said to be a *square-difference factor absorbing ideal* (*sdf-absorbing ideal*) of R if for $0 \neq a, b \in R$, whenever $a^2 - b^2 \in I$, then $a + b \in I$ or $a - b \in I$ [5].

Motivated by sdf-absorbing ideals, the aim of this paper is to introduce the notion of *cubes-difference factor absorbing ideals* (*cdf-absorbing ideals*) of R as a generalization of prime ideals and obtain some results similar to the results on sdf-absorbing ideals of a commutative ring given in [5]. We say that a proper ideal I of R is a *cubes-difference factor absorbing ideal* (*cdf-absorbing ideal*) of R if whenever $a^3 - b^3 \in I$ for $a, b \in R$, then $a - b \in I$ or $a^2 + ab + b^2 \in I$.

2 Main Results

We begin with the definition.

Definition 2.1. We say that a proper ideal I of R is a *cubes-difference factor absorbing ideal* (*cdf-absorbing ideal*) of R if whenever $a^3 - b^3 \in I$ for $a, b \in R$, then $a - b \in I$ or $a^2 + ab + b^2 \in I$.

Remark 2.2. This is clear by using the fact that $a^3 + b^3 = a^3 - (-b)^3$, an ideal I of R is a cubes-difference factor absorbing ideal of R if and only if whenever $a^3 + b^3 \in I$ for $a, b \in R$, then $a + b \in I$ or $a^2 - ab + b^2 \in I$.

Remark 2.3. Clearly, every prime ideal of R is a cdf-absorbing ideal of R. But the Example 2.14 (a) shows that the converse is not true in general.

The following result is analogous to [5, Theorem 2.2].

Proposition 2.4. Let I be a cdf-absorbing ideal of R. Then for each $a \in R$ we have that $a^3 \in I$ implies either $a^2 \in I$ or $a \in I$.

Proof. Let $a^3 \in I$. If $a = 0$, we are done. So, let $a \neq 0$. Then for each $i \in I$, we have $a^3 - i^3 \in I$. This implies that $a - i \in I$ or $a^2 + ai + i^2 \in I$ since I is a cdf-absorbing ideal of R. Hence $a \in I$ or $a^2 \in I$. □

The following example shows that an ideal I with the property that for each $a \in R$, $a^3 \in I$ implies either $a^2 \in I$ or $a \in I$ need not be a cdf-absorbing ideal. However, the converse of Proposition 2.4 does hold when $\text{char}(R) = 3$.

Example 2.5. Consider the ideal $35\mathbb{Z}$ of the ring $\mathbb{Z}$. Then for $a = 3$ and $b = -2$ of $\mathbb{Z}$ we have $a^3 - b^3 = 35 \in 35\mathbb{Z}$. But $a - b = 5 \notin 35\mathbb{Z}$ and $a^2 + ab + b^2 = 7 \notin 35\mathbb{Z}$. Thus, $35\mathbb{Z}$ is not a cdf-absorbing ideal of $\mathbb{Z}$. But since $35\mathbb{Z}$ is a radical ideal of the ring $\mathbb{Z}$, for each $a \in \mathbb{Z}$, $a^3 \in 35\mathbb{Z}$ implies either $a^2 \in 35\mathbb{Z}$ or $a \in 35\mathbb{Z}$.

The following result is analogous to [5, Theorem 2.4].

Theorem 2.6. Let I be an ideal of R with the property that for each $a \in R$, $a^3 \in I$ implies either $a^2 \in I$ or $a \in I$ and $\text{char}(R) = 3$. Then I is a cdf-absorbing ideal of R.

Proof. Let $a^3 - b^3 \in I$ for $a, b \in R$. Since $\text{char}(R) = 3$, we have $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 \in I$, and thus either $a - b \in I$ or $(a-b)^2 \in I$ by assumption. So, we have $(a-b)^2 \in I$. Since $\text{char}(R) = 3$, we have $a^2 - 2ab = a^2 + ab$. Thus $a^2 + ab + b^2 = a^2 - 2ab + b^2 = (a-b)^2 \in I$. Therefore, I is a cdf-absorbing ideal of R. □

The following result is analogous to [5, Theorem 3.6].

Corollary 2.7. Let R be a commutative von Neumann regular ring with $\text{char}(R) = 3$. Then every proper ideal of R is a cdf-absorbing ideal of R.

Proof. Every proper ideal I of a commutative von Neumann regular ring is a radical ideal and so for each $a \in R$ we have that $a^3 \in I$ implies either $a^2 \in I$ or $a \in I$. Now, the result follows from the Theorem 2.6. □

Example 2.8. (a) All proper non-zero ideals of $\mathbb{Z}_8$ are cdf-absorbing ideals.

(b) Consider the ideal $8\mathbb{Z}$ of the ring $\mathbb{Z}$. Then for $a = 4$ and $b = 2$ of $\mathbb{Z}$ we have $a^3 - b^3 = 56 \in 8\mathbb{Z}$. But $a - b = 2 \notin 8\mathbb{Z}$ and $a^2 + ab + b^2 = 28 \notin 8\mathbb{Z}$. Thus $8\mathbb{Z}$ is not a cdf-absorbing ideal of $\mathbb{Z}$. Also, for this a and b we have the zero ideal of $\mathbb{Z}_8$ is not cdf-absorbing ideal.

(c) For $a = (4, 0)$ and $b = (1, 0)$ of $\mathbb{Z}_9 \times \mathbb{Z}_9$ we have $a^3 - b^3 = (63, 0) = (0, 0) \in \{0\} \times \mathbb{Z}_9$. But $a - b = (3.0) \notin \{0\} \times \mathbb{Z}_9$ and $a^2 + ab + b^2 = (21, 0) \notin \{0\} \times \mathbb{Z}_9$. Thus the ideal $\{0\} \times \mathbb{Z}_9$ is not a cdf-absorbing ideal of $\mathbb{Z}_9 \times \mathbb{Z}_9$. Also, for this a and b, we get that the zero ideal of $\mathbb{Z}_9 \times \mathbb{Z}_9$ is not a cdf-absorbing ideal.

(d) Let $R = \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_9$. Then the ideal $I = \{0\} \times \{0\} \times \mathbb{Z}_9$ is not a cdf-absorbing ideal of R (to see this, let $a = (2, 1, 0), b = (8, 1, 0) \in R$).

(e) Consider the ideal $9\mathbb{Z}$ of the ring $\mathbb{Z}$. Then for $a = 1$ and $b = -2$ of $\mathbb{Z}$ we have $a^3 - b^3 = 9 \in 9\mathbb{Z}$. But $a - b = 3 \notin 9\mathbb{Z}$ and $a^2 + ab + b^2 = 3 \notin 9\mathbb{Z}$. Thus $9\mathbb{Z}$ is not a cdf-absorbing ideal of $\mathbb{Z}$.

The following result is analogous to [5, Theorem 2.5].

Theorem 2.9. Let I be a cdf-absorbing ideal of a commutative ring R. Then the following statements are equivalent:

(a) If $a^3 - b^3 \in I$ for $a, b \in R$, then $(a-b)^2 \in I$ and $a^2 + ab + b^2 \in I$;

(b) $3 \in I$;

(c) $\text{char}(R/I) = 3$.

Proof. $(a) \Rightarrow (b)$ Let $a = b = 1$. Then $a^3 - b^3 = 0 \in I$, and thus $3 = 1 + 1(1) + 1 = a^2 + ab + b^2 \in I$ by hypothesis.

$(b) \Rightarrow (a)$ Assume that $a^3 - b^3 \in I$ for $a, b \in R$. Then $a - b \in I$ or $a^2 + ab + b^2 \in I$ since I is a cdf-absorbing ideal of R. If $a - b \in I$, then $a^2 + ab + b^2 = (a-b)^2 + 3ab \in I$. If $a^2 + ab + b^2 \in I$, then $(a-b)^2 = a^2 - 2ab + b^2 = a^2 + ab + b^2 - 3ab \in I$. Therefore, $(a-b)^2, a^2 + ab + b^2 \in I$.

$(b) \Leftrightarrow (c)$ This is clear. □

Definition 2.10. We say that a proper ideal I of R is a $*$-*prime ideal* if whenever $b(a^2+ab+b^2) \in I$ for $a, b \in R$, then $b \in I$ or $a^2 + ab + b^2 \in I$.

Example 2.11. Consider the ideal $39\mathbb{Z}$ of the ring $\mathbb{Z}$. Then for $a = 1$ and $b = 3$ of $\mathbb{Z}$ we have $b(a^2 + ab + b^2) = 3(13) = 39 \in 39\mathbb{Z}$. But $b = 3 \notin 39\mathbb{Z}$ and $a^2 + ab + b^2 = 13 \notin 39\mathbb{Z}$. Thus $39\mathbb{Z}$ is not a $*$-prime ideal of the ring $\mathbb{Z}$. By Example 2.14 (a) $39\mathbb{Z}$ is a cdf-absorbing ideal of $\mathbb{Z}$.

The following result is analogous to [5, Theorem 2.6].

Theorem 2.12. Let I be an ideal of R and $3 \in U(R)$. Then I is a cdf-absorbing ideal of R if and only if I is a $*$-prime ideal of R.

Proof. First suppose that I is a cdf-absorbing ideal of R and $b(a^2 + ab + b^2) \in I$ for $a, b \in R$. First, assume that $b \neq a$ and $b \neq -a$. Let $x := (a + 2b)/3 \in R$ and $y := (a - b)/3 \in R$. Since $b \neq a$ and $b \neq -a$, we have $x^3 - y^3 = b(a^2 + ab + b^2)/3 \in I$. Thus $b \in I$ or $a^2 + ab + b^2 \in I$ since I is a cdf-absorbing ideal of R. Next, assume that $b = a$ or $b = -a$. Then $b^3 \in I$, and hence $b \in I$ or $b^2 \in I$ by Proposition 2.4.

Conversely, let I be a $*$-prime ideal of R and $a^3 - b^3 \in I$ for some $a, b \in R$. Set $x := (a + 2b)/3 \in R$ and $y := (a - b)/3 \in R$. Then we get that $b = x - y$ and $a = 2y + x$. Thus

$$9y(y^2 + xy + x^2) = (a - b)(a^2 + ab + b^2) = (a^3 - b^3) \in I.$$

Now, since $3 \in U(R)$, we have that $y(y^2+xy+x^2) \in I$ Hence by assumption, $(a-b)/3 = y \in I$ or $(a^2+ab+b^2)/3 = (y^2+xy+x^2) \in I$. Therefore, we get that $(a-b) \in I$ or $(a^2+ab+b^2) \in I$, as needed. □

The following result is analogous to [5, Theorem 2.7].

Theorem 2.13. Let I be a proper ideal of R. Then the following statements are equivalent:

(a) I is a cdf-absorbing ideal of R;

(b) If $b(a^2 + ab + b^2) \in I$ for $a, b \in R \setminus I$, then the system of linear equations $b = Y - X$, $a = Y + 2X$ has no non-zero solution in R (i.e., there are no $x, y \in R$ that satisfy both equations)

Proof. $(a) \Rightarrow (b)$ Suppose that I is a cdf-absorbing ideal of R, $b(a^2+ab+b^2) \in I$ for $a, b \in R \setminus I$, and the system of linear equations $b = Y - X$, $a = Y + 2X$ has a solution in R for some $x, y \in R$. Then $x^3 - y^3 = b(a^2 + ab + b^2) \in I$, but $a = 2x + y = a^2 + ab + b^2 \notin I$ and $y - x = b \notin I$, a contradiction.

$(b) \Rightarrow (a)$ Suppose that $x^3 - y^3 \in I$ for $x, y \in R$. Let $a = 2x + y$ and $b = y - x$. Then $b(a^2 + ab + b^2) = x^3 - y^3 \in I$, and the system of linear equations $2X + Y = a, Y - X = b$ has a solution in R for $x, y \in R$. Thus $2x + y = a^2 + ab + b^2 \in I$ or $y - x = b \in I$, and hence I is a cdf-absorbing ideal of R. □

We next give several examples of cdf-absorbing ideals.

Example 2.14. (a) One can easily verify that a proper ideal I of $\mathbb{Z}$ with the form $I = 3p\mathbb{Z}$, where $p \neq 3$ is prime, is a cdf-absorbing ideal of $\mathbb{Z}$.

(b) Let R be a boolean ring. Then every proper ideal of R is a cdf-absorbing ideal of R since $x^3 = x$ for every $x \in R$.

(c) The ideal $12\mathbb{Z}$ of the ring $\mathbb{Z}$ is a cdf-absorbing ideal of $\mathbb{Z}$.

(d) Let $R = K[X]$, where K is a field, and $I = (X^2 + X + 1)(X - 1)R$. Then since $X^3 - 1^3 = (X - 1)(X^2 + X + 1) \in I$ but $(X - 1) \notin I$ and $(X^2 + X + 1) \notin I$ we have that I is not a cdf-absorbing ideal of R..

The following result is analogous to [5, Theorem 2.9].

Theorem 2.15. Let I be a cdf-absorbing ideal of R and let S be a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then $S^{-1}I$ is a cdf-absorbing ideal of $S^{-1}R$.

Proof. Let $(a/s)^3-(b/t)^3 \in S^{-1}I$ for some $a/s, b/t \in S^{-1}R$. Then there exists $u \in S$ such that $ua^3t^3-ub^3s^3 \in I$. This implies that $(uat)^3-(ubs)^3 \in I$. Thus by assumption $uat-ubs \in I$ or $(uat)^2+u^2atbs+(ubs)^2 \in I$. Therefore, $a/s-b/t \in S^{-1}I$ or $(a/s)^2+(ab)/(st)+(b/t)^2 \in S^{-1}I$, as needed. □

The following result is analogous to [5, Theorem 2.10].

Theorem 2.16. Let $f : R \longrightarrow T$ be a homomorphism of commutative rings. Then we have the following.

(a) If J is a cdf-absorbing ideal of T, then $f^{-1}(J)$ is a cdf-absorbing ideal of R.

(b) If f is surjective and I is a cdf-absorbing ideal of R containing $ker(f)$, then $f(I)$ is a cdf-absorbing ideal of T.

Proof. (a) Let J be a cdf-absorbing ideal of T. Assume that $x, y \in R$ such that $x^3-y^3 \in f^{-1}(J)$. Then $f(x)^3-f(y)^3 \in J$. Thus by assumption, $f(x)-f(y) \in J$ or $f(x)^2+f(xy)+f(y)^2 \in J$. Hence, $f(x-y) \in J$ or $f(x^2+xy+y^2) \in J$. Therefore, $x-y \in f^{-1}(J)$ or $x^2+xy+y^2 \in f^{-1}(J)$, as needed.

(b) Let f be surjective and I be a cdf-absorbing ideal of R containing $ker(f)$. Assume that $x, y \in T$ such that $x^3-y^3 \in f(I)$. As f is surjective, we have $x = f(a), y = f(b)$ for some $a, b \in R$. Hence, $f(a^3-b^3) \in f(I)$. This implies that $a^3-b^3 \in I$ because $ker(f) \subseteq I$. Thus by assumption, $a-b \in I$ or $a^2+ab+b^2 \in I$. Therefore, $x-y \in f(I)$ or $x^2+xy+y^2 \in f(I)$, as required. □

The following result is analogous to [5, Corollary 2.11].

Corollary 2.17. (a) Let $R \subseteq T$ be an extension of commutative rings and J a cdf-absorbing ideal of T. Then $J \cap R$ is a cdf-absorbing ideal of R.

(b) If $J \subseteq I$, then I/J is a cdf-absorbing ideal of R/J if and only if I is a cdf-absorbing ideal of R.

Proof. This follows from Theorem 2.16. □

The following example shows that the "$ker(f) \subseteq I$" hypothesis is needed in Theorem 2.16 (b). Also, this example is analogous to [5, Example 2.12 (b)].

Example 2.18. Let $f : \mathbb{Z}[X] \longrightarrow \mathbb{Z}$ be the epimorphism given by $f(g(X)) = g(0)$. Then $I = (X+8)$ is a prime ideal, and thus a cdf-absorbing ideal, of $\mathbb{Z}[X]$, but $f((X+8)) = 8\mathbb{Z}$ is not a cdf-absorbing ideal of $\mathbb{Z}$ by Example 2.8 (b). Note that $ker(f) = (X) \not\subseteq (X+8) = I$; so the "$ker(f) \subseteq I$" hypothesis is needed in Theorem 2.16 (b).

Theorem 2.19. Let I_1, I_2 be non-zero proper ideals of the commutative rings R_1, R_2, respectively.

(a) If $I_1 \times I_2$ is a cdf-absorbing ideal of $R_1 \times R_2$, then I_1, I_2 are cdf-absorbing ideals of R_1, R_2, respectively. Also, if $3 \notin I_2$ and $\sqrt[3]{i+1} \in R_1$ for some $0 \neq i \in I_1$, then $\sqrt[3]{i+1}-1 \in I_1$.

(b) If I_1, I_2 are cdf-absorbing ideals of R_1, R_2, respectively, and $3 \in I_2$, then for each $(0,0) \neq a = (a_1, a_2), b = (b_1, b_2) \in R$ with $a^3-b^3 \in I$, we have $a^2+ab+b^2 \in I$ or $(a-b)^2 \in I$.

Proof. $(a) \Rightarrow (b)$ Let $I = I_1 \times I_2$ be a cdf-absorbing ideal of $R = R_1 \times R_2$. Then one can see that I_1, I_2 are cdf-absorbing ideals of R_1, R_2, respectively. Next, assume that $\sqrt[3]{i+1} \in R_1$ for some $0 \neq i \in I_1$ and $3 \notin I_2$. Let $a = (\sqrt[3]{i+1}, 1), b = (1,1) \in R$. Then $a^3-b^3 = (i, 0) \in I$; so $(\sqrt[3]{i+1}-1, 0) = a-b \in I$ or $(\sqrt[3]{(i+1)^2}+\sqrt[3]{i+1}+1, 3) = a^2+ab+b^2 \in I$. Since $3 \notin I_2$. we get that $\sqrt[3]{i+1}-1 \in I_1$.

$(b) \Rightarrow (a)$ Assume that $3 \in I_2$. Let $(0,0) \neq a = (a_1, a_2), b = (b_1, b_2) \in R$ with $a^3-b^3 \in I$. Then $a_2^3-b_2^3 \in I_2$, and thus $a_2-b_2 \in I_2$ or $a_2^2+a_2b_2+b_2^2 \in I_2$ since I_2 is a non-zero cdf-absorbing ideal of R_2. Since $3 \in I_2$, we have $a_2^2+a_2b_2+b_2^2, (a_2-b_2)^2 \in I_2$ by Theorem 2.9. Also, $a_1^3-b_1^3 \in I_1$; so $a_1+b_1 \in I_1$ or $a_1^2+a_1b_1+b_1^2 \in I_1$ since I_1 is a non-zero cdf-absorbing ideal of R_1. If $a_1^2+a_1b_1+b_1^2 \in I_1$, then $a^2+ab+b^2 \in I$. If $a_1-b_1 \in I_1$, then $(a_1-b_1)^2 \in I_1$. Therefore, $(a-b)^2 \in I$. □

In the following result, we determine the cdf-absorbing ideals in idealization rings. Recall that for a commutative ring R and R-module M, the *idealization of* R *and* M is the commutative ring $R(+)M = R \times M$ with identity $(1,0)$ under addition defined by $(r,m)+(s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs, rn+sm)$. For more on idealizations, see [8, 13]. Every ideal of $R(+)M$ has the form $I(+)N$ for I an ideal of R and N a submodule of M (see [13, Theorem 25.1(1)]).

The following result is analogous to [5, Theorem 4.16].

Theorem 2.20. Let I a non-zero proper ideal of R, M an R-module, and N a submodule of M. Then we have the following.

(a) If $I(+)N$ is a cdf-absorbing ideal of $R(+)M$, then I is a cdf-absorbing ideal of R and $\{im : i \in I \text{ and } m \in M \setminus N\} \subseteq N$.

(b) If I is a cdf-absorbing ideal of R, then $I(+)M$ is a cdf-absorbing ideal of $R(+)M$.

Proof. (a) Let $I(+)N$ be a cdf-absorbing ideal of $R(+)M$. Then it is easily verified that I is a cdf-absorbing ideal of R. Let $m \in M \setminus N$, $i \in I$, and $a := (i,0), b := (0,m) \in R(+)M$. Then $a^3 - b^3 = (i^3, 0) \in I(+)N$ implies that $a^2+ab+b^2 = (i^2, im) \in I(+)N$ or $a-b = (i,-m) \in I(+)N$. Since $m \in M \setminus N$, we have $a-b = (i,-m) \notin I(+)N$. Thus $a^2+ab+b^2 = (i^2, im) \in I(+)N$. Hence, $im \in N$. Therefore, $\{im : i \in I \text{ and } m \in M \setminus N\} \subseteq N$.

(b) Let I be a cdf-absorbing ideal of R. Suppose that $a^3 - b^3 \in I(+)M$ for $(0,0) \neq a, b \in R(+)M$, where $x = (a, m_1)$ and $y = (b, m_2)$. Since I is a non-zero cdf-absorbing ideal of R and $a^3 - b^3 \in I$, we have $a^2+ab+b^2 \in I$ or $a-b \in I$. If $a^2+ab+b^2 \in I$, then $x^2+xy+y^2 \in I(+)M$. If $a-b \in I$, then $x-y \in I(+)M$. Thus $I(+)M$ is a cdf-absorbing ideal of $R(+)M$. □

The following example shows that it is crucial that I be a non-zero ideal in Theorem 2.20.

Example 2.21. Let $R = \mathbb{Z}_8$, $M = N = \mathbb{Z}_8$, and $I = \{0\}$. Then $\{0\}$ is a cdf-absorbing ideal of $\mathbb{Z}_8$, but $\{0\}(+)\mathbb{Z}_8$ is not a cdf-absorbing ideal of $\mathbb{Z}_8(+)\mathbb{Z}_8$. To see this consider $x = (2,0), y = (0,2)$. Then $a^3 - b^3 = (8,0) = (0,0) \in \{0\}(+)\mathbb{Z}_8$ but $a - b = (2,-2) \notin \{0\}(+)\mathbb{Z}_8$ and $a^2+ab+b^2 = (4,4) \notin \{0\}(+)\mathbb{Z}_8$.

Recall that a proper ideal I of R is said to be *n-semi-absorbing* for the positive integer n if $x^{n+1} \in I$ implies that $x^n \in I$ for any $x \in R$ [4].

Next, we consider when $\{0\}(+)M$ is a cdf-absorbing ideal of $R(+)M$. The following result is analogous to [5, Remark 4.18 (b)].

Proposition 2.22. Let M be a non-zero R-module. Then $\{0\}(+)M$ is a cdf-absorbing ideal of $R(+)M$ if and only if $\{0\}$ is a 2-semi-absorbing ideal of R and $\{0\}$ is a cdf-absorbing ideal of R.

Proof. First assume that $\{0\}(+)M$ is a cdf-absorbing ideal of $R(+)M$. Suppose that $\{0\}$ is not a 2-semi-absorbing ideal of R. Then there exists $a \in R$ such that $a^3 = 0$ but $a^2 \neq 0$. Let $x = (a,0)$ and $y = (0,m)$ for some $m \in M$. Then $x^3 - y^3 = (a^3, 0) = (0,0) \in \{0\}(+)M$. Thus by assumption, $x - y = (a, -m) \in \{0\}(+)M$ or $x^2+xy+y^2 = (a^2, am) \in \{0\}(+)M$, which are contradictions. Thus $\{0\}$ is a 2-semi-absorbing ideal of R. Now, let $a^3 - b^3 \in \{0\}$. Then by consider $x = (a,0)$ and $y = (b,0)$ one can see that $\{0\}$ is a cdf-absorbing ideal of R. Conversely, Assume that $\{0\}$ is a cdf-absorbing ideal of R. Let $x^3 - y^3 \in \{0\}(+)M$ for $x, y \in R(+)M$, where $x = (a, m_1)$ and $y = (b, m_2)$. Since $\{0\}$ is a cdf-absorbing ideal of R and $a^3 - b^3 \in \{0\}$, we have $a^2+ab+b^2 \in \{0\}$ or $a-b \in \{0\}$. If $a^2+ab+b^2 \in \{0\}$, then $x^2+xy+y^2 \in \{0\}(+)M$. If $a-b \in \{0\}$, then $x-y \in \{0\}(+)M$. Thus $\{0\}(+)M$ is a cdf-absorbing ideal of $R(+)M$. □

In the next result, we study cdf-absorbing ideals in amalgamation rings. Let A, B be commutative rings, $f : A \longrightarrow B$ a homomorphism, and J an ideal of B. Recall that the *amalgamation of* A *and* B *with respect to* f *along* J is the subring $A \bowtie_J B = \{(a, f(a)+j) \mid a \in A, j \in J\}$ of $A \times B$ [11].

The following result is analogous to [5, Theorem 4.19.].

Theorem 2.23. Let A and B be commutative rings, $f : A \longrightarrow B$ a homomorphism, J an ideal of B, and I a non-zero proper ideal of A. Then $I \bowtie_J B$ is a cdf-absorbing ideal of $A \bowtie_J B$ if and only if I is a cdf-absorbing ideal of A.

Proof. If $I \bowtie_J B$ is a cdf-absorbing ideal of $A \bowtie_J B$, then it is easy to see that I is a cdf-absorbing ideal of A.

Conversely, assume that I is a non-zero cdf-absorbing ideal of A. Let $x = (a, f(a)+j_1), y = (b, f(b) + j_2) \in A \bowtie_J B$ such that $x^3 - y^3 \in I \bowtie_J B$. Since $a^3 - b^3 \in I$ and I is a non-zero cdf-absorbing ideal of A, we have $a^2 + ab + b^2 \in I$ or $a - b \in I$. If $a^2 + ab + b^2 \in I$, then

$$x^2 + xy + y^2 =$$

$$(a^2 + ab + b^2, f(a^2 + ab + b^2) + j_1(2f(a) + j_1 + f(b)) + j_2(2f(b) + j_2 + f(a))) \in I \bowtie_J B.$$

Similarly, if $a - b \in I$, then

$$x - y = (a - b, f(a - b) + j_1 - j_2) \in I \bowtie_J B.$$

Thus $I \bowtie_J B$ is a cdf-absorbing ideal of $A \bowtie_J B$. □

The following example shows that it is again crucial that I be a non-zero ideal in Theorem 2.23. Also, the following result is analogous to [5, Example 4.20].

Example 2.24. Let $A = B = J = \mathbb{Z}_8$, $f = 1_A : A \longrightarrow A$, and $I = \{0\}$. Then $\{0\}$ is a cdf-absorbing ideal of $\mathbb{Z}_8$. But $\{0\} \bowtie_{\mathbb{Z}_8} \mathbb{Z}_8$ is not a cdf-absorbing ideal of $\mathbb{Z}_8 \bowtie_{\mathbb{Z}_8} \mathbb{Z}_8$. To see this, consider $x = (2, 0), y = (0, 2)$. Then $a^3 - b^3 = (8, 8) = (0, 0) \in \{0\} \bowtie_{\mathbb{Z}_8} \mathbb{Z}_8$ but $a - b = (2, -2) \notin \{0\} \bowtie_{\mathbb{Z}_8} \mathbb{Z}_8$ and $a^2 + ab + b^2 = (4, 4) \notin \{0\} \bowtie_{\mathbb{Z}_8} \mathbb{Z}_8$.

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Author information

Faranak Farshadifar, Department of Mathematics Education,
Farhangian University, P.O. Box 14665-889, Tehran, Iran.
E-mail: f.farshadifar@cfu.ac.ir

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